Point-Source Limit of a Model Field Theory and the Question of Compositeness*

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The point-source limit is found for a previously proposed static model satisfying crossing symmetry and two-particle unitarity below three-particle thresholds. A numerical solution for this limit is obtained, and both renormalized and unrenormalized quantities are calculated. The renormalization constants are found to vanish; the relation of this to compositeness of the neutron in this model is discussed.

I. INTRODUCTION

In an earlier paper[1] (hereafter referred to as I), the equations for a variant of the charged scalar static model were solved. These equations determined off-shell scattering matrices for $\pi^-p\to\pi^-p$ and $\pi^+p\to\pi^+p$ which satisfied two-particle unitarity below the three-particle thresholds and were related by crossing symmetry. The πpn vertex was taken to be $g_0(2\pi^2/\alpha^2k^2+1)^{1/2}$ where k is the momentum of the pion, α^{-1} is a cutoff, and $\alpha=0$ corresponds to the point-source limit. These equations were solved in the bound-state region of the energy plane for $\alpha^{-1}=5$ and the entire range of the unrenormalized coupling constant g_0. The results of this calculation are discussed in I.

In this paper the point-source limit $\alpha\to 0$ of our model is studied. The prescription for obtaining this limit is given in Sec. II, while Sec. III is devoted to explaining the details of computation. In Sec. IV the nature of this limit is explored. In particular, we show that both the neutron wave function and vertex renormalization constants approach zero. This leads into a discussion of the nature of the neutron; i.e., whether it is "elementary" or "composite."

II. POINT-SOURCE LIMIT

The main feature of interest in this paper is the existence and nature of the point-source limit. It is well known that the charged scalar static model is renormalizable in the point-source limit; however, since the model of I is only an approximation to charged scalar theory, it is necessary to show explicitly that it has a point-source limit.

We remind the reader that there are three parameters in our theory: m_0, the unrenormalized neutron mass, g_0, the unrenormalized πpn coupling constant, and the cutoff α^{-1}. The mass of the pion is fixed at unity and the zero of the energy scale is taken at the proton mass. The unrenormalized neutron mass is determined by requiring that the physical mass of the neutron be equal to that of the proton. Therefore, we have two parameters left at our disposal.

The point-source limit is computed as follows. First, for fixed α we find the curve for the N^{++} isobar bound state $\omega_B=\omega_B(g^2,\alpha)$, where g^2 is the renormalized πpn coupling constant. This eliminates g_0^2 as a parameter in favor of the physical g^2. Only then is α taken to zero to yield a point interaction.

Since we do not know the functional form of $\omega_B(g^2,\alpha)$ for fixed cutoff, it is approximated by making least-square fits to the data using polynomials in g^2 up to fifth order. For each α,

$$\omega_B=\sum_i a_i(\alpha)g^{2i}. \qquad (2.1)$$

Figure 1 shows cubic fits to the isobar bound-state data for $\alpha^{-1}=30$ and $\alpha^{-1}=90$. Higher-order fits give similar results.

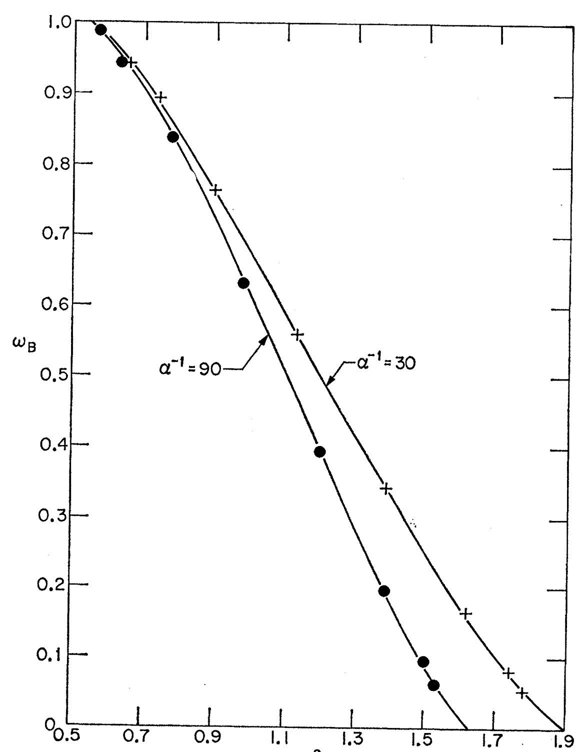

FIG. 1. Isobar bound-state energy ω_B versus renormalized coupling constant g^2 for fixed cutoff α.

