Relationship between Nonlinear and Linear Realizations of Chiral $SU(2)\times SU(2)$: Theory and Applications

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The problem of converting nonlinear realizations of $SU(2)\times SU(2)$ has been approached in three different ways. Weinberg has introduced a matrix $\Lambda(\boldsymbol{\pi})$ with well-defined properties under infinitesimal chiral transformations and has shown that it can be used to linearize nonlinear fields. Coleman, Wess, and Zumino have achieved the same result by treating the pion fields as parameters of finite chiral rotations; and the present author has made use of the fact that linear realizations are eigenstates of the Casimir operators and close upon themselves under the action of chiral operators. Here we show that the three approaches are equivalent to one another. We first calculate the most general expression for $\Lambda(\boldsymbol{\pi})$, and show that certain of its matrix elements have the desired properties with respect to the Casimir and chiral operators. We then show that the method of Coleman, Wess, and Zumino is a special case of $\Lambda(\boldsymbol{\pi})$. In the course of the analysis, we find that $\Lambda(\boldsymbol{\pi})$ is manifestly covariant under redefinitions of the pion field. To illustrate the usefulness of converting nonlinear fields to linear forms, we calculate π-π scattering lengths and construct weak currents for meson decay.

I. INTRODUCTION

ONE question that arises in the theory of chiral symmetry is the relationship between linear and nonlinear realizations[1–3] of $SU(2)\times SU(2)$. Nonlinear realizations are designed to take into account the distinctive feature of chirality, namely, the correlation of processes involving different numbers of soft pions, and to provide a method for constructing Lagrangians which reproduce the results of current algebra. They differ from the usual linear representations in that the action of chiral operators upon any field is described by a nonlinear function of the pion field. Nevertheless, it is possible to construct linear realizations out of nonlinear ones.

Working with infinitesimal transformations, Weinberg[1] has shown that there exists a matrix $\Lambda(\boldsymbol{\pi})$ which is a function of the pion field and which can be used to convert nonlinear realizations into linear ones. He established the chiral properties of $\Lambda(\boldsymbol{\pi})$, but did not determine its specific form. Coleman, Wess, and Zumino[2] have considered more general groups from a global point of view, and have shown that the construction of linear realizations can be achieved by using the pion fields themselves as parameters of pure chiral transformations. The present writer,[4] approaching the problem from another point of view, has shown that linear realizations can be constructed by redefining the nonlinear pion field in a clearly prescribed way. Here we wish to show that these three approaches are all equivalent to one another.

We also wish to draw attention to two advantages of working with linear realizations constructed from nonlinear ones. The first is that linear realizations and all of their space-time derivatives behave in the same way under chiral transformations. Thus, when constructing Lagrangians, we do not need covariant derivatives.[1] The second advantage is that linear realizations are manifestly covariant under redefinitions of the pion field. Consequently, all physical results obtained from them in the tree approximation are independent of the specific form chosen for the action of chiral operators upon the pion field. To illustrate these points, we calculate S-wave π-π scattering lengths, and construct the weak hadronic currents for both strangeness-conserving and strangeness-violating decays.

Our approach to the problem of constructing linear realizations can be illustrated in the following way. Coleman, Wess, and Zumino[2] have proved that, if the linear representations of $SU(2)\times SU(2)$ are characterized by the "spins" (j^+,j^-) of the two commuting $SU(2)$ subgroups, then the ones that can be realized from a nonlinear pion field belong to the class for which $j^+=j^-=j$. A representation of this kind has an isospin spectrum running from $T=0$ to $T=2j$ in unit steps, and its Casimir eigenvalues are simple functions of j. When isospin operators act upon any state of the representation, they may alter its T_3 eigenvalue, but they cannot change its total isospin T. Chiral operators, on the other hand, act as raising and lowering operators for both T_3 and T. Since the isospin spectrum is bounded at $T=2j$, the successive application of chiral operators to any state must either reproduce the original state at some stage, or annihilate it.

Now a set of "fields" with $T=0, 1, \ldots, 2j$ can always be constructed from n-fold products of the pion field with itself. Furthermore, the action of the nonlinear chiral operator upon one of these "fields" with $T=n$ converts it to an admixture of "fields" with $T=n-1$, n, and $n+1$. Thus it is possible for these product fields to provide a basis for a linear representation of the type (j,j). To ensure that they do in fact form such a basis, we must force the products to have the correct Casimir

* Supported in part by the U. S. Atomic Energy Commission.
[1] S. Weinberg, Phys. Rev. 166, 1568 (1968).
[2] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969).
[3] See, for example, J. Schwinger, Phys. Letters 24B, 473 (1967); P. Chang and F. Gürsey, *ibid.* 164, 1752 (1967); W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969); C. J. Isham, Nuovo Cimento 61A, 729 (1969).
[4] S. P. Rosen, Phys. Rev. 188, 2542 (1969).

eigenvalues, and to behave as a closed system under the successive application of chiral operators. This can be achieved by multiplying each n-fold product by an isoscalar function $h_n(\boldsymbol{\pi}^2)$, which is determined by the Casimir and chiral operator conditions.[4] We shall see in Sec. II that the modified products are manifestly covariant under redefinitions of the pion field.

Fields other than the pion can be converted to linear realizations with $j^+ \neq j^-$ in much the same way. Basis "fields" are constructed from the original field and n-fold pion products. They are then multiplied by functions of π^2 in such a way that they have common Casimir eigenvalues and close under action of chiral operators. The process can be carried out in an economical way with the aid of Weinberg's matrix $\Lambda(\boldsymbol{\pi})$.

To show that our approach is equivalent to the ones of Weinberg,[1] and of Coleman, Wess, and Zumino,[4] we first describe it in detail in Sec. II. We then determine the exact form of $\Lambda(\boldsymbol{\pi})$ in Sec. III, and show that certain of its matrix elements are identical to the linear realizations obtained in Sec. II. We also find that $\Lambda(\boldsymbol{\pi})$ is manifestly covariant under redefinitions of the pion field. In Sec. IV we show that the result of Coleman, Wess, and Zumino[2] is a special case of $\Lambda(\boldsymbol{\pi})$ with a particular definition of the pion field.

In Sec. V we use the matrix $\Lambda(\boldsymbol{\pi})$ to construct the kinematic part of the pion Lagrangian; it is chirally symmetric. The symmetry-breaking mass term is obtained from the analysis of Sec. II. We then expand the Lagrangian in powers of the pion field and thus obtain the S wave of π-π scattering lengths. Weak hadronic currents are discussed in Sec. VI, and the Callan-Treiman relation[5] is shown to hold under fairly general assumptions.

II. PION FIELD

The algebra of $SU(2)\times SU(2)$ consists of isospin operators T_a and chiral operators K_a which obey the standard commutation rules

$$[T_a,T_b]=[K_a,K_b]=i\epsilon_{abc}T_c\,,\qquad [T_a,K_b]=i\epsilon_{abc}K_c\,. \tag{2.1}$$

In general, linear representations are characterized by the "spins" (j^+,j^-) of the commuting $SU(2)$ subgroups

$$J_a^{\pm}=\tfrac{1}{2}(T_a\pm K_a)\,. \tag{2.2}$$

The isospin spectrum for such a representation is given by

$$T=(j^+ + j^-)\,,\quad (j^+ + j^- -1),\ldots,|j^+ - j^-|\,, \tag{2.3}$$

and the eigenvalues of the Casimir operators are

$$C_1\equiv \mathbf{T}^2+\mathbf{K}^2=2j^+(j^+ +1)+2j^-(j^- +1)\,,\qquad C_2\equiv \mathbf{T}\cdot\mathbf{K}=\mathbf{K}\cdot\mathbf{T}=(j^+ - j^-)(j^+ + j^- +1)\,. \tag{2.4}$$

[5] C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).

In the particular class of representations for which $j^+=j^-=j$, the isospin spectrum runs from zero to $2j$, and the Casimir eigenvalues are

$$C_1=4j(j+1)\,,\quad C_2=0\,. \tag{2.5}$$

It can be shown from the commutation rules of Eq. (2.1) that if we apply the operator $K_+=K_1+iK_2$ to the isoscalar member of this representation n times, we obtain the state with $T=T_3=n$. Since the spectrum has an upper bound at $n=2j$, it follows that

$$(K_+)^{2j+1}S=0\,, \tag{2.6}$$

where S is the isoscalar.

Equations (2.5) and (2.6) are the two criteria we shall use to construct linear realizations out of a nonlinearly transforming pion field.[4] For if we can find an isoscalar function of the pion field which satisfies Eq. (2.6) and whose Casimir eigenvalues are those of Eq. (2.5), then the function must belong to a linear realization (j,j). Since K_+ commutes with C_1 and C_2, and since the eigenvalues are independent of the pion field, every "field" obtained by applying K_+ to the initial function n times must also belong to (j,j). Thus we obtain all the member fields with $T=T_3=n$ $(n=0,\ldots,2j)$; the remaining fields can be obtained from them by using the isospin lowering operator $T_-=T_1-iT_2$.

