Photon-Photon Scattering close to the Forward Direction*

HUNG CHENG†
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

TAI TSUN WU
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

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The scattering amplitude for photon-photon elastic scattering in quantum electrodynamics is explicitly obtained for the case where the energy is very large while the momentum transfer is small compared with the rest energy of the electron (divided by c). This amplitude has no infrared divergence and, at high energies, the fourth- and sixth-order contributions are negligible. The explicit answer for the eighth order is remarkably simple. In terms of helicity states, it is found that the amplitudes $RR \to RR$ and $RL \to RL$ are the same, while the amplitude $RR \to LL$ is different but of the same order of magnitude. By the optical theorem, the total cross section for $\gamma+\gamma \to e^-+e^-+e^++e^+$ is found to be 6.5 μb at high energies.

I. INTRODUCTION

In our previous discussion of high-energy behavior[1] of all the two-body elastic scattering amplitudes in quantum electrodynamics with zero-mass photons, attention is directed exclusively to the case where the momentum transfer is fixed and *nonzero* while the energy approaches infinity. This can also be seen to be the case in the detailed considerations[2–4] and in the later developments of simpler methods, through either suitable momentum variables[5] or the impact picture.[6,7]

Because of the long-range nature of the Coulomb field, the behavior of the above-mentioned amplitudes at high energies is in general very much more complicated near the forward direction. In the special case of Delbrück scattering, this point has been discussed in detail before.[4] The same complications also occur in the case of Compton scattering. In both cases, there are two distinct scales for momentum transfer, namely, the electron mass m and m^2/ω, where ω is the photon energy in the center-of-mass (c.m.) system.

It is the purpose of the present paper to point out that this complication does *not* occur for photon-photon scattering, which is, incidentally, the most involved case in the previous treatments.[8] Even in the exactly forward direction, the impact-factor representation[1,2] holds for photon-photon scattering. More precisely, for photon-photon scattering, there is only one scale for momentum transfer, namely, the mass of the electron.

II. IMPACT-FACTOR REPRESENTATION

We first consider the case where the momentum transfer Δ is exactly zero. In this case, we apply the impact-factor representation to obtain[1,2]

$$\mathfrak{M}_0^{(\gamma\gamma)} \sim is(2\pi)^{-2}\int d^2q_\perp\, (\mathbf{q}_\perp^2)^{-2}\mathcal{G}_{ij}^{\gamma}(0,\mathbf{q}_\perp)\mathcal{G}_{i'j'}^{\gamma}(0,\mathbf{q}_\perp) \quad (1)$$

at high energy, where s is the square of the total energy in the c.m. system, i and i' (j and j') are the polarization indices for the incoming (outgoing) photons, and $\mathcal{G}_{ij}^{\gamma}$ is the photon impact factor given by[9]

$$\mathcal{G}_{ij}^{\gamma}(0,\mathbf{q}_\perp) = -8\alpha^2\int_0^1 d\beta\, d\beta'\, \delta(1-\beta-\beta')\int_0^1 dx \times[\mathbf{q}_\perp^2 x(1-x)+m^2]^{-1}\{2\beta\beta' x(1-x)q_{\perp i}q_{\perp j} -\mathbf{q}_\perp^2\delta_{ij}[\tfrac{1}{4}-2\beta\beta'(x-\tfrac{1}{2})^2]\} \quad (2)$$

for this special case. The Feynman diagrams taken into account are shown in Fig. 1. Up to e^8, these are the only important diagrams at high energies. Unlike the Delbrück or Compton case, here it is meaningful to set $\Delta=0$ in the impact-factor representation because, as shown later in this section, the integral on the right-hand side of (1) is convergent and can be explicitly evaluated.

For this case of the forward direction, the photon impact factor is especially simple, because the β

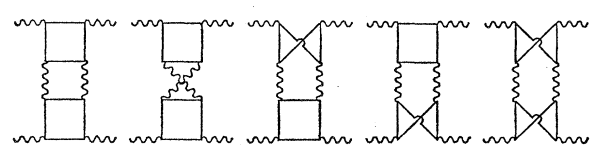

FIG. 1. The five eighth-order Feynman diagrams that contribute at high energies. (Up to the eighth order, the contributions of other diagrams are smaller roughly by a factor s^{-1}.)

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[2] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969).
[3] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1868 (1969).
[4] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1873 (1969).
[5] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1899 (1969).
[6] H. Cheng and T. T. Wu, Phys. Rev. Letters **23**, 1311 (1969).
[7] H. Cheng and T. T. Wu, Phys. Rev. D 1, 1069 (1970); 1, 1083 (1970).
[8] See, for example, Eqs. (4.4)–(4.8) of Ref. 2.
[9] See Eq. (3.4) of Ref. 2.

integration in (2) can be trivially evaluated to give

$$\mathcal{J}_{ij}^{\gamma}(0,\mathbf{q}_\perp)=\tfrac{2}{3}\alpha^2\int_0^1 dx\,[\mathbf{q}_\perp^2x(1-x)+m^2]^{-1}$$
$$\times\{\mathbf{q}_\perp^2\delta_{ij}[3-(1-2x)^2]-4x(1-x)q_{\perp i}q_{\perp j}\}.\quad(3)$$

