Behavior of Commutator Matrix Elements at Small Distances. I. Existence and Structure of Equal-Time Limits

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The behavior of commutator matrix elements at small distances is investigated in the framework of general quantum field theory. The equivalence between current density commutators and the commutators of a finite number of charge moments and a current density at equal times is proved by means of microcausality but without using the spectrum condition. If N is the (finite) order of the commutator, then the equal-time limits of the commutators between all charge moments of degree higher than $2N+m-1$ (m fixed between zero and N) and a current density vanish. The equal-time commutators between the first $2N+m-1$ charge moments and a current density exist if and only if the equal-time limit of the corresponding current densities exists and is a sum of $2N+m-1$ derivatives of the δ function in the space variables $\mathbf{x}$. The coefficients of the individual δ functions are identical to the equal-time limits of the charge moments and one density. If the spectrum condition holds in addition, then m is equal to zero.

I. INTRODUCTION

In the last decade, much effort has been devoted to the analysis of the asymptotic behavior in space and time of matrix elements of field operators.[1,2] Important results such as the cluster decomposition, various asymptotic conditions,[1,2] and (last but not least) the weak asymptotic series of field operators[3] have been derived. All these results emerged from an ingenious exploitation of a few general principles. On the other hand, very little is known about the behavior of matrix elements at small distances. The reason obviously is that this region is much more sensitive to the unknown dynamics of interacting systems than is the asymptotic region, where these systems are greatly separated from each other.

Successful investigations of broken symmetries are based on the postulate of equal-time commutation relations between the electromagnetic and weak hadron currents $j^{\mu}{}_{\alpha}(x)$ as well as their generalized charges[4,5]:

$$Q_{\alpha}(x^0)=\int d^3x\, j^0{}_{\alpha}(x). \tag{1}$$

The following three sets of commutation relations proposed by Gell-Mann[6] combine assumptions on the behavior of current matrix elements in small space-time regions with algebraic structures related to an underlying broken symmetry group:

$$\lim_{x^0\to 0}\langle\Psi|[Q_{\alpha}(x^0),Q_{\beta}(0)]|\Phi\rangle^T=ic^{\alpha\beta\gamma}\langle\Psi|Q_{\gamma}(0)|\Phi\rangle, \tag{2}$$

$$\lim_{x^0\to 0}\langle\Psi|[Q_{\alpha}(x^0),j^{\mu}{}_{\beta}(0)]|\Phi\rangle^T=ic^{\alpha\beta\gamma}\langle\Psi|j^{\mu}{}_{\gamma}(0)|\Phi\rangle, \tag{3}$$

$$\lim_{x^0\to 0}\langle\Psi|[j^0{}_{\alpha}(x),j^{\mu}{}_{\beta}(0)]|\Phi\rangle^T =ic^{\alpha\beta\gamma}\langle\Psi|j^{\mu}{}_{\gamma}(0)|\Phi\rangle\delta(\mathbf{x}). \tag{4}$$

The assumption on the structure of the matrix elements in space-time is contained in the existence of the limits and the δ function in Eq. (4). The algebraic structures are represented by the structure constants $c^{\alpha\beta\gamma}$ of the broken symmetry group occurring on the right-hand side. Ψ and Φ are state vectors from the Hilbert space of physical states. T means subtraction of the vacuum expectation values of the commutator, which removes all possible c-number terms.

A considerable amount of work has been invested in the analysis of the algebraic aspects of this scheme,[4,5] whereas the analysis of the space-time structure has been largely ignored. Serious difficulties which occur in the purely algebraic treatment of the relations[4,7,8] seem to disappear if one takes into account also the space-time structures—especially microcausality—of these relations.[9]

The great success of this scheme rests on the occurrence of the charges $Q_{\alpha}(x^0)$ on the left-hand side of (2) and (3). They can be replaced via Gauss's theorem[10,11]

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[1] R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, R. I., 1965). References 1 and 2 contain a complete bibliography of the original literature.

[2] K. Hepp, *Axiomatic Field Theory and Particle Symmetries* (Gordon and Breach, New York, 1965), Vol. 1.

[3] H. Araki and R. Haag, Commun. Math. Phys. **4**, 77 (1967).

[4] B. Renner, *Current Algebras and their Applications* (Pergamon, New York, 1968).

[5] S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[6] M. Gell-Mann and J. Ne'eman, *The Eightfold Way* (Benjamin, New York, 1964).

[7] I. T. Grodski and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

[8] S. F. Chang, R. Dashen, and L. O'Raifeartaigh, Phys. Rev. **182**, 1805 (1969).

[9] U. Völkel and A. H. Völkel, Nuovo Cimento **63**, 203 (1969).

[10] B. Schroer and P. Stichel, Commun. Math. Phys. **3**, 258 (1966).

[11] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40A**, 1171 (1965).

by a four-dimensional integral over the divergences of the corresponding currents $\partial_\mu j^\mu{}_\alpha(x)$, which according to Haag[12] can be considered as interpolating fields of spin-zero particles. Partial conservation of axial-vector current (PCAC) and dispersion relations finally lead to scattering cross sections on the left-hand side of (2) and (3). In order to treat the density commutators in the same way, one would like to replace them by a *complete equivalent* set of relations containing a charge or related objects like charge moments

$$Q_\alpha(x^0,m) =: \int d^3x \Big(\prod_{i=1}^{3} (x^i)^{m_i} \Big) j^0{}_\alpha(x)\,, \tag{5}$$

$$m_i \geq 0 \text{ integer}\,, \quad \mathbf{m} =: (m_1,m_2,m_3)\,,$$

where the equality sign with the colon means "is by definition."