A. Construction of Linear Realizations

Following Weinberg,[1] we consider an isovector pion field π_a $(a=1,2,3)$, which behaves in the standard way under isospin,

$$T_a\pi_b=i\epsilon_{abc}\pi_c\,, \tag{2.7}$$

but which transforms under chiral operators according to the rule

$$K_a\pi_b=-i[\delta_{ab}f(\boldsymbol{\pi}^2)+\pi_a\pi_b g(\boldsymbol{\pi}^2)]\,, \tag{2.8}$$

where $\boldsymbol{\pi}^2=\sum \pi_a\pi_a$. In order for the commutator of two chiral operators to yield an isospin operator, the function $g(\boldsymbol{\pi}^2)$ must be

$$g=\frac{1+2ff'}{f-2\boldsymbol{\pi}^2 f'}\,, \tag{2.9}$$

where the prime denotes differentiation with respect to $\boldsymbol{\pi}^2$. A useful form of this relation is

$$2f'(f+\boldsymbol{\pi}^2 g)=fg-1\,. \tag{2.10}$$

If we apply a succession of chiral operators to the pion field using Eq. (2.8), we will obtain, in general, a sequence of "fields" which neither terminates nor repeats itself. To avoid this difficulty, we apply the chiral operators not to the pion field itself but to the modified "field"

$$\hat{\pi}_a=h_1(\boldsymbol{\pi}^2)\pi_a\,, \tag{2.11}$$

where h_1 is a function yet to be specified. We can then show from Eq. (2.8) that

$$(iK_+)^n h_1(\pi^2)\pi_+ = h_{n+1}(\pi^2)(\pi_+)^{n+1} \quad (n=1, 2, 3, \ldots), \tag{2.12}$$
$$h_{n+1}(\pi^2) = ngh_n(\pi^2) + 2h_n'(f+\pi^2 g),$$

where $\pi_+ = (\pi_1 + i\pi_2)$. In order that the "fields" in Eq. (2.12) belong to a linear (j,j) realization, the sequence of functions $h_n(\pi^2)$ must terminate at $n=2j$. Thus

$$h_{2j+1}(\pi^2) = 0. \tag{2.13}$$

We can best examine the implications of Eq. (2.13) for the function $h_1(\pi^2)$ by writing[4]

$$h_n(\pi^2) = \nu_n(u)/\sigma^n, \quad \sigma = (f^2+\pi^2)^{1/2}, \quad u = -f/\sigma. \tag{2.14}$$

The recurrence relation in Eq. (2.12) then takes a very simple form,

$$\nu_{n+1}(u) = \frac{d}{du}\nu_n(u), \tag{2.15}$$

and the requirement of Eq. (2.13) becomes

$$\nu_{2j+1}(u) = \frac{d^{2j}}{du^{2j}}\nu_1(u) = 0. \tag{2.16}$$

Thus $\nu_1(u)$ must be a polynomial of degree $(2j-1)$ in the new variable u.

In order to satisfy the Casimir eigenvalue conditions of Eq. (2.5), we define the function S to be

$$S = \tfrac{1}{3}i \sum_a K_a[h_1(\pi^2)\pi_a]. \tag{2.17}$$

Since $h_1(\pi^2)\pi_a$ is an isovector "field," the function S must be an isoscalar, i.e.,

$$T_b S = 0 \quad (b=1, 2, 3),$$

and hence it must be an eigenstate of C_2 [see Eq. (2.4)] with zero as its eigenvalue:

$$C_2 S = 0.$$

The requirement that S be an eigenfunction of C_1 [see Eq. (2.4)] with eigenvalue as in Eq. (2.5) is

$$C_1 S \equiv (\sum K_a K_a) S = 4j(j+1)S. \tag{2.18}$$

Now the action of a single chiral operator upon S must give rise to a "field" with isospin equal to 1. Since the pion field is the only isovector available, we must have

$$K_a S = -i\mu h_1(\pi^2)\pi_a. \tag{2.19}$$

In general, μ is a function of π^2; however, given the definition of S [Eq. (2.17)], we see that the simplest way of satisfying Eq. (2.18) is to require that μ be a constant with value

$$\mu = -4j(j+1)/3. \tag{2.20}$$

We use this result to obtain another restriction on $h_1(\pi^2)$.

From Eqs. (2.17) and (2.8), we find that

$$S = \tfrac{1}{3}(3fh_1 + \pi^2 h_2) \tag{2.21}$$

and

$$K_a S = -\tfrac{1}{3}i(\pi^2 h_3 + 5fh_2 - 3h_1)\pi_a, \tag{2.22}$$

where h_2 and h_3 belong to the sequence of functions defined in Eq. (2.12). To satisfy Eqs. (2.19) and (2.20), we must have

$$\pi^2 h_3 + 5fh_2 + (2j-1)(2j+3)h_1 = 0. \tag{2.23}$$

In terms of the functions ν_n of Eqs. (2.14) and (2.15), this becomes a differential equation for ν_1:

$$(1-u^2)\frac{d^2\nu_1}{du^2} - 5u\frac{d\nu_1}{du} + (2j-1)(2j+3)\nu_1 = 0. \tag{2.24}$$

Solving this differential equation by a power series expansion

$$\nu_1(u) = \sum b_n u^n, \tag{2.25}$$

we obtain a recurrence relation for the coefficients b_n:

$$(2r+1)(2r)b_{2r+1} = [(2r-1)(2r+3) - (2j-1)(2j+3)]b_{2r-1}. \tag{2.26}$$

When j is an integer, the coefficients of the odd powers of u terminate at $n=2j-1$, but those of the even powers continue *ad infinitum*. In order to satisfy Eq. (2.16), we must choose the odd-power solution. When j is a half-integer, the coefficients of the even powers terminate, again at $n=2j-1$, and so we must use this solution instead of the odd-power one. Thus the polynomial solution of Eq. (2.24) is

$$\nu_1(u) = \sum_r b_{2r-1}u^{2r-1}, \tag{2.27}$$

where b_{2r-1} satisfies Eq. (2.26) and the summation over r is 1, 2, 3, ..., j, for j an integer, and $\frac{1}{2}$, $\frac{3}{2}$, ..., j, for j a half-integer.

We have now succeeded in finding the function $h_1(\pi^2) = \sigma^{-1}\nu_1(u)$, such that S, the isoscalar function defined in Eq. (2.17), satisfies Eq. (2.6) and has the Casimir eigenvalues of Eq. (2.5). By operating on it with the operators K_+ and T_-, we can now generate a set of "fields" which span the linear realization (j,j) for any value of j. The "fields" with isospin $T=T_3=n$ are simply

$$\Theta(T=T_3=n) \equiv h_n(\pi^2)(\pi_+)^n. \tag{2.28}$$

B. Redefining Pion Field

Suppose that we define a new pion field by means of the relation

$$\pi^*{}_a = \pi_a \Phi(\pi^2). \tag{2.29}$$

The action of the chiral operator upon $\pi^*{}_a$ is then

$$K_b \pi^*{}_a = -i[\delta_{ba}F(\pi^{*2}) + \pi^*{}_b\pi^*{}_a G(\pi^{*2})], \tag{2.30}$$

where[1]

$$F(\pi^{*2}) = f(\pi^2)\Phi(\pi^2),$$
$$G(\pi^{*2}) = [g\Phi + 2\Phi'(f+\pi^2 g)]/\Phi^2. \tag{2.31}$$

We now define new variables σ^* and u^* analogous to those of Eq. (2.14),

$$\sigma^*=(F^2+\boldsymbol{\pi}^{*2})^{1/2}=\Phi\sigma\,,\quad u^*=-F/\sigma^*=u\,,\qquad(2.32)$$

and find from Eqs. (2.14), (2.29), and (2.32) that

$$h_n(\boldsymbol{\pi}^2)(\pi_+)^n=h_n(\boldsymbol{\pi}^{*2})(\pi^*{}_+)^n.\qquad(2.33)$$

Therefore the linearized "fields" $\Theta(T=T_3=n)$ of Eq. (2.28) are manifestly covariant with respect to redefinitions of the pion field. It then follows that all "fields" of the linear realization are covariant.