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[1] D. F. Freeman, G. R. North, and M. H. Rubin, Phys. Rev. 188, 2426 (1969).

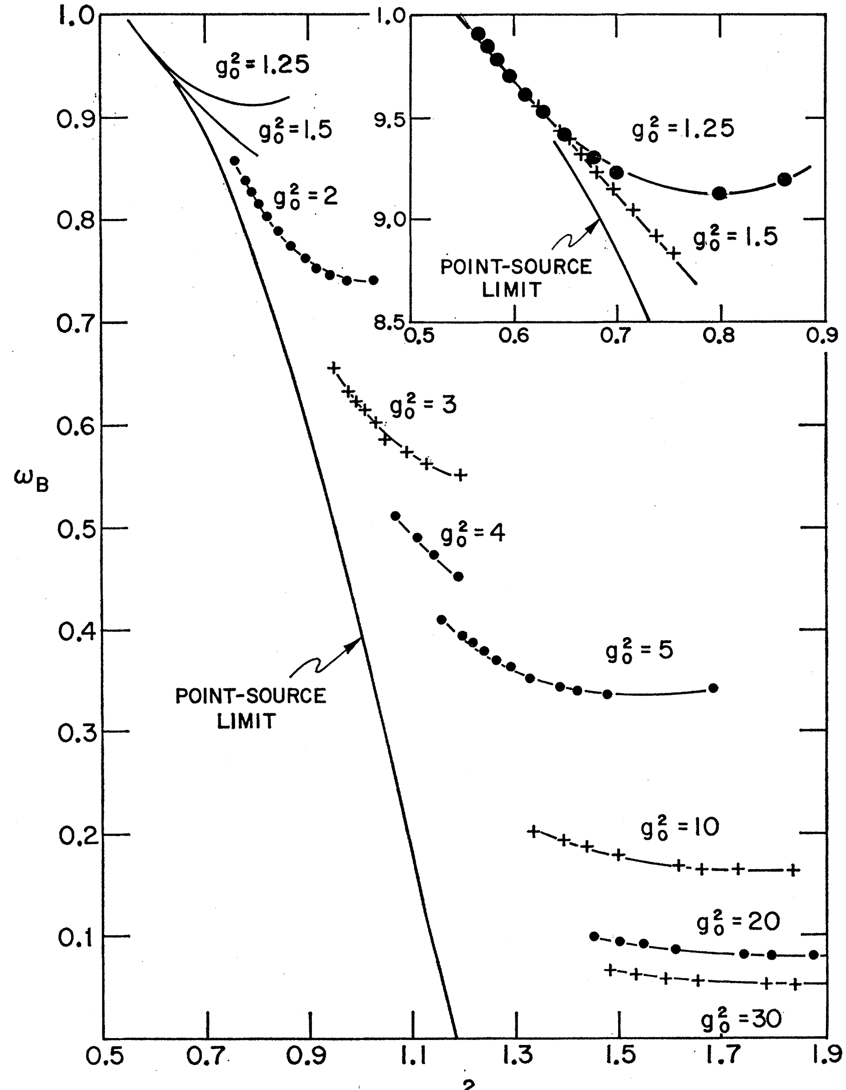

FIG. 2. Isobar bound-state energy ω_B versus renormalized coupling constant g^2 for fixed unrenormalized coupling constant g_0^2. The point-source limit is explained in the text.