That such an equivalent set of relations may exist was indicated by a recent publication.[9] There, among other things, it has been shown by consistent use of microcausality, current conservation (or what corresponds to it for nonconserved currents), and some further technical assumptions (such as high-energy behavior, as well as interchanges of limits and integrations) that the density commutator between one-particle states has the form

$$\lim_{x^0\to 0} \langle \Psi | [j^0{}_\alpha(x), j^0{}_\beta(0)] | \Phi \rangle^T = \sum_{\nu_i=1}^{3} A_{\alpha\beta}{}^{(\nu_1,\nu_2,\nu_3)} \prod_{i=1}^{3} \frac{\partial^{\nu_i}}{(\partial x^i)^{\nu_i}} \delta(\mathbf{x})\,, \tag{6}$$

with

$$A^\nu{}_{\alpha\beta} = \lim_{x^0\to 0} \langle \Psi | [Q_\alpha(x^0,\nu), j^0{}_\beta(0)] | \Phi \rangle^T .$$

In the present paper we will give a proof for the equivalence between the equal-time commutator of two current densities and the equal-time commutators of a *finite* number of charge moments and one current density. The number depends on the order of the density commutator. The proof rests on the general principles of quantum field theory, especially microcausality. Neither spectrum condition nor additional technical assumptions will be imposed.

The proof emerges from the following two simple observations. (i) By inspection of the right-hand side of (4), it is obvious that the equal-time limit of densities is to be considered as a limit in the sense of generalized functions.[13,14] This means that for every member $h(\mathbf{x})$ from a certain class of test functions, we have

$$\lim_{x^0\to 0} \langle \Psi | \Big[\int d^3x\, j^0{}_\alpha(x) h(\mathbf{x}), j^\mu{}_\beta(0) \Big] | \Phi \rangle^T = i c^{\alpha\beta\gamma} \langle \Psi | j^\mu{}_\gamma(0) | \Phi \rangle h(\mathbf{0})\,. \tag{7}$$

Now the main difference between (3) and (4) or (7) is that in the case of the charge-density commutator one demands the existence of the limit for only one single test function $h_0(\mathbf{x}) \equiv 1$, whereas for the density-density commutator one requires the existence for a whole class of functions $h(\mathbf{x})$ with the same right-hand side. Furthermore, we obtain the commutator of a charge moment $Q_\alpha(x^0,\mathbf{m})$ and a density by choosing the special function

$$h_m(\mathbf{x}) = \prod_{i=1}^{3} (x^i)^{m_i}$$

in (7). So the problem is reduced to whether the equal-time limit of a current commutator is completely known when we know its value for a finite number of functions of the form

$$h_{\mathbf{m}}(\mathbf{x}) = B(\mathbf{m}) \Big(\prod_{i=1}^{3} (x^i)^{m_i} \Big)\,, \quad m_i = 0, 1, \ldots, M\,. \tag{8}$$

This property we expect from microcausality and finite order of commutators. (ii) Microcausality tells us that the current commutator vanishes for spacelike arguments $(x^0)^2 - \mathbf{x}^2 < 0$ (Fig. 1). For sufficiently small values of $|x^0| < \epsilon$ the commutator "sees" only the local behavior of the test functions in the neighborhood of the origin and is independent of their global behavior. In the neighborhood of the origin, we can approximate the test functions by linear combinations of (8) to arbitrary precision. If the commutator is not too singular at the origin (is of finite order in the language of distribution theory[13,14]), we expect the limit $x^0 \to 0$ to vanish for all $h_{\mathbf{m}}(\mathbf{x})$ with sufficiently high m_i since the support in $\mathbf{x}$ is reduced to the point $\mathbf{0}$ in this limit.

In Sec. II we give a precise formulation of what is meant by charges and equal-time limits, thereby making more precise the meaning of Eqs. (1)–(5). We collect some results of general quantum field theory and construct a representation of commutator matrix elements which we need for the proofs.

In Sec. III we first prove that the equal-time limit vanishes for all C^∞ functions $h(\mathbf{x})$ with the property

$$\lim_{\epsilon\to 0} \epsilon^{-(2N+m)} h(\epsilon\mathbf{x}) < \infty\,, \tag{9}$$

where N is the order of the commutator and m is a certain number with $0 \leq m \leq N$. All the functions (8) with $M \geq 2N+m$ have this property. This fact is the key to all our conclusions. Since every test function $h(\mathbf{x})$ can be split into a polynomial of degree $2N+m$ and a remainder $\hat{h}(\mathbf{x})$ with the property (9),[15]

$$h(\mathbf{x}) = \sum_{n=0}^{2N+m} \frac{1}{n!} \Big(\sum_{i=1}^{3} x^i \frac{\partial}{\partial z^i} \Big)^n h(\mathbf{z}) \Big|_{\mathbf{z}=0} + \hat{h}(\mathbf{x})\,, \tag{10}$$

[12] R. Haag, Phys. Rev. **112**, 669 (1958).

[13] L. Schwartz, *Théorie des Distributions* (Herrmann, Paris, 1957/59).

[14] I. M. Gelfand and G. E. Schilow, *Verallgemeinerte Funktionen (Distributionen)* (Deutscher Verlag der Wissenschaften, Berlin, 1960/62).

[15] The expression

$$\Big(\sum_{i=1}^{3} x^i \frac{\partial}{\partial z^i} \Big)^n$$

means to take the nth power and apply $(\partial/\partial z^i)^n$ to the function on the right.

it immediately follows that if the equal-time limits of the commutators between the first $2N+m-1$ charge moments and a current density exist, then the equal-time limit of the commutator of two densities exists and is a sum of at most $2N+m-1$ derivatives of the δ function, and conversely. Of course, this is true for arbitrary local fields of finite order.