As a special case, we may take $\pi^*{}_a$ to be the field $\hat{\pi}_a$ of Eq. (2.11). The function $\Phi(\boldsymbol{\pi}^2)$ is then $h_1(\boldsymbol{\pi}^2)$, and we find from Eqs. (2.31) and (2.12) that

$$F(\hat{\pi}^2)=fh_1\,,\quad G(\hat{\pi}^2)=h_2/h_1{}^2.\qquad(2.34)$$

Using the relation between h_2 and h_1 in Eq. (2.12), we can easily show that the functions F and G of Eq. (2.34) satisfy a condition analogous to Eq. (2.9). Thus the construction of the linear realization is equivalent to redefining the pion field as in Eqs. (2.11) and (2.34). Furthermore, Eq. (2.11), together with the particular case of Eq. (2.33) in which $n=1$ and $\pi^*{}_a\equiv\hat{\pi}_a$, implies that

$$h_1(\hat{\pi}^2)=1.\qquad(2.35)$$

This equation serves to determine F as a function of $\hat{\pi}^2$.

C. Anomalous Solution

Up to now we have tacitly assumed that the quantity $\sigma=(f^2+\boldsymbol{\pi}^2)^{1/2}$ does not vanish. Our justification is that in order for the chiral operators to be Hermitian the function $f(\boldsymbol{\pi}^2)$ must be real. If σ is zero, f must be imaginary, and the nonlinear realization is not unitary. There is, however, an interesting solution to the linearization problem in this case.

When σ vanishes, we find that

$$f=i(\boldsymbol{\pi}^2)^{1/2}\,,\qquad(2.36)$$

and hence that

$$1+2ff'=f-\pi^2f'=0.\qquad(2.37)$$

It follows from Eq. (2.9) that the function $g(\boldsymbol{\pi}^2)$ is indeterminate; we are therefore free to choose it in any way we please. In particular, if g is such that

$$fg=-(2j-1)\,,\qquad(2.38)$$

then we obtain a linear realization (j,j) for all integral and half-integral values of j except $j=0$. To show this, we form the function $S=\frac{1}{3}i(\sum K_a\pi_a)$ and then use Eqs. (2.36) and (2.38) to prove that S satisfies Eq. (2.6) and has the Casimir eigenvalues of Eq. (2.5). Thus we have a linear realization which is not unitary, but of finite dimension.

III. OTHER FIELDS

Let Ψ be any field other than the pion with transformation properties

$$T_a\Psi=-t_a\Psi\,,\quad K_a\Psi=v_{ab}(\boldsymbol{\pi})t_b\Psi\,,\qquad(3.1)$$

where t_a is some matrix representation, not necessarily irreducible, of $SU(2)$, i.e.,

$$[t_a,t_b]=i\epsilon_{abc}t_c.\qquad(3.2)$$

Weinberg[1] has shown that in order for the chiral algebra to be satisfied, the function $v_{ab}(\boldsymbol{\pi})$ must be

$$v_{ab}(\boldsymbol{\pi})=\epsilon_{abc}\pi_c v(\boldsymbol{\pi}^2)\,,\quad v(\boldsymbol{\pi}^2)=1/[f+(f^2+\boldsymbol{\pi}^2)^{1/2}]=1/\sigma(1-u).\qquad(3.3)$$

Suppose that there exists a set of matrices x_a which, together with the set t_a, forms a linear representation of $SU(2)\times SU(2)$, i.e.,

$$[x_a,x_b]=i\epsilon_{abc}t_c\,,\quad [t_a,x_b]=i\epsilon_{abc}x_c.\qquad(3.4)$$

Then, given a matrix $\Lambda(\boldsymbol{\pi})$ which depends upon the pion field and which satisfies the equations

$$T_a\Lambda(\boldsymbol{\pi})=-[t_a,\Lambda(\boldsymbol{\pi})]\qquad(3.5)$$

and[1]

$$K_a\Lambda(\boldsymbol{\pi})=-x_a\Lambda(\boldsymbol{\pi})-\Lambda(\boldsymbol{\pi})v_{ab}t_b\,,\qquad(3.6)$$

we can easily show that

$$T_a\Lambda(\boldsymbol{\pi})\Psi=-t_a\Lambda(\boldsymbol{\pi})\Psi\,,\quad K_a\Lambda(\boldsymbol{\pi})\Psi=-x_a\Lambda(\boldsymbol{\pi})\Psi.\qquad(3.7)$$

These two equations imply that $\Lambda(\boldsymbol{\pi})\Psi$ forms a basis for the linear representation spanned by the matrices t_a and x_b.

Weinberg[1] has proved that $\Lambda(\boldsymbol{\pi})$ must exist, and here we intend to derive its specific form. Before doing so, however, we wish to comment on the linear realizations which can be obtained from Ψ. We see from Eqs. (3.1) and (3.7) that $\Lambda(\boldsymbol{\pi})\Psi$ has the same spectrum of isospin as Ψ itself. Since $\Lambda(\boldsymbol{\pi})\Psi$ is a linear representation, this spectrum must be of the form given in Eq. (2.3).

Now there are two possibilities for the field Ψ: Either it is an irreducible representation of $SU(2)$ corresponding to a single isospin t, or else it is reducible and contains several different isospins. In the irreducible case, the only linear realizations of $SU(2)\times SU(2)$ to which $\Lambda(\boldsymbol{\pi})\Psi$ can belong are those that contain a single isospin, namely,

$$j^+=t\,,\quad j^-=0\qquad(3.8a)$$

and

$$j^+=0\,,\quad j^-=t.\qquad(3.8b)$$

The corresponding matrices x_a are

$$x_a=+t_a\qquad(3.9a)$$

for $(t,0)$, and

$$x_a=-t_a\qquad(3.9b)$$

for $(0,t)$. In the reducible case, the matrices x_a, and hence $\Lambda(\boldsymbol{\pi})$ itself, exist only if the isospin spectrum of Ψ is given by Eq. (2.3). If the spectrum differs from Eq. (2.3) then Ψ cannot be converted to a linear realization.

There is one other point concerning the irreducible case. We can construct "fields" with isospin

$$t,\,t+1,\,\ldots,\,t+n\qquad(3.10)$$

by combining Ψ with appropriate products of the pion field. These "fields" will form a basis for a linear realization with $j^+ + j^- = t+n$ and $|j^+ - j^-| = t$, provided that they can be modified by functions of π^2 in such a way as to satisfy the Casimir and chiral operator conditions discussed in the Introduction. One way of finding these functions is to convert the pion field to a linear realization, and then combine this realization with $\Lambda(\pi)\Psi$ by means of the usual Clebsch-Gordan techniques. Thus, once the matrix $\Lambda(\pi)$ is known, we can construct all the allowed linear realizations out of Ψ and the pion field.

A. Determination of $\Lambda(\pi)$

It is easy to show from Eq. (2.7) and the commutation rules of Eqs. (3.2) and (3.4) that if

$$T_a\Psi = -t_a\Psi,$$

then

$$\begin{aligned} T_a(\mathbf{t}\cdot\pi)\Psi &= -t_a(\mathbf{t}\cdot\pi)\Psi, \\ T_a(\mathbf{x}\cdot\pi)\Psi &= -t_a(\mathbf{x}\cdot\pi)\Psi, \end{aligned} \tag{3.11}$$

where

$$(\mathbf{t}\cdot\pi) = \sum t_a\pi_a, \quad (\mathbf{x}\cdot\pi) = \sum x_a\pi_a. \tag{3.12}$$

Therefore, if $\Lambda(\pi)$ is not to alter the isospin of Ψ, it can depend upon the matrices t_a and x_b only through the combinations $(\mathbf{t}\cdot\pi)$ and $(\mathbf{x}\cdot\pi)$.

From Eq. (3.6), we find that

$$-i\frac{\partial\Lambda}{\partial\pi_c}(\delta_{ac}f+\pi_a\pi_c g) = -x_a\Lambda - \Lambda v_{ab}t_b. \tag{3.13}$$

If we now multiply this equation by π_a and sum over a, we obtain

$$-i(f+\pi^2 g)\pi_a\frac{\partial\Lambda}{\partial\pi_a} = -(\mathbf{x}\cdot\pi)\Lambda, \tag{3.14}$$

where Eq. (3.3) has been used to show that $\sum \pi_a v_{ab}$ vanishes. The only way to ensure that Eq. (3.14) is satisfied is to make $\Lambda(\pi)$ a function of $(\mathbf{x}\cdot\pi)$ alone, and not of $(\mathbf{t}\cdot\pi)$.