The point-source limit is obtained by extrapolating the coefficients in Eq. (2.1) to $\alpha=0$. Since here we must extrapolate rather than interpolate, our lack of knowledge of the functional form of $\omega_B(g^2,\alpha)$ is more serious. Consequently we have used simple models to suggest the form

$$a_i(\alpha)\approx p_i+q_i/\ln\alpha \tag{2.2}$$

for small α. p_i and q_i are found by making least-square fits to the a_i obtained from Eq. (2.1) for several α's. For purposes of comparison, the forms

$$a_i(\alpha)\approx p_i+q_i\alpha \tag{2.3}$$

and

$$a_i(\alpha)\approx p_i+q_i\alpha^2 \tag{2.4}$$

for small α were also used. We believe that Eq. (2.2), when used to extrapolate, leads to a reasonable point. source limit. Our reasons for this are discussed in Sec-IV.

III. DETAILS OF CALCULATION

The numerical computation at fixed cutoff was performed as indicated in I, with a few changes: (1) The calculation of g^2 was improved by using a better extrapolation to zero energy, and by calculating Z_1^{-1} and Z_2 as the primary quantities. (2) Redundant calculations were eliminated to reduce the running time to 0.6 min per iteration. (3) The interpolation parameter (see I, Appendix B) q was set to 0.25 at the strongest coupling used; this gave better convergence than the much smaller values for q indicated in I. (4) Up to 26 iterations were needed with the strongest

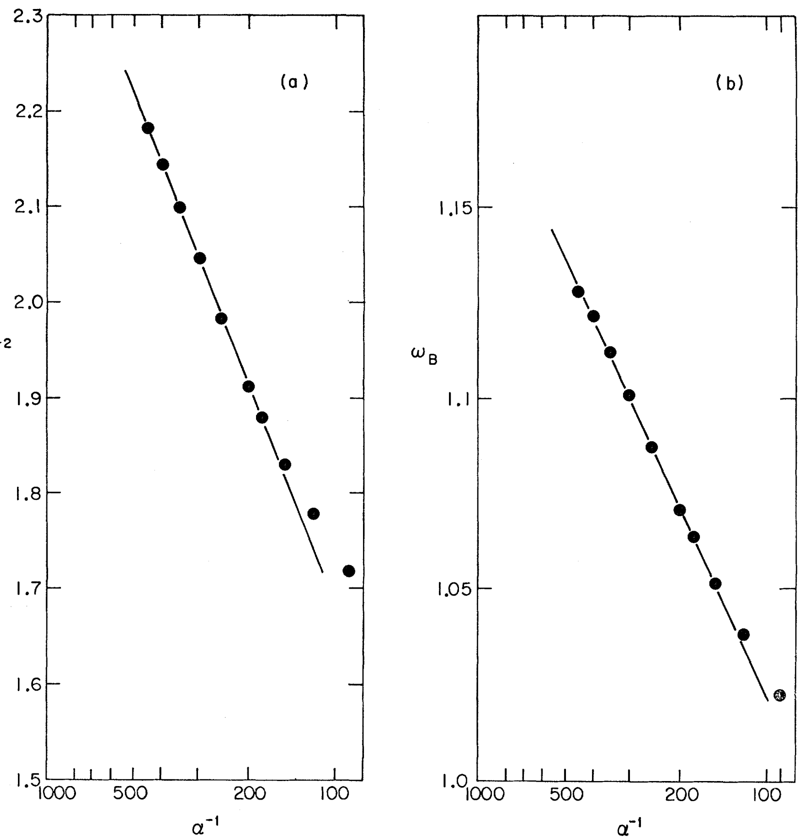

FIG. 3. (a) Renormalized coupling constant g^2 versus cutoff α for $g_0^2=1.25$. (b) Isobar bound-state energy ω_B versus cutoff α for $g_0^2=1.25$.

coupling to obtain satisfactory convergence; the results of I had larger errors than we realized.

For the above reasons, the exact numbers from I are not to be relied upon. However, we have found no reason to change any of the qualitative conclusions to which they led us.