Finally, in Sec. IV we draw some conclusions, for instance, on the structure of one-particle contributions to equal-time commutators.

II. POSTULATES AND MATHEMATICAL CONSEQUENCES

We consider the currents $j^\mu{}_\alpha$ to be members of a polynomial algebra of fields which satisfy the usual postulates of Wightman fields except for the spectrum condition.[1,16,17] In detail, we require for the fields (I) translational invariance; (II) local commutativity (microcausality); (III) that the fields

$$j^\mu{}_\alpha(f)=\int d^4x\, f(x)j^\mu{}_\alpha(x)\,, \tag{11}$$

smeared with test functions $f(x)$ from[18] S_4,[13,14] are operators with a dense domain in a Hilbert space H. From this assumption it follows[16,17] that the matrix elements of the fields are distributions of finite order N.[13,14] They can be represented as a Nth-order derivative of a continuous function $\mu(x)$:

$$\langle\Psi|j(x)|\Phi\rangle=D_x{}^{N,m}\mu(x)$$

$$=\frac{\partial^{N-m}}{\partial(x_0)^{N-m}}(\partial^{|m|}/\prod_{i=1}^{3}(\partial x^i)^{m_i})\mu(x)\,, \tag{12}$$

$$|m|=\sum_{i=1}^{3}m_i.$$

$\mu(x)$ is polynomial bounded:

$$\mu(x)/[1+\sum_{i=0}^{4}(x^i)^2]^\kappa<\infty \tag{13}$$

for some finite integer κ. The support of $\mu(x)$ is contained in an arbitrarily small neighborhood of the support of $\langle\Psi|j(x)|\Phi\rangle$.

If, in addition, the spectrum condition holds, then according to Borchers,[19] m can be set equal to zero. This is the only reason for the slight changes in our results when the spectrum condition holds.

[16] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

[17] L. Gårding and A. S. Wightman, Arkiv Fysik **28**, 129 (1964).

[18] For technical simplicity we restrict ourselves to tempered distributions. By a slightly larger amount of technicalities the proofs can be extended to D_4 and even more general function spaces characterized in Sec. IV.

[19] H. J. Borchers, Nuovo Cimento **33**, 1600 (1964).

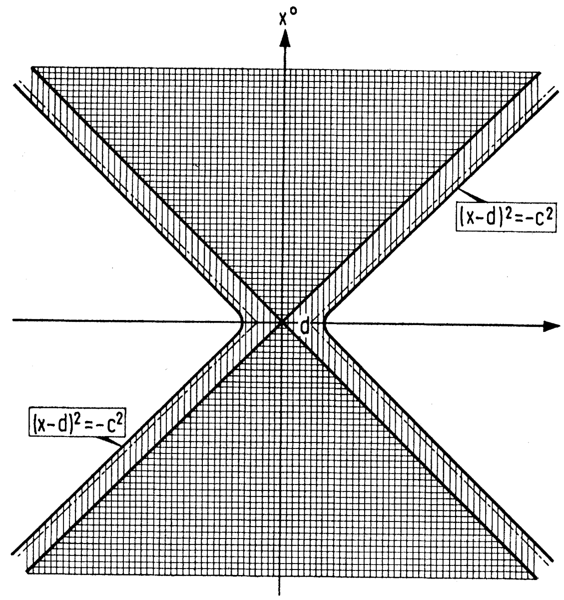

FIG. 1. Support of $F(x)$ (▦) and $\mu(x)$ (|||).

Since the matrix elements of the fields are distributions, neither of the charges (1), (5) exists as an operator with dense domain (because of misbehavior at infinity) unless it is conserved; nor do the equal-time limits exist in the usual sense (because of misbehavior at the origin).[10] As is well known[9,10] the recipe is to consider (1)–(5) as limiting processes of the following smeared-out objects:

$$j^0{}_\alpha(\psi_\epsilon;\,f_R(\mathbf{m}))$$

$$=:\int d^4x\,\psi_\epsilon(x^0)(\prod_{i=1}^{3}(x^i)^{m_i})f_R(\mathbf{x})j^\mu{}_\alpha(x)\,, \tag{14}$$

where (i) $f_R(\mathbf{x})$ is a C^∞ function with the property

$$\begin{aligned}f_R(\mathbf{x})&=1 \quad\text{for}\quad |\mathbf{x}|\leq R\\ &=0 \quad\text{for}\quad |\mathbf{x}|>R+\Delta R,\end{aligned} \tag{15}$$

(ii) ψ_ϵ is a sequence of type δ, i.e.,

$$\psi_\epsilon(x^0)=:(1/\epsilon)\psi(x^0/\epsilon)\,, \tag{16}$$

where the $\psi(x^0)$ are C^∞ functions with the properties

$$\text{supp}\psi(x^0)\in[-a,a] \quad [\psi\in D_1(a)],$$

$$\int_{-a}^{+a}dx^0\psi(x^0)=1\,. \tag{17}$$

If we formally take the limits $\epsilon\to0$ and $R\to\infty$ in (14), we get back the charges (1) and their moments (5) at the point $x^0=0$.