We now assume that $\Lambda(\pi)$ can be expanded as a power series in $(x\cdot\pi)$:

$$\Lambda(\pi) = \sum_{n=0}^{\infty} a_n(\pi^2)(i\mathbf{x}\cdot\pi)^n. \tag{3.15}$$

Substituting this expression in Eq. (3.14) and equating the coefficients of equal powers of $(\mathbf{x}\cdot\pi)$, we obtain a differential equation for a_n:

$$(f+\pi^2 g)(2\pi^2 a_n' + na_n) = -a_{n-1}. \tag{3.16}$$

To solve this equation, we assume that

$$a_n = \frac{1}{n!}\left(\frac{\lambda}{\sqrt{\pi^2}}\right)^n, \tag{3.17}$$

where λ is a function of u [see Eq. (2.14)]. From Eqs. (3.16) and (2.9), we find that

$$\frac{d\lambda}{du} = -\frac{1}{(1-u^2)^{1/2}}. \tag{3.18}$$

The solution of Eq. (3.18) is

$$\cos\lambda = -u, \quad \sin\lambda = -(1-u^2)^{1/2}, \tag{3.19}$$

and hence the general expression for $\Lambda(\pi)$ is

$$\Lambda(\pi) = \exp[i(\lambda/\sqrt{\pi^2})(\mathbf{x}\cdot\pi)], \tag{3.20}$$

with λ as in Eq. (3.19). Notice that the right-hand side of Eq. (3.20) is manifestly covariant under redefinitions of the pion field.

Having derived this expression for $\Lambda(\pi)$ from Eq. (3.14), we must now go back and show that it satisfies our original requirement, namely, Eq. (3.13). To do this, we first calculate the derivative of $\Lambda(\pi)$ with respect to π_c:

$$\frac{\partial\Lambda}{\partial\pi_c} = 2i\pi_c\left(\frac{\lambda}{\sqrt{\pi^2}}\right)'(\mathbf{x}\cdot\pi)\Lambda + \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\lambda}{\sqrt{\pi^2}}\right)^n\left[\sum_{r+s=n-1}(\mathbf{x}\cdot\pi)^r x_c(\mathbf{x}\cdot\pi)^s\right], \tag{3.21}$$

where the prime denotes differentiation with respect to π^2. We now write

$$x_c \equiv (1/\pi^2)\{\pi_c(\mathbf{x}\cdot\pi) + [[x_c,(\mathbf{x}\cdot\pi)],(\mathbf{x}\cdot\pi)]\} \tag{3.22}$$

and use this identity to reexpress Eq. (3.21) as

$$\frac{\partial\Lambda}{\partial\pi_c} = 2i\pi_c\frac{\lambda'}{\sqrt{\pi^2}}(\mathbf{x}\cdot\pi)\Lambda + \frac{1}{\pi^2}[[x_c,(\mathbf{x}\cdot\pi)],\Lambda]. \tag{3.23}$$

Using Eqs. (3.18) and (3.23), we can write the left-hand side of Eq. (3.13) as

$$(1/\pi^2)\{-\pi_c(\mathbf{x}\cdot\pi)\Lambda + f\epsilon_{abc}\pi_a[t_b,\Lambda]\}. \tag{3.24}$$

To show that this is equal to the right-hand side of Eq. (3.13), we must use the expression for v_{ab} in Eq. (3.3) together with the identity

$$\Lambda t_b = \left[\cos\lambda t_b - \frac{\sin\lambda}{\sqrt{\pi^2}}(\mathbf{x}\times\pi)_b + \frac{2\sin^2(\frac{1}{2}\lambda)}{\pi^2}\pi_b(\mathbf{t}\cdot\pi)\right]\Lambda, \tag{3.25}$$

which is proved in Appendix A. The required result then emerges after some tedious algebra.

Since we have made no assumptions about the representation spanned by t_a and x_b, the expression for $\Lambda(\pi)$ in Eq. (3.20) is valid for all representations. However, we shall give two useful special cases. When Ψ is a field of isospin $t=\frac{1}{2}$, we have

$$\Lambda=-\left(\frac{1-u}{2}\right)^{1/2}\left[1-\frac{i}{\sigma(1-u)}(\mathbf{x}\cdot\boldsymbol{\pi})\right], \quad (3.26)$$

where $\mathbf{x}=\boldsymbol{\tau}$ for the representation $(\frac{1}{2},0)$ and $\mathbf{x}=-\boldsymbol{\tau}$ for $(0,\frac{1}{2})$. ($\boldsymbol{\tau}$ are the Pauli matrices.) When Ψ is a field with isospin equal to 1, we have

$$\Lambda=\left[1-\frac{i}{\sigma}(\mathbf{x}\cdot\boldsymbol{\pi})-\frac{(1+u)}{\boldsymbol{\pi}^2}(\mathbf{x}\cdot\boldsymbol{\pi})^2\right], \quad (3.27)$$

where again $\mathbf{x}$ is either $(+1)$ or (-1) times the isospin matrices.

B. Relation Between $\Lambda(\boldsymbol{\pi})$ and Pion Field

To investigate the relation between the matrix $\Lambda(\boldsymbol{\pi})$ and the method used for converting the pion field into a linear realization, we consider the special case in which the matrices t_a and x_b span a (j,j) representation of $SU(2)\times SU(2)$. The matrices are

$$\tfrac{1}{2}(t_a+x_a)=\hat{t}_a\otimes I\,,\quad \tfrac{1}{2}(t_a-x_a)=I\otimes\hat{t}_a\,, \quad (3.28)$$

where $\hat{t}_a$ are a set of $(2j+1)\times(2j+1)$ spin matrices and I is a unit matrix of the same dimensions. The basis states of this representation are $|T,T_3\rangle$, where the isospin quantum numbers run from zero to $2j$.

Now the operators T_a and K_a act only upon the pion field and do not affect the matrices t_a and x_b, or the states $|T,T_3\rangle$. Consequently, when we apply them to matrix elements of $\Lambda(\boldsymbol{\pi})$, we find that

$$\begin{aligned}T_a\langle T,T_3|\Lambda(\boldsymbol{\pi})|T',T_3'\rangle &=\langle T,T_3|T_a\Lambda(\boldsymbol{\pi})|T',T_3'\rangle\\ &=-\langle T,T_3|[t_a,\Lambda(\boldsymbol{\pi})]|T',T_3'\rangle\,, \quad (3.29)\\ K_a\langle T,T_3|\Lambda(\boldsymbol{\pi})|T',T_3'\rangle &=\langle T,T_3|K_a\Lambda(\boldsymbol{\pi})|T',T_3'\rangle\\ &=-\langle T,T_3|x_a\Lambda(\boldsymbol{\pi})+\Lambda(\boldsymbol{\pi})v_{ab}t_b|T',T_3'\rangle\,. \quad (3.30)\end{aligned}$$

When $|T',T_3'\rangle$ is the isosinglet state (i.e., $T'=T_3'=0$), the second terms on the right-hand sides of Eqs. (3.29) and (3.30) both vanish. From the completeness of the states $|T,T_3\rangle$ in the space of the matrices t_a and x_b, we then obtain

$$\begin{aligned}T_a\langle T,T_3|\Lambda(\boldsymbol{\pi})|0,0\rangle &=-\langle T,T_3|t_a|T'',T_3''\rangle\langle T'',T_3''|\Lambda(\boldsymbol{\pi})|0,0\rangle\,,\\ K_a\langle T,T_3|\Lambda(\boldsymbol{\pi})|0,0\rangle &=-\langle T,T_3|x_a|T'',T_3''\rangle\langle T'',T_3''|\Lambda(\boldsymbol{\pi})|0,0\rangle\,.\end{aligned} \quad (3.31)$$

Equation (3.31) implies that the matrix elements $\langle T,T_3|\Lambda(\boldsymbol{\pi})|0,0\rangle$ form a linear (j,j) realization of $SU(2)\times SU(2)$, each matrix element having isospin quantum numbers (T,T_3). For $T=T_3$ they are, in fact, the functions $h_n(\boldsymbol{\pi}^2)(\pi_+)^n$ which were obtained in Sec. II above.

As an example, we consider the special case $j=\frac{1}{2}$. The matrices $\hat{t}_a$ of Eq. (3.28) are given by Pauli matrices

$$\hat{t}_a=\tfrac{1}{2}\tau_a\,, \quad (3.32)$$

and $\Lambda(\boldsymbol{\pi})$ is the Kronecker product

$$\Lambda(\boldsymbol{\pi})=\exp\left(\frac{i\lambda}{2\sqrt{\boldsymbol{\pi}^2}}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right)\otimes\exp\left(\frac{-i\lambda}{2\sqrt{\boldsymbol{\pi}^2}}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right). \quad (3.33)$$

The basis states are also Kronecker products

$$|T,T_3\rangle=C_{T_3mn}{}^{T\frac{1}{2}\frac{1}{2}}\chi_m{}^{(+)}\otimes\chi_n{}^{(-)}\,, \quad (3.34)$$

where χ_m are the usual spin-$\frac{1}{2}$ column vectors. Using Eq. (3.19) for λ, we find that

$$\begin{aligned}\langle 0,0|\Lambda(\boldsymbol{\pi})|0,0\rangle &=f/(f^2+\boldsymbol{\pi}^2)^{1/2}=-u\,,\\ \langle 1,a|\Lambda(\boldsymbol{\pi})|0,0\rangle &=\pi_a/(f^2+\boldsymbol{\pi}^2)^{1/2}=\pi_a/\sigma\,.\end{aligned} \quad (3.35)$$

Thus the "fields" which transform according to the linear realization $(\frac{1}{2},\frac{1}{2})$ are those on the right-hand side of Eq. (3.35).