In order to see how close our numerical solutions are to the solutions of the actual integral equations of our model, we varied our approximation methods and checked for stability. We changed the fixed set of energies $-\infty < E_i < 1$ (described in I) at which calculations were carried out by slightly varying the existing points and adding others. All results of the two calculations agreed to within 0.3%. In addition, the accuracy of our Gaussian quadrature was checked by replacing the method used for the bulk of the computations, based on a ten-point Gaussian, by one based on five two-point Gaussians. Some hand computations have convinced us that the latter is more reliable. Direct comparison indicates that g^2 and ω_B agree to within 3%, and both are consistently larger in our more accurate calculation. We therefore believe that our data, as presented, have an over-all systematic error, and that the true renormalized coupling should be slightly larger and the N^{++} isobar slightly less bound.[2] Because of the immense amount of computer time involved, we have not redone the calculations.

[2] This is for fixed g_0^2 and α. The curves $\omega_B=\omega_B(g^2,\alpha)$ will remain the same shape but will be displaced slightly to the right.

IV. DISCUSSION

A. Point-Source Limit

In Sec. II we gave a prescription for the limit $\alpha \to 0$ in which all results are expressed in terms of α and the renormalized coupling constant g^2, thereby eliminating g_0^2. We would now like to show that our prescription gives a nontrivial limit in the sense that the scattering matrix is not identically 1, to discuss the features of that limit, and to indicate to what extent our numerical results for it, indicated in Fig. 2, are reliable.

In discussing our results, it is convenient to consider first the weakly bound isobar state ($\omega_B \approx 1$). In this region, Fig. 1 shows that $\omega_B(g^2,\alpha)$ does not vary greatly with α for cutoffs $\alpha^{-1} \geq 30$. This is consistent with the intuitive notion that the short-range part of the "potential," or far-away singularities of the scattering matrix, do not contribute significantly in weak-binding situations. This is emphasized by the data of Fig. 2, where there are two cases where markedly different values of g_0^2 and α give the same values of g^2 and ω_B to better than 1%. We conclude that insofar as calculating g^2 and ω_B is concerned, we have reached the point-source limit. Since we have nonzero coupling and a real bound state in this region, this limit is not trivial. Furthermore, any extrapolation to $\alpha=0$ in this region should be satisfactory; in practice, Eqs. (2.2), (2.3), and (2.4) did give very nearly equal results.

In this same weak-binding region the curves $g_0^2=\text{const}$ (Fig. 2) are clearly asymptotic to the point-source-limit curve. As $\alpha\to 0$, ω_B increases and g^2 decreases. In Fig. 3 we see that for $g_0^2=1.25$ and $\alpha\to 0$, g^{-2} and ω_B increase approximately as $\ln\alpha$. Therefore at the limit $\alpha=0$, the πpn coupling vanishes, and the real part of the pole corresponding to the $N^{++}\to\infty$, this is clearly a trivial theory. Figure 2 suggests that the same result holds for any fixed g_0^2, inasmuch as the fixed g_0^2 curves show similar forms. We conclude that the prescription $g_0^2=\text{const}$, $\alpha\to 0$ leads to a trivial theory for any finite g_0^2; therefore, our point-source limit must correspond to infinite g_0^2.

Similar arguments show that in our point-source limit, m_0 is infinite, and Z_1 and Z_2 are zero.[3,4] For example, contours of constant $Z_1^{-1}<\infty$ on the ω_B-versus-g^2 graph behave in much the same way the constant g_0^2 contours behave. As $\alpha\to 0$ along a constant Z_1 contour, we do not obtain any physically understandable limit. Since we do not obtain the point-source limit, we conclude Z_1^{-1} must be infinite in this latter limit.

Outside the region of weakly bound states, we must extrapolate to reach the point-source limit. As indicated in Sec. II, this was done in three different ways, corresponding to the use of Eqs. (2.2), (2.3), and (2.4). These three methods gave markedly different forms for $\omega_B=\omega_B(g^2)$ outside the weak-binding region, but had approximately the same goodness of fit to the polynomial coefficients. The limits based on Eqs. (2.3) and (2.4) were quite close to the curve $\omega_B=\omega_B(g^2,\alpha)$ for the largest cutoff employed; that based on Eq. (2.2) lay further away. We noted that the $g_0^2=\text{const}$ curves were asymptotic to the point-source limit. Clearly the curve $g_0^2=30$ in Fig. 2 is only slowly turning up, and the point-source-limit curve must be markedly displaced to the left of it. Only our logarithmic extrapolation satisfied this criterion.