Now the precise meaning of the equal-time limits (3) and (4) is[20]

$$\lim_{\epsilon\to 0}\lim_{R\to\infty}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon;f_R(0)),j^\mu{}_\beta(0)]|\Phi\rangle^T = ic^{\alpha\beta\gamma}\langle\Psi|j^\mu{}_\gamma(0)|\Phi\rangle\,, \quad (18)$$

$$\lim_{\epsilon\to 0}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon,h),j^\mu{}_\beta(0)]|\Phi\rangle^T = ic^{\alpha\beta\gamma}\langle\Psi|j^\mu{}_\gamma(0)|\Phi\rangle h(\mathbf{0}) \quad (19)$$

for all $h(\mathbf{x})$ from S_3. What we have to show is

(a) $$\lim_{\epsilon\to 0}\lim_{R\to\infty}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon,h),j^\mu{}_\beta(0)]|\Phi\rangle^T=0 \quad (20)$$

for all $h(\mathbf{x})\in S_3$ with the property (9);

(b) $$\lim_{\epsilon\to 0}\lim_{R\to\infty}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon,f_R(\mathbf{v})),j^\mu{}_\beta(0)]|\Phi\rangle^T$$

exists for all integers $v_i\geq 0$ with

$$\sum_{i=1}^{3} v_i\leq 2N+m-1$$

if and only if

$$\lim_{\epsilon\to 0}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon;h),j^\mu{}_\beta(0)]|\Phi\rangle^T = \sum_{\nu=0}^{2N+m-1}\sum_{i=0}^{\nu}\sum_{j=0}^{\nu-i}\frac{A(i,j,\nu-i-j)}{i!j!(\nu-i-j)!} \times\frac{\partial^\nu}{\partial z_1{}^i\partial z_2{}^j\partial z_3{}^{\nu-i-j}}h(\mathbf{z})\bigg|_{\mathbf{z}=0} \quad (21)$$

for all $h(\mathbf{x})\in S_3$;

(c)

$$A_{\alpha\beta}(\mathbf{v})=\lim_{\epsilon\to 0}\lim_{R\to\infty}\langle\Psi|[j^0{}_\alpha(\psi_\epsilon;f_R(\mathbf{v})),j^\mu{}_\beta(0)]|\Phi\rangle^T. \quad (22)$$

None of the indices $\{\alpha,\beta,\mu\}$ occurring in the expression above are essential for our further investigations. Only the order N depends on μ. We will drop them all and consider the simple expression

$$F(\psi_\epsilon,h)\equiv\int d^4x\,\psi_\epsilon(x^0)h(\mathbf{x})F(x) =:\langle\Psi|[j(\psi_\epsilon,h),g(0)]|\Phi\rangle^T. \quad (23)$$

From microcausality,

$$F(x)=0 \quad\text{for}\quad x^2<0\,,$$

it follows that

$$F(\psi_\epsilon;\mathbf{x})=:\int d^3x^0\,\psi_\epsilon(x^0)F(x^0,\mathbf{x})$$

vanishes for $\mathbf{x}^2>a^2$, since the support of ψ_ϵ is contained in $[-a,a]$. Therefore we can perform the limit $R\to\infty$ in (18)–(22) and forget the cutoff in $\mathbf{x}$. Moreover we can admit arbitrary C^∞ functions $h(\mathbf{x})$ in (23).

Our final task in this section is to construct a representation for $F(\psi,h)$ containing microcausality and the property of finite order N. A tempered distribution $F(x)$ vanishes for spacelike arguments x if and only if[18,19] it is the Fourier transform of the limit

$$\tilde{F}(\tilde{f},\mathbf{p})=\lim_{\lambda\to 0}\tilde{U}(\tilde{f},\mathbf{p},\lambda) \quad (24)$$

of a tempered solution of the five-dimensional wave equation

$$\Box_5\tilde{U}(p,\lambda)\equiv\left(\frac{\partial^2}{\partial p^{02}}-\sum_{i=1}^{3}\frac{\partial^2}{\partial(p^i)^2}-\frac{\partial^2}{\partial\lambda^2}\right)\tilde{U}(p,\lambda)=0 \quad (25)$$

with the boundary condition

$$\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\tilde{U}(\tilde{f},p,\lambda)=0\,. \quad (26)$$

According to a theorem of Gårding and Malgrange,[21] every tempered solution of the wave equation has the properties

$$\int dp^0\,f(p^0)\tilde{U}(p^0,\mathbf{p},\lambda)\in O_{M4}\,,$$
$$\int d^3p d\lambda\,g(\mathbf{p},\lambda)\tilde{U}(p^0,\mathbf{p},\lambda)\in O_{M1} \quad (27)$$

for all $\tilde{f}$ and $\tilde{g}$ from S_1 and S_4, respectively. Here O_M is the space of all C^∞ functions of polynomial increase.[13] Therefore, the limits (24) and (26) are well defined. Moreover, from (24)–(26) we obtain

$$\lim_{\lambda\to 0}\frac{\partial^{2n+1}}{\partial\lambda^{2n+1}}\int dp^0\,\tilde{f}(p^0)\tilde{U}(p^0,\mathbf{p},\lambda)=0\,,$$

$$\lim_{\lambda\to 0}\frac{\partial^{2n}}{\partial\lambda^{2n}}\int dp^0\,\tilde{f}(p^0)\tilde{U}(p^0,p,\lambda) =\int dp^0\,F(p)\left(\frac{\partial^2}{(\partial p^0)^2}-\sum_{i=1}^{3}\frac{\overleftarrow{\partial}^2}{(\partial p_i)^2}\right)^n\tilde{f}(p^0)\,. \quad (28)$$

If $F(x)$ is of the order N, then Eqs. (24)–(26) have the unique solution[22]

$$\tilde{U}(p,\lambda)=\frac{1}{(2\pi)^{5/2}}\int d^4x\,e^{ipx}F(x)\left\{\sum_{k=0}^{N}(-1)^k\frac{(x^2)^k\lambda^{2k}}{(2k)!} +\Theta(x^2)\left[\cos(\lambda\sqrt{x^2})-\sum_{k=0}^{N}(-1)^k\frac{(x^2)^k\lambda^{2k}}{(2k)!}\right]\right\}\,. \quad (29)$$

[20] In the present article we are not concerned with the charge algebra (2).