If we use the approach of Sec. II, we find from Eq. (2.27) that ν_1 is a constant for $j=\frac{1}{2}$ and ν_2 is zero. Thus h_1 is proportional to $1/(f^2+\boldsymbol{\pi}^2)^{1/2}$ and h_2 vanishes. From Eqs. (2.21) and (2.11), we then see that S and $\hat{\pi}_a$ are exactly the same as the quantities obtained in Eq. (3.35) from the matrix $\Lambda(\boldsymbol{\pi})$.

The particular case $j=\frac{1}{2}$ corresponds to the nonlinear σ model of Gell-Mann and Lévy.[6] The matrices $\Lambda(\boldsymbol{\pi})$ for $j>\frac{1}{2}$ therefore represent a generalization of the σ model to higher isospins.

IV. CONNECTION WITH GLOBAL TRANSFORMATIONS

In the global approach of Coleman, Wess, and Zumino,[2] nonlinear transformations are obtained by writing the product of any element g_0 of the group $SU(2)\times SU(2)$ with a finite chiral transformation as

$$g_0e^{i\boldsymbol{\xi}\cdot\mathbf{K}}=e^{i\boldsymbol{\xi}'\cdot\mathbf{K}}e^{i\mathbf{u}'\cdot\mathbf{T}}. \quad (4.1)$$

The new fields ξ_a' and u_b' $(a,b=1,2,3)$ are functions of g_0 and ξ_a, and the nonlinear transformation of ξ_a is

$$\xi_a\to\xi_a'. \quad (4.2)$$

If any other field Ψ transforms under isospin according to

$$\Psi\to e^{i\boldsymbol{\theta}\cdot\mathbf{t}}\Psi\,, \quad (4.3)$$

where t_a are the isospin matrices of Eq. (3.2), then its nonlinear transformation is

$$\Psi\to e^{i\mathbf{u}'\cdot\mathbf{t}}\Psi\,, \quad (4.4)$$

where u_b' are the functions appearing in Eq. (4.1).

[6] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1900); S. Weinberg, *ibid.* 18, 188 (1967).

Coleman, Wess, and Zumino have shown that the fields $\boldsymbol{\xi}$ and Ψ can be converted to linear realizations by means of the matrix $L=e^{i\boldsymbol{\xi}\cdot\mathbf{x}}$, where x_a are the chiral matrices of Eq. (3.4). For the field ξ_a itself, they prove that each column of L yields a linear realization, and for Ψ they prove that $L\Psi$ transforms linearly. If the ξ_a represent the pion field, then this result is of almost the same form as the one obtained from the Weinberg approach. The only difference is that while the parameters of L are the fields ξ_a, the parameters of $\Lambda(\boldsymbol{\pi})$ are

$$(\lambda/\sqrt{\boldsymbol{\pi}^2})\pi_a$$

[see Eq. (3.20)]. What we now show is that L is the special case of $\Lambda(\boldsymbol{\pi})$ in which

$$\pi_a \rightarrow \xi_a\,, \quad \lambda=(\boldsymbol{\xi}^2)^{1/2}. \tag{4.5}$$

We begin by extracting the infinitesimal form of the Coleman, Wess, and Zumino transformation[2] from the finite transformations of Eqs. (4.1) and (4.2). From this we obtain the functions $F(\boldsymbol{\xi}^2)$ and $G(\boldsymbol{\xi}^2)$ appropriate to the case of Eq. (4.1); then with the aid of Eqs. (2.14) and (3.19) we can calculate λ, and hence the matrix $\Lambda(\boldsymbol{\pi})$.

For a pure chiral transformation, we have

$$g_0=e^{i\boldsymbol{\xi}^0\cdot\mathbf{K}}. \tag{4.6}$$

Now we show in Appendix B that

$$e^{i\boldsymbol{\xi}^0\cdot\mathbf{K}}e^{i\boldsymbol{\xi}\cdot\mathbf{K}}=e^{i\rho\mathbf{k}\cdot\mathbf{K}}e^{ih\boldsymbol{\xi}^0\times\boldsymbol{\xi}\cdot\mathbf{T}}, \tag{4.7}$$

where

$$\begin{aligned}\cos\rho&=\cos\eta\,\cos\eta^0-\mathbf{n}\cdot\mathbf{n}^0\,\sin\eta\,\sin\eta^0,\\ \mathbf{k}&=(1/\sin\rho)\{\mathbf{n}^0[\cos\eta\,\sin\eta^0\\ &\qquad-\mathbf{n}\cdot\mathbf{n}^0\,\sin\eta\,(1-\cos\eta^0)]+\mathbf{n}\,\sin\eta\},\\ \boldsymbol{\xi}^0&=\eta^0\mathbf{n}^0,\quad \boldsymbol{\xi}=\eta\mathbf{n},\quad (\mathbf{n}^0)^2=\mathbf{n}^2=1.\end{aligned} \tag{4.8}$$

The function h is given in Appendix B.

Comparing Eqs. (4.8) and (4.1), we find that

$$\xi_a'=\rho k_a \quad (a=1,2,3). \tag{4.9}$$

When $\boldsymbol{\xi}^0$ vanishes, ξ_a' is equal to ξ_a, and when $\boldsymbol{\xi}^0$ is an infinitesimal vector, we have

$$\xi_a'=\xi_a+\left[\frac{\partial\xi_a'}{\partial\xi_b{}^0}\right]\xi_b{}^0, \tag{4.10}$$

where the differential coefficient must be evaluated at $\boldsymbol{\xi}^0=0$. From Eqs. (4.8) and (4.9), we obtain

$$\left[\frac{\partial\xi_a'}{\partial\xi_b{}^0}\right]=-(\delta_{ab}F(\boldsymbol{\xi}^2)+\xi_a\xi_bG(\boldsymbol{\xi}^2)), \tag{4.11}$$

where

$$\begin{aligned}F(\boldsymbol{\xi}^2)&=-(\boldsymbol{\xi}^2)^{1/2}\cot((\boldsymbol{\xi}^2)^{1/2}),\\ G(\boldsymbol{\xi}^2)&=-(1+F(\boldsymbol{\xi}^2))/\boldsymbol{\xi}^2.\end{aligned} \tag{4.12}$$

It is not difficult to show that the functions F and G of Eq. (4.12) satisfy the Weinberg condition of Eq. (2.9). The Coleman, Wess, and Zumino transformation is therefore a special case of Eq. (2.8); it can, in fact, be obtained directly from Eq. (2.8) by redefining the pion field [see Eqs. (2.29) and (2.31)] as

$$\xi_a=\pi_a\Phi(\boldsymbol{\pi}^2)\,, \quad \Phi(\boldsymbol{\pi}^2)=\lambda/\sqrt{\boldsymbol{\pi}^2}. \tag{4.13}$$

Since the function λ is covariant with respect to redefinitions of the pion field, we can write [see Eqs. (2.14) and (3.19)]

$$\cot\lambda=u/(1-u^2)^{1/2}=-F/\sqrt{\boldsymbol{\xi}^2}. \tag{4.14}$$

Given the form of F in Eq. (4.12), we find that the expression for λ obtained from Eq. (4.14), namely,

$$\lambda=\sqrt{\boldsymbol{\xi}^2}, \tag{4.15}$$

is consistent with Eq. (4.13). Thus the matrix $L=e^{i\boldsymbol{\xi}\cdot x}$ is a special case of $\Lambda(\boldsymbol{\pi})$ corresponding to a pion field defined as in Eq. (4.12).

V. π-π INTERACTION

The pion Lagrangian consists of two parts, a kinematic term and a mass term. The kinematic term preserves chiral symmetry, but the mass term does not. Since isospin is conserved, the mass term is restricted to $SU(2)\times SU(2)$ realizations of the type (j,j). Here we assume that it belongs to one such realization rather than an admixture of them.

For a nonlinear pion field, the isoscalar member of the (j,j) realization is given by the function S of Eq. (2.21). In general, S contains terms of the first and higher orders in $\boldsymbol{\pi}^2$. In the tree approximation, the first-order term corresponds to the pion mass, and the second-order one accounts for π-π scattering. Higher-order terms describe processes involving larger numbers of pions.

The kinematic term can be derived from the matrix $\Lambda(\boldsymbol{\pi})$ of Eq. (3.20), and like the function S, it can also be expanded in powers of the pion field. The quadratic term is identified with the kinetic energy $\frac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}$, and the quartic term contributes to π-π scattering in the tree approximation. Higher-order terms correspond to higher-order processes, for example, $2\pi\rightarrow 4\pi$.