Also, the results in Fig. 3 show logarithmic dependence on the cutoff very clearly. We take this to be a confirmation of the predictions of our simple models.[5] For these reasons we believe that the extrapolation based on Eq. (2.2) gives the best result for our point-source limit, shown in Fig. 2.

[3] It is widely believed that any solution to a realistic field theory has infinite g_0^2, m_0, Z_1^{-1}, and Z_2^{-1}. See L. D. Landau, in *Theoretical Physics in the Twentieth Century*, edited by M. Fierz and V. F. Weisskopf (Interscience, New York, 1960).

[4] In connection with this it is interesting to note that while the physical quantities have attained their point-source-limit values, the unphysical quantities m_0, Z_1, and Z_2 have not. In particular, when α^{-1} and g_0^2 are increased so that ω_B and g^2 remain approximately constant, m_0, Z_1^{-1}, and Z_2^{-1} increase.

[5] The limit involved is not the same as the point-source limit, but we nevertheless expect that it indicates the correct asymptotic dependence on cutoff.

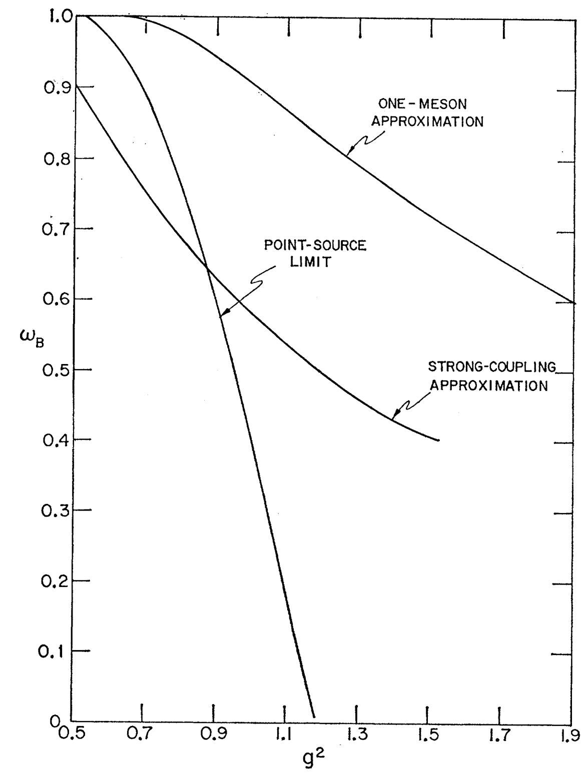

FIG. 4. Isobar bound-state energy ω_B versus renormalized coupling constant g^2: comparison of point-source limit with two approximations to charged scalar theory (see text).

It is not easy to estimate the error in our point-source-limit curve, except in the weakly bound region, where we know it to be small. Thus, it is possible that there is a large error in the mean slope of the limit curve; it is almost certain that some details of the shape are incorrect. However, these are not our primary interest[6]; rather, we concern ourselves largely with the properties discussed above, which we are confident are true for our model independent of numerical uncertainties in our work.

We also mention a second limiting procedure that gives a nontrivial result: $g_0^2\to\infty$ for $\alpha>0$, followed by $\alpha\to 0$. The first step gives $g^2=g_{\max}^2(\alpha)$ and $\omega_B=0$, independent of α. (See I for a full discussion of this limit.) The second step yields $g^2=g_{\max}^2(0)=g_{\max}^2$ and $\omega_B=0$; this is simply one point of our point-source limit. Some features of this second limit are discussed below.

B. Comparison with Charged Scalar Theory

Figure 4 shows, in addition to our point-source limit, two approximate solutions to the charged scalar static model with point interactions. The one-meson approxi-

[6] Of course the limit could in principle be reached directly by the use of very large cutoffs, if such matters were of interest; however, the expense in terms of computation time would be prohibitive.

mation provides a rigorous upper bound. The strong-coupling approximation takes a result of Goebel,[7] valid asymptotically in g^2, and extends it to the range shown. The comparison is discussed at length in I. Briefly, the strong-coupling calculation includes inelastic effects omitted in our model. The fact that our model shows stronger binding for most couplings means that the inelastic binding effects we include are greater than those of the exact theory.