[21] A. S. Wightman, *Dispersion Relations and Elementary Particles* (Hermann, Paris, 1960).

[22] V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables* (MIT Press, Cambridge, Mass., 1966).

Since

$$G(x,\lambda)=:\Theta(x^2)\left[\cos(\lambda\sqrt{x^2})-\sum_{k=0}^{N}(-1)^k\frac{(x^2)^k\lambda^{2k}}{(2k)!}\right] \quad (30)$$

is N-times continuous differentiable, the right-hand side of (29) is well defined.

The next step is to represent $\tilde{U}(p,\lambda)$ by its Cauchy data on the spacelike surface $p^0=0$. This representation will later be combined with (29). The essential trick is that in this way the smearing in x^0 in the commutator $F(\psi,f)$ will be shifted in part to a smearing over the space variables $\mathbf{x}$—the key for our proofs in Sec. III.

The Cauchy representation reads[2,18,19]

$$\int dp^0\tilde{\psi}(p^0)\tilde{U}(p,\lambda)=\lim_{p^{0\prime}\to 0}\int d^3p'd\lambda'\int dp^0\tilde{\psi}(p^0)$$
$$\times\left[\tilde{\Delta}_5(p^0,\mathbf{p}-\mathbf{p}',\lambda-\lambda')\frac{\partial}{\partial p^{0\prime}}\tilde{U}(p^{0\prime},\mathbf{p}',\lambda')\right.$$
$$\left.+\frac{\partial}{\partial p^0}\tilde{\Delta}_5(p^0,\,\mathbf{p}-\mathbf{p}',\,\lambda-\lambda')\tilde{U}(p^{0\prime},\mathbf{p}',\lambda')\right], \quad (31)$$

with

$$\tilde{\Delta}_5(p,\lambda)=-\frac{i}{(2\pi)^4}\int d^4xds\; e^{i(px-\lambda s)}\epsilon(x^0)\delta(x^2-s^2)$$

and (32)

$$\tilde{\psi}(p^0)=:\int dx^0e^{-ip^0x^0}\psi(x^0)\,.$$

Because of the following Lemma 1,

$$\tilde{\psi}^0{}_{\Delta_5}(\mathbf{p},\lambda)=:\int dp^0\tilde{\psi}(p^0)\tilde{\Delta}_5(p^0,\mathbf{p},\lambda)$$

is from S_4 if $\tilde{\psi}\in S_1$; it has compact support in $(\mathbf{p},\lambda)$ if $\tilde{\psi}$ has compact support. Therefore, according to (27), the right-hand side of (31) is well defined with the integrations and limits in the order written.

Let us consider the functions

$$\psi^n{}_{\Delta_5}(\mathbf{x},s)=:-2\pi i\int dx^0(x^0)^n\psi(x^0)\epsilon(x^0)\delta(x^2-s^2)$$
$$=-\pi i[(\mathbf{x}^2+s^2)^{1/2}]^{n-1}[\psi((\mathbf{x}^2+s^2)^{1/2})$$
$$-(-1)^n\psi(-(\mathbf{x}^2+s^2)^{1/2})]. \quad (33)$$

For the special cases $n=0,\ 1$, these are the Fourier transforms of the two expressions occurring in (31) after the p^0 integration is performed. Since $\psi^n{}_{\Delta_5}(\mathbf{x},s)$ is an even function of $(\mathbf{x}^2+s^2)^{1/2}$, the following lemma easily follows from (33).

Lemma 1: If $\psi(x^0)\in S_1$, then for any finite integer $n\geq 0$, $\psi^n{}_{\Delta_5}(\mathbf{x},s)$ is from S_4. If $\psi(x^0)$ has compact support in $|x^0|\leq a$, then also $\Psi^n{}_{\Delta_5}(\mathbf{x},s)$ has compact support contained in $\mathbf{x}^2+s^2\leq a^2$.

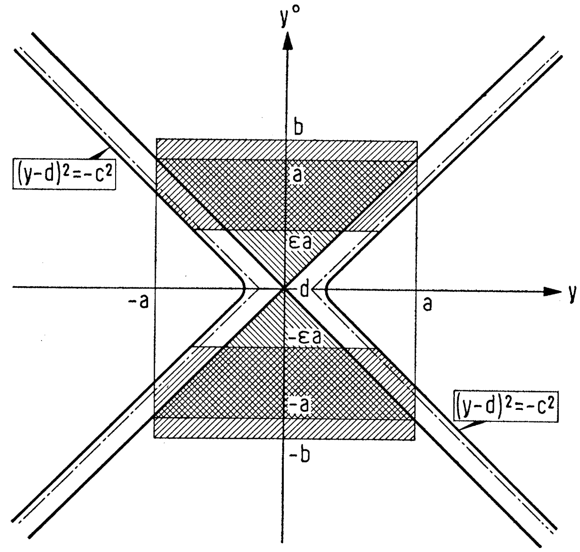

FIG. 2. Domains of integration for $\tilde{\Pi}$(⧹) and $\tilde{\Pi}$(⫽).