Putting the mass and kinematic terms together, we obtain the general pion Lagrangian. From it we can then calculate a_0 and a_2, the S-wave scattering lengths in the $T=0$ and $T=2$ states, respectively. Because S and $\Lambda(\boldsymbol{\pi})$ are manifestly covariant with respect to redefinitions of the pion field [see Eqs. (2.29) and (2.31)], the formulas for a_0 and a_2 are the same for all choices of the functions $f(\boldsymbol{\pi}^2)$ and $g(\boldsymbol{\pi}^2)$ in Eq. (2.8). They do, however, depend upon the parameter j which describes the way in which the mass part of the Lagrangian breaks the chiral symmetry.[1]

A. Symmetry-Breaking Mass Term

We can show with the aid of Eqs. (2.14), (2.15), (2.21), and (2.24) that S is a function of u obeying the differential equation

$$(1-u^2)\frac{d^2S}{du^2}-3u\frac{dS}{du}+4j(j+1)S=0. \quad (5.1)$$

To obtain the terms of first and second order in $\boldsymbol{\pi}^2$, we develop a Taylor series for S noting that [see Eq. (2.14)]

$$u=-1 \quad \text{when} \quad \boldsymbol{\pi}^2=0. \quad (5.2)$$

Therefore

$$S=[S]+\left[\frac{dS}{du}\frac{du}{d\boldsymbol{\pi}^2}\right]\boldsymbol{\pi}^2 +\frac{1}{2}\left[\frac{d^2S}{du^2}\left(\frac{du}{d\boldsymbol{\pi}^2}\right)^2+\frac{dS}{du}\frac{d^2u}{d(\boldsymbol{\pi}^2)^2}\right](\boldsymbol{\pi}^2)^2+\cdots, \quad (5.3)$$

where quantities in square brackets must be evaluated at $\boldsymbol{\pi}^2=0$. From Eq. (5.1) and its first derivative with respect to u, we find that

$$\left[\frac{dS}{du}\right]=-\frac{4j(j+1)}{3}[S],$$
$$\left[\frac{d^2S}{du^2}\right]=\frac{(4j(j+1)-3)4j(j+1)}{3\times 5}[S]. \quad (5.4)$$

We now choose $[S]$ in such a way that the coefficient of $\boldsymbol{\pi}^2$ is $-\frac{1}{2}m_\pi{}^2$: thus

$$S=\frac{3m_\pi{}^2}{8j(j+1)u'}-\tfrac{1}{2}m_\pi{}^2\boldsymbol{\pi}^2 +\frac{m_\pi{}^2}{4u'}\left[\frac{4j(j+1)-3}{5}(u')^2-u''\right](\boldsymbol{\pi}^2)^2, \quad (5.5)$$

where u' and u'' are the first and second derivatives of u at $\boldsymbol{\pi}^2=0$. The constant term in Eq. (5.5) represents a zero-point energy and has no further physical significance.

B. Symmetry-Preserving Kinematic Term

To calculate the symmetry-preserving kinematic part of the Lagrangian, we use the form of $\Lambda(\boldsymbol{\pi})$ appropriate to a (j',j') realization (see Sec. III B). In such a realization the matrices t_a and x_a are given by Eq. (3.28), and the basis for the space spanned by these matrices is a set of column vectors $|T,T_3\rangle$ ($T=0, 1, \ldots, 2j'$) which are independent of the pion field. From Eq. (3.7), the matrix $\Lambda(\boldsymbol{\pi})$ and its space-time derivative $\partial_\mu\Lambda(\boldsymbol{\pi})$ satisfy the property

$$T_aM|0\rangle=-t_aM|0\rangle, \quad K_aM|0\rangle=-x_aM|0\rangle, \quad M\equiv\Lambda,\ \partial_\mu\Lambda \quad (5.6)$$

where $|0\rangle$ is the isoscalar column vector.

Now it follows from Eq. (5.6) that the quantity

$$U(M)\equiv\langle 0|M^\dagger M|0\rangle \quad (5.7)$$

(where † denotes Hermitian adjoint) is invariant under $SU(2)\times SU(2)$. The quantity $U(\Lambda)$ is of no physical interest because

$$\Lambda^\dagger\Lambda=1 \quad (5.8)$$

[see Eq. (3.20)], but $U(\partial_\mu\Lambda)$ does provide us with a chirally symmetric function of the pion field and its space-time derivatives. We shall therefore assume that the kinetic energy part of the Lagrangian is given by $gU(\partial_\mu\Lambda)$, where g is a coupling constant yet to be fixed.

To calculate $U(\partial_\mu\Lambda)$, we use

$$\partial_\mu\Lambda=\frac{\partial\Lambda}{\partial\pi_c}\partial_\mu\pi_c, \quad (5.9)$$

where $\partial\Lambda/\partial\pi_c$ is given in Eq. (3.23), and

$$\langle 0|t_a|0\rangle=0,$$
$$\langle 0|x_ax_b|0\rangle=\tfrac{1}{3}\delta_{ab}\langle 0|\mathbf{x}^2|0\rangle=\tfrac{4}{3}j'(j'+1)\delta_{ab}. \quad (5.10)$$

We then find that

$$gU(\partial_\mu\Lambda)=g\frac{4j'(j'+1)}{3}\left\{\frac{1-u^2}{\boldsymbol{\pi}^2}(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}) +\left[\frac{4}{1-u^2}\left(\frac{du}{d\boldsymbol{\pi}^2}\right)^2-\frac{(1-u^2)}{(\boldsymbol{\pi}^2)^2}\right](\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\right\}. \quad (5.11)$$

The only dependence of $gU(\partial_\mu\Lambda)$ upon the representation to which t_a and x_a have been assigned occurs in the factor $4j'(j'+1)$ on the right-hand side of Eq. (5.11); the remaining factor is a universal function common to all (j',j') representations. It is not difficult to show that this function is exactly the one obtained from the method of covariant derivatives.[1,7]

We now expand $gU(\partial_\mu\Lambda)$ in powers of the pion field and choose g so that the coefficient of $\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}$ is $(-\frac{1}{2})$. Up to fourth order we obtain

$$gU(\partial_\mu\Lambda)=-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}-\frac{[u''-(u')^2]}{4u'}\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}) -\frac{[u''+(u')^2]}{2u'}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2+\cdots, \quad (5.12)$$

where u' and u'' are again the first and second derivatives of u evaluated at $\boldsymbol{\pi}^2=0$.

C. π-π Scattering Lengths

Omitting the constant term in the expansion of S [see Eq. (5.5)], we can now write the pion Lagrangian as

$$\mathcal{L}=-\tfrac{1}{2}m_\pi{}^2\boldsymbol{\pi}^2-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}+\mathcal{L}_{\pi-\pi}. \quad (5.13)$$

[7] L. Bessler, Phys. Rev. **184**, 1523 (1969).

From Eqs. (5.5) and (5.12), the π-π interaction is

$$\mathfrak{L}_{\pi\text{-}\pi}=\tfrac{1}{4}u'\{\tfrac{1}{5}[4j(j+1)-3]m_\pi^2(\boldsymbol{\pi}^2)^2+\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})-2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2\}-(u''/4u')[m_\pi^2(\boldsymbol{\pi}^2)^2+\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2]. \quad (5.14)$$

The coefficient of u'' is proportional to the four-divergence of $\boldsymbol{\pi}^2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})$ and hence it makes no contribution to π-π scattering.[7] Thus the scattering lengths are proportional to u'. From Eq. (2.14), we find that

$$u'=2\alpha^2,\quad \alpha^{-1}=2f_0, \quad (5.15)$$

where f_0 is the value of $f(\boldsymbol{\pi}^2)$ at $\boldsymbol{\pi}^2=0$. Equation (5.14) then yields exactly the same formulas for a_0 and a_2 as those of Weinberg[1,8] and of Bessler[7]:

$$a_0=[j(j+1)+1](m\alpha^2/2\pi),\quad a_2=\tfrac{2}{5}[j(j+1)-2](m\alpha^2/2\pi). \quad (5.16)$$

Since the method we have used to derive Eq. (5.16) is covariant under redefinitions of the pion field, these formulas for a_0 and a_2 are valid for all definitions of the pion field and all choices of the function $f(\boldsymbol{\pi}^2)$ in Eq. (2.8). However, the numerical value of f_0 is determined by the ratio g_A/g_V for neutron β decay,[1,7] and so our freedom to redefine the pion field is subject to a practical limitation: The redefinition must always be such that $F(\boldsymbol{\pi}^{*2}=0)$ is equal to f_0. Consequently the function $\Phi(\boldsymbol{\pi}^2)$ of Eqs. (2.29) and (2.31) must have the property that

$$\Phi(\boldsymbol{\pi}^2=0)=1. \quad (5.17)$$

This result is equivalent to the recent argument by Bessler[7] concerning the uniqueness of the S matrix.