C. Structure of Neutron

It is of some interest to examine the behavior of the neutron-propagator renormalization constant Z_2 and the πpn-vertex renormalization constant Z_1. The vanishing of these constants is a much studied criterion for the compositeness of a particle.[8] In our model there are two limits in which Z_1 and Z_2 vanish: infinite g_0^2 for fixed α, and the point-source limit.

Finite Cutoff

For fixed cutoff, the vanishing of the renormalization constants as g_0^2 goes to infinity is very similar to the behavior of the usual field-theoretic models used to study compositeness.[8] In I it was shown that for fixed cutoff in the infinite-g_0^2 limit: g^2 is uniquely determined, $g^2=g_{\max}^2(\alpha)$,[9] the N^{++} isobar and the neutron become degenerate, and the scattering in the cross and direct channels become equal. Since the N^{++} is a bound state in our model, this equivalence argues strongly for the neutron being a bound state of a p and a π^-, the intuitive meaning of composite. Ida's definition of a composite particle[8,10] as having $Z_1=0$, $Z_2=0$, and its mass and coupling constants uniquely determined is also satisfied here.

At this point we would like to indicate a conjecture about our model in the limit we have been discussing: If we replace the Born term, Eq. (2.7) of I, by its limiting form,

$$V_{\omega'\omega}(E)=-\lambda f(\omega')f(\omega)\,,\qquad \lambda=\lim_{g_0^2\to\infty} g_0^2/m_0\,, \tag{4.1}$$

then the solution of Eqs. (2.3)–(2.6) of I will give just our $g_0^2\to\infty$ limit. Two points support our conjecture: First, Eq. (4.1) clearly makes the direct and cross channels identical, since $V=V^\times$. This is observed in our computations, as noted above. Second, a Born term such as given in Eq. (4.1) presupposes a Fermi $\pi p\pi p$ coupling in the interaction Hamiltonian. The replacement of Yukawa coupling by Fermi coupling is a well-known feature of the $Z_2=0$ limit in simple models.[8,11]

Point-Source Limit

We now turn to the point-source limit. Since this describes a theory without cutoffs, we expect it to be closer to a realistic field theory and hopefully also to the real world. As noted above, in this limit Z_1 and Z_2 vanish and g_0^2 is infinite. However, the isobar bound state is not degenerate with the neutron, except at the point of maximum coupling, and the renormalized πnp coupling g^2 remains a free parameter of the theory, with the restriction $g^2\leq g_{\max}^2$. Insofar as we can determine, in the range $g^2<g_{\max}^2$ the neutron does not fit into any of Ida's classifications.[10] $Z_1=0$ and $Z_2=0$ in the usual models fixes both the mass and coupling of the particle under consideration, and gives a Lagrangian with the neutron removed and *no* compensating Fermi interaction present. In our model, the neutron coupling g^2 is not fixed; moreover, the Yukawa coupling through the neutron is the whole of the interaction, so that its removal would yield a trivial theory, in contradiction to our results. Again, $Z_2=0$ ($Z_1\neq 0$) in the usual models yields the Fermi coupling mentioned above. However, as discussed for our $g_0^2\to\infty$ limit, this makes the direct and crossed channels identical, and the neutron and N^{++} isobar degenerate, again in contradiction to our results.