Introducing the Fourier transforms in the right-hand side of (31), we get

$$\int dp^0\tilde{\psi}(p^0)\tilde{h}(p)\tilde{U}(p,0)=\frac{i}{(2\pi)^{5/2}}\lim_{p^0\to 0}\int d^4xds\; e^{ip^0x^0}h(\mathbf{x})$$
$$\times[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]U(x,s)\,, \quad (34)$$

with

$$U(x,s)=\frac{1}{(2\pi)^{5/2}}\int d^4pd\lambda\; e^{-i(px-\lambda s)}\tilde{u}(p,\lambda)$$
$$=F(x)\left(\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}\delta(s)+\Theta(x^2)\left[(\sqrt{x^2})\delta(x^2-s^2)\right.\right.$$
$$\left.\left.-\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}\delta(s)\right]\right). \quad (35)$$

Now (27) implies $\int d^3xds\, g(\mathbf{x},s)U(x^0,\mathbf{x},s)$ to be from O_{C1}', the space of strongly decreasing distributions (Fourier space of O_{M1}) [13] for every $g(\mathbf{x},s)\in S_4$. Therefore, the limit can be performed if we keep the prescription that the x^0 integration has to be done at the end. Performing the s integrations by means of the δ functions in (35), we get

$$F(\psi,h)=i\int d^4x\,F(x)h(\mathbf{x})$$
$$\times\left\{\sum_{k=0}^{N}\frac{1}{(2k)!}(x^2)^k\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\right|_{s=0}$$
$$+\Theta(x^2)\left[\psi^0{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})x^0+\psi^1{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})-\sum_{k=0}^{N}\frac{1}{(2k)!}(x^2)^k\right.$$
$$\left.\left.\times\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\right|_{s=0}\right]\right\}. \quad (36)$$

$F(x)$ is the Nth derivative of a continuous polynomial-bounded function $\mu(x)$ [see Eqs. (12) and (13)]:

$$F(x)=D_x{}^{N,m}\mu(x)=\frac{\partial^{N-m}}{(\partial x^0)^{N-m}}(\partial^{|m|}/\prod_{i=1}^{3}\partial(x^i)^{m_i})\mu(x)\,,\qquad |m|=\sum_{i=1}^{3}m_i\leq N. \tag{37}$$

The support of $\mu(x)$ is contained in an arbitrary neighborhoood of the support of $F(x)$ (Fig. 1):

$$\operatorname{supp}\mu(x)\subseteq(x-d)^2>-c^2,\quad d=(0,\mathbf{d}). \tag{38}$$

Since the support of $\psi^n{}_{\Delta_5}(\mathbf{x},s)$ is contained in $x^2+s^2\leq a^2$, the region of integration in (36) is bounded by a cylinder $|\mathbf{x}|\leq a$ (Fig. 2). Therefore the distribution

$$T(x^0)=:\int d^3x\,\mu(x^0,\mathbf{x})D_x{}^{v,m}\Bigg\{h(\mathbf{x})\Bigg[\sum_{k=0}^{N}\frac{(x^2)^k}{(2k!)}\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\Bigg|_{s=0}$$
$$+\Theta(x^2)\Bigg(\psi^0{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})x^0+\psi^1{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})-\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\Bigg|_{s=0}\Bigg)\Bigg]\Bigg\} \tag{39}$$

has compact support in $|x^0|\leq b=a+\Delta$ for all $v\leq N$ and $h(\mathbf{x})\in C^\infty$. For $|x^0|>b$, the two sums over k in (39) cancel each other whereas the support of the remaining term is given by $\psi_{\Delta_5}{}^n(\mathbf{x},\sqrt{x^2})=-\pi i(x^0)^{n-1}[\psi(x^0)-(-1)^n\psi(-x^0)]$.

If we furthermore observe that the Θ function and the differential operator $D_x{}^{v,m}$ can be interchanged for all $v\leq N$, we finally obtain the desired representation

$$F(\psi,h)=i(-1)^n\int_{-b}^{+b}dx^0\int_{\mathbf{x}^2\leq a^2}d^3x\,\mu(x^0,\mathbf{x})D^{N,m}{}_x\Bigg\{h(x)\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\Bigg|_{s=0}\Bigg\}$$
$$+i(-1)^N\int_{-b}^{+b}dx^0\int_{\mathbf{x}^2\leq(x^0)^2}d^3x\,\mu(x^0,\mathbf{x})D_x{}^{N,m}\Bigg\{h(\mathbf{x})\Bigg(\psi^0{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})x^0+\psi^1{}_{\Delta_5}(\mathbf{x},\sqrt{x^2})$$
$$-\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}[\psi^0{}_{\Delta_5}(\mathbf{x},s)x^0+\psi^1{}_{\Delta_5}(\mathbf{x},s)]\Bigg|_{s=0}\Bigg)\Bigg\}\,. \tag{40}$$

The important points in this representation are (i) an integration of continuous functions over finite regions, and (ii) the fact that the original smearing in the time variable x^0 has been shifted in part to a smearing in the space variable $\mathbf{x}$. By means of this representation we will prove our statements in Sec. III.

III. STRUCTURE OF EQUAL-TIME LIMITS

We begin with some further properties of the test functions characterized by condition (9).