When (3) is substituted into (1), the angular integration over $\mathbf{q}_\perp$ can be carried out:

$$q_{\perp i}q_{\perp j}\rightarrow\tfrac{1}{2}q_\perp^2\delta_{ij}\quad(4)$$

and

$$q_{\perp i}q_{\perp j}q_{\perp i'}q_{\perp j'}\rightarrow\tfrac{1}{8}(\delta_{ij}\delta_{i'j'}+\delta_{ii'}\delta_{jj'}+\delta_{ij'}\delta_{i'j})(\mathbf{q}_\perp^2)^2.\quad(5)$$

Thus the factor $(\mathbf{q}_\perp^2)^{-2}$ in the integrand of (1) is canceled. This is the reason why the integral of (1) is convergent. After this cancellation, the radial integral over $\mathbf{q}_\perp$ is next carried out. With the variable $y=2x-1$, the result is

$$\mathfrak{M}_0^{(\gamma\gamma)}\sim\delta_{ij}\delta_{i'j'}A+\tfrac{1}{2}(\delta_{ii'}\delta_{jj'}+\delta_{ij'}\delta_{i'j}-\delta_{ij}\delta_{i'j'})A_{\mathrm{ex}},\quad(6)$$

where

$$A=is\alpha^4(9\pi m^2)^{-1}\int_0^1 dy\,dy'\,\frac{(5-y^2)(5-y'^2)}{y'^2-y^2}\ln\frac{1-y^2}{1-y'^2}\quad(7)$$

and

$$A_{\mathrm{ex}}=is\alpha^4(9\pi m^2)^{-1}\int_0^1 dy\,dy'\,\frac{(1-y^2)(1-y'^2)}{y'^2-y^2}\times\ln\frac{1-y^2}{1-y'^2}.\quad(8)$$

It is not very difficult to evaluate the two integrals that appear in (7) and (8). First, interpreted as principal-value integrals, the logarithmic factor in each case can be replaced by $2\ln(1-y^2)$. Secondly, after this replacement, the y' integration can be performed. Remembering that[10]

$$\int_0^1 dy\,y^{-1}[\ln(1-y)]^2=2\zeta(3)\quad(9)$$

and

$$\int_0^1 dy\,y^{-1}[\ln(1+y)]^2=\tfrac{1}{4}\zeta(3),\quad(10)$$

where $\zeta(3)\sim1.2020569$ is the value of the Riemann zeta function at 3, we get

$$A=is\alpha^4(36\pi m^2)^{-1}[175\zeta(3)-38]\quad(11)$$

and

$$A_{\mathrm{ex}}=is\alpha^4(36\pi m^2)^{-1}[7\zeta(3)-6].\quad(12)$$

III. RESULTS

We recall that, in the case of forward Compton scattering from a spin-$\frac{1}{2}$ target, there are just two independent scattering amplitudes, one for the case of parallel spins and the other for antiparallel spins in the direction of motion. In the present case of forward photon-photon scattering, there are three independent amplitudes: one for parallel spins, a second one for elastic scattering with antiparallel spins, and a third one for exchange scattering with antiparallel spins in the direction of motion. More precisely, let R and L denote the two possible helicity states for a photon, the three amplitudes in the forward direction are, respectively,

$$RL\rightarrow RL,\quad RR\rightarrow RR,\quad\text{and}\quad RR\rightarrow LL.$$

Expressed in this way, the first two amplitudes are both given by the A of (11), while the third one is given by the A_{ex} of (12) in the limit of high energies. Note that *this third amplitude*, which is of the same order of magnitude as (although numerically smaller than) the other two amplitudes, *implies the exchange of two units of helicity.*

Although we have so far considered only the case $\Delta=0$, the result (6) with (11) and (12) actually holds for all momentum transfers that satisfy the condition

$$\Delta\ll m.\quad(13)$$

In other words, in the case of photon-photon scattering at high energies, the second scale for the momentum transfer, namely, $m^2/\sqrt{s}$, is of no importance. This is very different from the cases of Delbrück scattering and Compton scattering.

The total cross section σ for photon-photon scattering is related to the photon-photon scattering amplitude $\mathfrak{M}^{(\gamma\gamma)}$ in the forward direction by the optical theorem

$$\sigma(s)=s^{-1}\,\mathrm{Im}\mathfrak{M}^{(\gamma\gamma)}.\quad(14)$$

Thus, up to the eighth order of e, we have

$$\lim_{s\to\infty}\sigma(s)\sim\frac{\alpha^4}{36\pi m^2}[175\zeta(3)-38]\sim6.5\,\mu\mathrm{b},\quad(15)$$

independent of s as well as the helicities of the incoming photons. The right-hand side of (15) is also the total cross section of $\gamma+\gamma\rightarrow e^-+e^-+e^++e^+$.

[10] These integrals have previously appeared in the calculation of the high-energy behavior in quantum electrodynamics. See, in particular, Appendix G of Ref. 4.