It is interesting to note some of the properties of the scattering lengths in Eq. (5.16). The combination

$$2a_0-5a_2=6(m\alpha^2/2\pi) \quad (5.18)$$

is independent of the symmetry-breaking parameter j, and the combination

$$4a_0+5a_2=6j(j+1)(m\alpha^2/2\pi) \quad (5.19)$$

is directly proportional to it. When $j=1$, a_2 vanishes[9]; by contrast a_0 can never vanish. Since $(m\alpha^2/2\pi)\approx 0.115m_\pi^{-1}$, the scattering lengths for $j=\frac{1}{2}$, namely,

$$a_0\approx 0.20,\quad a_2\approx -0.06,\quad j=\tfrac{1}{2} \quad (5.20)$$

are the same as those calculated by Weinberg.[1]

VI. WEAK CURRENTS

As another example of the use of linear realizations, we construct the weak strangeness-conserving current for the pion field, and the strangeness-violating one for the K-meson field. We then find that parameter f_0 is related to the pion decay constant, and that the Callan-Treiman relation[5] holds for K_{l3} decay.

A. $\Delta S=0$ Current

The weak strangeness-conserving current transforms according to the (1,0) representation of $SU(2)\times SU(2)$. Since linear realizations constructed from the pion field are of the type (j,j), the only way in which we can obtain a (1,0) current is to take the product of two such realizations and carry out the reduction

$$(j,j)\otimes(j,j)\supset(1,0). \quad (6.1)$$

From Eq. (5.6) we find that this process leads to a current of the form

$$J_\mu{}^a=G\langle 0|\Lambda^\dagger(t_a+x_a)\partial_\mu\Lambda|0\rangle, \quad (6.2)$$

where Λ, t_a, and x_a are matrices appropriate to the (j,j) representation (see Sec. III B), and G is a constant to be determined below.

Evaluating the matrix element in Eq. (6.2) with the techniques described in previous sections, we find that

$$J_\mu{}^a=iG\frac{4j(j+1)}{3}\frac{1}{f^2+\boldsymbol{\pi}^2}\times[(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})_a+f\partial_\mu\pi_a-\pi_a\partial_\mu f]. \quad (6.3)$$

As in the case of the kinematic pion Lagrangian (see Sec. V B), the only dependence of the current on the representation (j,j) occurs in the factor $4j(j+1)$; the remaining factor is a function of the pion field common to all such representations. Furthermore, we can show that the axial-vector part of $J_\mu{}^a$ can be derived from the kinematic Lagrangian $gU(\partial_\mu\Lambda)$ of Eqs. (5.7), (5.11), and (5.12) by means of Noether's theorem[10] and Eq. (3.13). If we use this property to fix G, we obtain

$$J_\mu{}^a=\frac{if_0^2}{f^2+\boldsymbol{\pi}^2}[(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})_a+f\partial_\mu\pi_a-\pi_a\partial_\mu f], \quad (6.4)$$

where f_0 is the value of $f(\boldsymbol{\pi}^2)$ at $\boldsymbol{\pi}^2=0$.

On expanding $J_\pi{}^a$ in powers of the pion field, we find that the leading term in the vector current is $i(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})$, and the leading one in the axial-vector current is $if_0\partial_\mu\pi_a$. We therefore choose

$$f_0=f_\pi, \quad (6.5)$$

where f_π is the pion decay constant. With this restriction, the current of Eq. (6.4) is valid for all definitions of the pion field. In particular it yields the same current as the σ model of Gell-Mann and Lévy[6,11] when $f=(f_\pi^2-\boldsymbol{\pi}^2)^{1/2}$.

[8] S. Weinberg, Ref. 6; Phys. Rev. Letters **17**, 616 (1966).

[9] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1702 (1967). These authors calculate a_0 and a_2 by means of dispersion theory and find that a_2 is very small. Within experimental error their value for a_0 is consistent with Eq. (5.16) for $j=1$.

[10] S. Weinberg, Ref. 1, Eq. (4.11).

[11] W. A. Bardeen and B. W. Lee, in *Nuclear and Particle Physics*, edited by B. Margolis and C. S. Lam (Benjamin, New York, 1968), p. 273.

B. $\Delta S=1$ Current

The strangeness-violating current belongs to the $(\frac{1}{2},0)$ representation of $SU(2)\times SU(2)$. Now, as shown in Sec. III, a nonlinear K-meson field with isospin equal to $\frac{1}{2}$ can be converted to the linear realizations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. Since a linear realization and its space-time derivatives behave in the same way under chiral symmetry transformation, we can immediately write down a current

$$J_\mu{}^{(1)}=\partial_\mu[\Lambda(\tfrac{1}{2},0)\Psi_K] =[\partial_\mu\Lambda(\tfrac{1}{2},0)]\Psi_K+\Lambda(\tfrac{1}{2},0)\partial_\mu\Psi_K\,, \quad (6.6)$$

where Ψ_K denotes the isospin-$\frac{1}{2}$ K-meson field and $\Lambda(\frac{1}{2},0)$ is the matrix of Eq. (3.26) with $\mathbf{x}=\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ are the Pauli matrices.

Equation (6.6) is not the only current we can construct. Another possibility is to take the $(0,\frac{1}{2})$ linear realization of the K-meson field and combine it with the space-time derivative of the $(\frac{1}{2},\frac{1}{2})$ linear realization of the pion field:

$$(0,\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\supset(\tfrac{1}{2},0)\,. \quad (6.7)$$

To do this, we note from Eq. (3.33) that the $(\frac{1}{2},\frac{1}{2})$ realization of the pion field can be written as the Kronecker product

$$\Lambda(\tfrac{1}{2},\tfrac{1}{2})|0\rangle=\Lambda(\tfrac{1}{2},0)\otimes\Lambda(0,\tfrac{1}{2})|0\rangle\,, \quad (6.8)$$

where $\Lambda(0,\frac{1}{2})$ is given by Eq. (3.26) with $\mathbf{x}=-\boldsymbol{\tau}$, and the state $|0\rangle$ is the $T=0$ state of Eq. (3.34). We can then write another current as

$$J_\mu{}^{(2)}=\tilde{\Psi}_k\tilde{\Lambda}(0,\tfrac{1}{2})C\partial_\mu\Lambda(\tfrac{1}{2},\tfrac{1}{2})|0\rangle\,, \quad (6.9)$$

where $\sim$ denotes transpose, C is the charge conjugation matrix $i\tau_2$, and the inner product in Eq. (6.9) is taken with respect to the $(0,\frac{1}{2})$ representation. After some tedious algebra, we obtain

$$J_\mu{}^{(2)}=\tfrac{1}{2}\sqrt{2}\{\partial_\mu\Lambda(\tfrac{1}{2},0) +\Lambda(\tfrac{1}{2},0)[\partial_\mu\Lambda(\tfrac{1}{2},0)]\Lambda(0,\tfrac{1}{2})\}\Psi_k\,. \quad (6.10)$$

If we expand $J_\mu{}^{(1)}$ and $J_\mu{}^{(2)}$ up to second order in the pion field, the currents become

$$J_\mu{}^{(1)}=-\partial_\mu\Psi_K+\frac{i}{2f_0}\partial_\mu[\boldsymbol{\pi}\cdot\boldsymbol{\tau}\Psi_K] +\frac{1}{4f_0{}^2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\Psi_K+\frac{\boldsymbol{\pi}^2}{8f_0{}^2}\partial_\mu\Psi_K\,, \quad (6.11)$$

$$J_\mu{}^{(2)}=\frac{i}{\sqrt{2}f_0}(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi})\Psi_K+\frac{1}{2\sqrt{2}f_0{}^2}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi})\Psi_K\,. \quad (6.12)$$

We therefore choose the total strangeness-violating current to be

$$J_\mu(\Delta S=1)=-f_K(J_\mu{}^{(1)}-\sqrt{2}g_2J_\mu{}^{(2)})\,, \quad (6.13)$$

where f_K is the K-meson decay constant and g_2 is another coupling constant.

To determine g_2, we note that in the tree approximation, the matrix element of $J_\mu(\Delta S=1)$ for K_{l3} decay is

$$\langle\pi_a q|J_\mu(\Delta S=0)|K,p\rangle =(f_K/f_0)[g_2(p+q)_\mu+(1-g_2)(p-q)_\mu](\tfrac{1}{2}\tau_a)\Psi_K\,, \quad (6.14)$$

where p_μ and q_μ are the four-momenta of the K meson and pion, respectively. From Eqs. (6.14) and (6.5), we see that g_2 is related to the usual form factors $f_\pm$,

$$g_2=1/(1+\xi)\,,\quad \xi=f_-/f_+ \quad (6.15)$$

and that the Callan-Treiman relation[5] is satisfied,[11] i.e.,

$$f_++f_-=f_K/f_\pi\,. \quad (6.16)$$

Further applications of these methods to weak interactions will be considered elsewhere.