Definition: A C^∞ function $h(\mathbf{x})$ is said to be from class $P_3{}^m$ if

$$\lim_{\epsilon\to0}\epsilon^{-m}h(\epsilon\mathbf{x})<\infty\,.$$

From the definition it is obvious that

$$P_3{}^m\subset P_3{}^{m-1}. \tag{41}$$

Moreover, developing $h(\mathbf{x})$ in a MacLaurin series up to a finite order, we obtain from the well-known estimates on the remainder of this series[23]

Lemma 2: If $h(\mathbf{x})\in P_3{}^m$, then[15]

(i) $$\left(\sum_{i=1}^{3}x^i\frac{\partial}{\partial Z^i}\right)^v h(\mathbf{z})\Bigg|_{\mathbf{z}=0}=0\quad\text{for}\quad 0\leq v\leq m-1,$$

(ii) $$h^{|m|}(x)=:D_x{}^{|n|}h(\mathbf{x})\equiv(\partial^{|n|}/\prod_{i=1}^{3}(\partial x^i)^{n_i})h(\mathbf{x})\in P_3{}^{m-n}$$

for all $n_i\geq0$ with

$$\sum_{i=1}^{3}n_i=n.$$

Lemma 3: If $h(x)$ is an arbitrary C^∞ function, then

$$\tilde{h}_m(\mathbf{x})=:h(\mathbf{x})-\sum_{n=0}^{m-1}\left(\sum_{i=1}^{3}x^i\frac{\partial}{\partial Z^i}\right)^n h(\mathbf{z})\Bigg|_{\mathbf{z}=0}$$

belongs to $P^m{}_3$. Now our main theorem is

Theorem I: If the commutator $F(x)$ is of order N, then (the equal-time limit)

$$\lim_{\epsilon\to0}\int d^4x\,\psi_\epsilon(x^0)h(\mathbf{x})F(x)=0$$

for all $h(\mathbf{x})\in P_3{}^{2N+m}$ and a fixed m with $0\leq m\leq N$. If in addition the spectrum condition holds, then m is equal to zero.

Since all charge moments

$$h(\mathbf{x},\mathbf{n})=\prod_{i=1}^{3}(x^i)^{n_i}$$

[23] R. Courant, *Vorlesungen über Differential- und Integralrechnung* (Springer, Berlin, 1955), Vol. 2.

of degree

$$|n|=\sum_{i=1}^{3} n_i \geq 2N+m$$

are from class $P_3{}^{2N+m}$, we obtain, furthermore,

Corollary: If N is the order of the (density) commutator $F(x)$, then the equal-time commutators between the corresponding charge moments of degree $|n|\geq 2N+m$ and one current density vanish:

$$\lim_{\epsilon\to 0}\int d^4x\,\psi_\epsilon(x^0)h(\mathbf{x},\mathbf{n})F(x)=0. \tag{42}$$

If the spectrum condition holds, then m is zero.

Before we present the proof of Theorem I, let us draw some conclusions from it. Theorem I, together with Lemma 3 and Eq. (40), tells us that

$$\lim_{\epsilon\to 0}\int d^4x\,\psi_\epsilon(x^0)h(x)F(x)=\lim_{\epsilon\to 0}\sum_{n=0}^{2N+m-1}\int d^4x\,\psi_\epsilon(x^0)\times\left(\sum_{i=1}^{3}x^i\frac{\partial}{\partial Z^i}\right)^n h(\mathbf{z})\Big|_{\mathbf{z}=0}F(x). \tag{43}$$

However, on the right-hand side of (43) there occur the equal-time limits of the commutators of only the first $2N+m-1$ charge moments. Therefore the equal-time limit of two current densities exists if and only if the equal-time limits of the corresponding commutators between the first $2N+m-1$ charge moments and a current density exist. Moreover, since the support of the commutator shrinks to the point $\mathbf{x}=0$ in the limit, and the limit is unequal to zero only for the first $2N+m-1$ charge moments, it is at most a sum of $2N+m-1$ derivatives of the δ function $\delta(\mathbf{x})$.

Last but not least, it is obvious from the structure of the right-hand side of (43) that the coefficients of these δ functions are identical to the equal-time limits of the commutators of the corresponding charge moments and one current density.[24]

Collecting these arguments, we obtain a further theorem.

Theorem II: Let N be the order of the commutator matrix element $F(x)$. Then the equal-time limits of the first $2N+m-1$ charge moments,

$$\lim_{\epsilon\to 0}\int d^4x\,\psi_\epsilon(x^0)\left[\prod_{i=1}^{3}(x_i)^{n_i}\right]F(x),$$

with

$$0\leq|n|=\sum_{i=1}^{3}n_i\leq 2N+m-1$$

(m fixed between zero and N), exist if and only if the equal-time limit of the corresponding density commutator exists and is a sum of at most the first $2N+m-1$ derivatives of the δ function:

$$\lim_{\epsilon\to 0}\int d^4x\,\psi_\epsilon(x^0)F(x)h(\mathbf{x})=\sum_{\nu=0}^{2N+m-1}\sum_{i=0}^{\nu}\sum_{j=0}^{\nu-i}\frac{A(i,j,\nu-i-j)}{i!j!(\nu-i-j)!}\times\frac{\partial^\nu}{\partial Z_1{}^i\partial Z_2{}^j\partial Z_3{}^{\nu-i-j}}h(\mathbf{z})\Big|_{\mathbf{z}=0}. \tag{44}$$

The coefficients $A(\mathbf{v})$ are identical to the charge moments

$$A(\mathbf{v})\equiv\lim_{\epsilon\to 0}\int d^4x\,\psi_\epsilon(x^0)\left[\prod_{i=1}^{3}(x^i)^{\nu_i}\right]F(x). \tag{45}$$

If the spectrum condition holds, then m is equal to zero.