We now indicate on general grounds why we do not expect the usual classifications to be helpful in treating our model. The treatments of Lagrangian models set $Z_1=0$, and $Z_2=0$ for finite cutoffs and only then go to a point-source limit. The justification for such a procedure is the hope that the results obtained at finite cutoff are independent of the cutoff, and hence the limits can be taken in this order without doing violence to the theory. Our model leads us to believe that this is not the case: This procedure gives a single point (the intersection of our $g_0^2\to\infty$ and point-source limits) with $g^2=g_{\max}^2$ and $\omega_B=0$, i.e., the N^{++} isobar degenerate with the neutron. While the classifications seem to treat this single point adequately, it seems clear that they cannot apply to our model for $g^2<g_{\max}^2$. Dispersion-relation calculations seem limited by assumptions about the convergence of integrals along the left-hand cut. The assumption that the discontinuity across the left-hand cut is "sufficiently bounded" is an assumption about the short-range forces and therefore through crossing about the multiparticle intermediate states along the right-hand cut. These states have not received a satisfactory treatment and our model indicates that their effects are large. Although our model overestimates such effects, we believe it to be a reliable guide to their importance. We conjecture that the same objections to the use of these

[7] C. J. Goebel, Phys. Rev. **109**, 1846 (1958).

[8] See the review articles by K. Hayashi *et al.*, Fortschr. Physik **15**, 625 (1967); and H. Osborn, Ann. Phys. (N. Y.) **47**, 308 (1968).

[9] This implies $Z_2Z_1^{-2}$ vanishes in this limit.

[10] M. Ida, Progr. Theoret. Phys. (Kyoto) **34**, 92 (1962); **34**, 990 (1965).

[11] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

classifications hold for any point-source theory with "realistic" features such as crossing.

It should be noted that even if our neutron is in some sense a composite particle, our solution is not a bootstrap solution, since the proton is never modified. An extension of our model in which the proton and neutron are treated identically is now in progress.

V. CONCLUSION

The model of I has been shown to have a point-source limit. The main features of this model are that it satisfies off-shell two-particle unitarity below the three-particle thresholds and that it is crossing symmetric. These properties mean that the elastic scattering amplitudes have intermediate states containing large numbers of particles. This feature is not included in most nontrivial soluble models studied to date and provides us the opportunity to examine the effects of these states. The main drawbacks of the model are that it is a static model and that it fails to satisfy unitarity above the three-particle thresholds. We believe these drawbacks do not vitiate the insights into "realistic" field theories gained from our model.

In particular, certain features of the point-source limit of our model lead us to question the cogency of much of the work previously done on the question of compositeness of strongly interacting particles. The force of many of these investigations has been to show that the vanishing of renormalization constants is a sufficient condition to ensure that a particle is composite. We have discussed the conclusions that our model suggests in Sec. IV C.

Obtaining an adequate definition of compositeness seems to be the main difficulty. For example, it is generally argued that the vanishing of the propagator or wave-function renormalization constant (Z_2 in this paper) is a reasonable criterion for compositeness in that it corresponds to the intuitive notion of a particle composed of elementary constituents (we avoid the additional subtleties of bootstrap situations). This is formally stated through a sum rule for the spectral density function.[12] This argument is certainly valid for the simple models whose cogency we have questioned; it seems necessary to establish that this connection is useful in general. That is, is it possible to measure the spectral density function? This appears to be a nontrivial problem.[13]

It is sometimes argued that the renormalization constants will always vanish for a "realistic" field theory. If this is true then it is difficult to see that vanishing renormalization constants is a useful definition of compositeness, since it is simply a statement about the solutions to "realistic" field theories.

There seems to be some possibility that various definitions of the wave-function renormalization are inequivalent. For example, in the neutral scalar static model[12] Z_2 is defined as the overlap between the bare and physical nucleon state, and its vanishing in the point-source limit has nothing whatsoever to do with compositeness in the intuitive sense of a particle composed of *physical* elementary particles. Furthermore, it seems that the definition of Z_2 based on the spectral density should give $Z_2=1$ since in the point-source limit this theory is trivial.

ACKNOWLEDGMENTS

We wish to thank Dr. Michael Whippman and Dr. Ralph D. Amado for useful discussions about the problems associated with treating composite particles, and Dr. Robert Powers for discussions about these problems from the point of view of axiomatic field theory.

[12] See, for example, G. Barton, *Introduction to Advanced Field Theory* (Wiley, New York, 1963).

[13] A review of attempts to treat this problem is given in G. Barton, *Introduction to Dispersion Techniques in Field Theory* (Benjamin, New York, 1965), Chap. 8.