VII. SUMMARY

We have shown that the methods developed by Weinberg,[1] by Coleman, Wess, and Zumino,[2] and by the present author[4] are all equivalent to one another. As a by-product, we have also derived the specific relationship between Weinberg's treatment of nonlinear fields and that due to Coleman, Wess, and Zumino. The main feature of our approach is the introduction of the variable $u=-f/(f^2+\boldsymbol{\pi}^2)^{1/2}$. In terms of u, the differential equations we have to solve take relatively simple forms, and their solutions are independent of the definition of the pion field.

To illustrate the usefulness of our approach, we have considered the examples of π-π scattering and weak currents for meson decay. Other examples are not hard to find and we hope to discuss them in a later publication.

We anticipate that the nonlinear theory of chiral $SU(3)\times SU(3)$ can be approached in much the same way as above. The practical details, however, are much more complicated[12] because the product of two octets contains two independent octets, and because there are two invariants in $SU(3)$ as opposed to one in $SU(2)$. Thus the analog of Eq. (2.8) for $SU(3)$ may contain more than two terms, and the invariant functions associated with them will depend, in general, upon two variables instead of one.[12]

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[12] A. J. Macfarlane and P. H. Weisz, Nuovo Cimento **55A**, 853 (1968); A. J. Macfarlane, A. Sudbery, and P. H. Weisz, Commun. Math. Phys. **11**, 77 (1968).

APPENDIX A

In order to prove Eq. (3.25), which gives a closed expression for $\Lambda t_a \Lambda^{-1}$, we use the formula for the rotation of a vector operator:

$$e^{i\theta\mathbf{n}\cdot\mathbf{J}}\mathbf{r}e^{-i\theta\mathbf{n}\cdot\mathbf{J}}=\cos\theta\,\mathbf{r}+(1-\cos\theta)(\mathbf{r}\cdot\mathbf{n})\mathbf{n}-\sin\theta\,\mathbf{r}\times\mathbf{n}\,, \quad \text{(A1)}$$

where $\mathbf{n}$ is a unit vector, J_a $(a=1, 2, 3)$ is a set of $SU(2)$ generators, and $\mathbf{r}$ is a vector operator.

Using the matrices t_a and x_a of Eqs. (3.2) and (3.4), we can construct two sets of $SU(2)$ matrices which commute with one another:

$$t_a^{\pm}=\tfrac{1}{2}(t_a\pm x_a)\,. \quad \text{(A2)}$$

With the matrix Λ given in Eq. (3.20), we find that

$$\Lambda t_a\Lambda^{-1}=e^{i\Phi\boldsymbol{\pi}\cdot\mathbf{t}^+}t_a^+e^{-i\Phi\boldsymbol{\pi}\cdot\mathbf{t}^+}+e^{-i\Phi\boldsymbol{\pi}\cdot\mathbf{t}^-}t_a^-e^{i\Phi\boldsymbol{\pi}\cdot\mathbf{t}^-}\,,$$
$$\Phi=\lambda/\sqrt{\boldsymbol{\pi}^2}\,. \quad \text{(A3)}$$

We now apply the general formula of Eq. (A1) to each term in Eq. (A3), and on recombining them we obtain

$$\Lambda t_a\Lambda^{-1}=\cos\lambda\; t_a+\frac{(1-\cos\lambda)}{\boldsymbol{\pi}^2}(\mathbf{t}\cdot\boldsymbol{\pi})\pi_a-\frac{\sin\lambda}{\sqrt{\boldsymbol{\pi}^2}}(\mathbf{x}\times\boldsymbol{\pi})_a\,. \quad \text{(A4)}$$

This result leads to Eq. (3.25) because $(1-\cos\lambda)=2\sin^2(\tfrac{1}{2}\lambda)$. In the same way, we can also show that

$$\Lambda x_a\Lambda^{-1}=\cos\lambda\; x_a+\frac{(1-\cos\lambda)}{\boldsymbol{\pi}^2}(\mathbf{x}\cdot\boldsymbol{\pi})\pi_a-\frac{\sin\lambda}{\sqrt{\boldsymbol{\pi}^2}}(\mathbf{t}\times\boldsymbol{\pi})_a\,. \quad \text{(A5)}$$

APPENDIX B

In order to prove Eqs. (4.7) and (4.8), we consider the product

$$P=e^{i\theta\mathbf{n}\cdot\mathbf{J}}e^{i\Phi\mathbf{m}\cdot\mathbf{J}}e^{-ih(\mathbf{n}\times\mathbf{m})\cdot\mathbf{J}}\,, \quad \text{(B1)}$$

where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors, θ and Φ are two angles of rotation, and h is an as-yet undetermined function of these quantities. We consider the special case of P for which the operators J_a are represented by Pauli matrices:

$$J_a=\tfrac{1}{2}\sigma_a\,. \quad \text{(B2)}$$

Using standard properties of the Pauli matrices, we can express P in the form

$$P=a+ib\boldsymbol{\sigma}\cdot\mathbf{n}+ic\boldsymbol{\sigma}\cdot\mathbf{m}+id\boldsymbol{\sigma}\cdot\mathbf{n}\times\mathbf{m}\,, \quad \text{(B3)}$$

where the coefficients a, b, c, and d are functions of θ, Φ, h, and $\mathbf{n}\cdot\mathbf{m}$. We now choose h so that the coefficient d vanishes; this means that

$$\cos(\tfrac{1}{2}h\sin\Psi)=R^{-1}\cos\Omega\,,$$
$$\sin(\tfrac{1}{2}h\sin\Psi)=-R^{-1}\sin\Psi\,\sin\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\Phi\,, \quad \text{(B4)}$$

where

$$\cos\Psi=\mathbf{n}\cdot\mathbf{m}\,,$$
$$\cos\Omega=\cos\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\Phi-\cos\Psi\,\sin\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\Phi\,, \quad \text{(B5)}$$
$$R^2=\cos^2\Omega+(\sin\Psi\,\sin\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\Phi)^2\,.$$

With this choice of h, we find that

$$P=\cos\tfrac{1}{2}\rho+i\sin\tfrac{1}{2}\rho\;\boldsymbol{\sigma}\cdot\mathbf{k}\,,$$
$$\cos\rho=\cos\theta\,\cos\Phi-\cos\Psi\,\sin\theta\,\sin\Phi\,, \quad \text{(B6)}$$
$$\mathbf{k}=[2R(1-R^2)^{\frac{1}{2}}]^{-1}\{\mathbf{n}[\cos\Phi\,\sin\theta-\cos\Psi\,\sin\Phi(1-\cos\theta)]+\mathbf{m}\,\sin\Phi\}\,.$$

It follows from Eq. (B6) that for the spin-$\frac{1}{2}$ representation of J_a, we have

$$e^{i\theta\mathbf{n}\cdot\mathbf{J}}e^{i\Phi\mathbf{n}\cdot\mathbf{J}}=e^{i\rho\mathbf{k}\cdot\mathbf{J}}e^{ih\mathbf{n}\times\mathbf{m}\cdot\mathbf{J}}\,, \quad \text{(B7)}$$

where h is given by Eq. (B4). Since both sides of Eq. (B7) are products of group elements of the $SU(2)$ generated by J_a $(a=1, 2, 3)$, the equation must be valid for all representations if it is valid for one. We have shown that it is true for the spin-$\frac{1}{2}$ case, and hence it must hold for all spins.

In order to apply this result to the left-hand side of Eq. (4.7), we express K_a in terms of the operators $J_a^{\pm}$ of Eq. (2.2) and use the fact that J_a^+ and J_a^- commute with one another. We then find that

$$e^{i\boldsymbol{\xi}^0\cdot\mathbf{K}}e^{i\boldsymbol{\xi}\cdot\mathbf{K}}=(e^{i\boldsymbol{\xi}^0\cdot\mathbf{J}^+}e^{i\boldsymbol{\xi}\cdot\mathbf{J}^+})(e^{-i\boldsymbol{\xi}^0\cdot\mathbf{J}^-}e^{-i\boldsymbol{\xi}\cdot\mathbf{J}^-})\,. \quad \text{(B8)}$$

Applying Eq. (B7) to each factor of Eq. (B8), and then recombining them, we obtain the expression on the right-hand side of Eq. (4.7). The variables θ, Φ, $\mathbf{n}$, and $\mathbf{m}$ of Eq. (B6) are, of course, replaced by the corresponding ones of Eq. (4.8).