All that remains to be done is to prove Theorem I. In view of the representation (40) for the commutator $F(\psi_\epsilon,h)$, it is sufficient to show that the limit $\epsilon\to 0$ of the following two expressions vanishes for all $h(\mathbf{x})\in P_3{}^{2N+m}$ and all $0\leq m\leq\nu\leq N$; $n\geq 0$, $n'\geq 0$:

$$I_{\nu,m}{}^{n,n'}(\epsilon)=:\int_{-b}^{+b}dx^0\int_{\mathbf{x}^2\leq a^2}d^3x\,\mu(x^0,\mathbf{x})(x^0)^{n'}D_x{}^{\nu,m}\left\{\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}\psi^n{}_{\Delta_5}(\mathbf{x},s)^\epsilon\Big|_{s=0}h(\mathbf{x})\right\}, \tag{46}$$

$$\mathrm{II}^{n,n'}{}_{\nu,m}(\epsilon)=:\int_{-b}^{+b}dx^0\int_{\mathbf{x}^2\leq(x^0)^2}d^3x\,\mu(x^0,\mathbf{x})(x^0)^{n'}\frac{\partial^{\nu-m}}{\partial(x^0)^{\nu-m}}D_{\mathbf{x}}{}^{|m|}\times\left[h(\mathbf{x})\left(\psi^n{}_{\Delta_5}(x,\sqrt{x^2})^\epsilon-\sum_{k=0}^{N}\frac{(x^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}\psi^n{}_{\Delta_5}(\mathbf{x},s)^\epsilon\Big|_{s=0}\right)\right], \tag{47}$$

with $\psi^n{}_{\Delta_5}(\mathbf{x},s)^\epsilon$ according to (16) and (33) given by

$$\begin{aligned}\psi^n{}_{\Delta_5}(\mathbf{x},s)^\epsilon&=-(\pi i/\epsilon)[(x^2+s^2)^{1/2}]^{n-1}[\psi(\epsilon^{-1}(\mathbf{x}^2+s^2)^{1/2})-(-1)^n\psi(-\epsilon^{-1}(\mathbf{x}^2+s^2)^{1/2})]\\&=\epsilon^{n-2}\psi_{\Delta_5}(x/\epsilon,s/\epsilon).\end{aligned} \tag{48}$$

We restrict ourselves to demonstrating the vanishing of (47) for $\epsilon\to 0$. The proof for (46) follows with exactly the

[24] A different rigorous argument for this connection under the assumption of the existence of the limit as a tempered distribution in x has been found by Roepstorff and Stichel (private communication).

same arguments. Changing the integration variables, we get

$$\Pi^{n,n'}{}_{v,m}(\epsilon)=\epsilon^{n+n'-v+2}\int_{-b/\epsilon}^{b/\epsilon}dy^0\int_{\mathbf{y}^2\leq(y^0)^2}d^3y(y^0)^{n'}\mu(\epsilon y^0,\epsilon\mathbf{y})\frac{\partial^{v-m}}{\partial(y^0)^{v-m}}D_{\mathbf{y}}{}^{|m|}$$

$$\times\left[h(\epsilon\mathbf{y})\left(\psi^n{}_{\Delta_5}(\mathbf{y},\sqrt{y^2})-\sum_{k=0}^{N}\frac{(y^2)^k}{(2k)!}\frac{\partial^{2k}}{\partial s^{2k}}\psi^n{}_{\Delta_5}(\mathbf{y},s)\bigg|_{s=0}\right)\right].\qquad(49)$$

According to the support properties of $\psi^n{}_{\Delta_5}(\mathbf{y},s)$, the exact region of integration is given by (Fig. 2)

1. First term:

$$\mathbf{y}^2\leq(y^0),\quad -\min\{b/\epsilon,a\}\leq y\leq\min\{b/\epsilon,a\}.$$

2. Sum over k:

$$\mathbf{y}^2\leq\min\{(y^0)^2,a^2\},\quad -b/\epsilon\leq y^0\leq b/\epsilon.$$

For sufficiently small ϵ we have certainly $b>\epsilon a$. Therefore we can split the integral into two parts:

$$\Pi^{n,n'}{}_{v,m}(\epsilon)=\hat{\Pi}^{n,n'}{}_{v,m}(\epsilon)-\tilde{\Pi}^{n,n'}{}_{v,m}(\epsilon),\qquad(50)$$

with

$$\hat{\Pi}_{v,m}{}^{n,n'}(\epsilon)=:\epsilon^{n+n'-v+2}\int_0^a dy^0\int_{\mathbf{y}^2\leq(y^0)^2}d^3y\hat{\mu}(\epsilon y^0,\epsilon\mathbf{y})(y^0)^{n'}\frac{\partial^{v-m}}{\partial(y^0)^{v-m}}D_y{}^{|m|}[h(\epsilon\mathbf{y})\psi^n{}_{\Delta_5}(\mathbf{y},\sqrt{y^2})]\qquad(51)$$

and

$$\tilde{\Pi}_{v,m}{}^{n,n'}(\epsilon)=\epsilon^{n+n'-v+2}\int_a^{b/\epsilon}dy^0\int_{\mathbf{y}^2\leq a}d^3y\,\hat{\mu}(\epsilon y^0,\epsilon\mathbf{y})(y^0)^{n'}\frac{\partial^{v-m}}{\partial(y^0)^{v-m}}\sum_{k=0}^{N}\frac{1}{(2k)!}\sum_{t=0}^{m}\binom{m}{t}O_{\mathbf{y}}{}^{|m-t|}h(\epsilon\mathbf{y})$$

$$\times\hat{O}_{\mathbf{y}}{}^{|t|}\left([(y^0)^2-\mathbf{y}^2]^k\frac{\partial^{2k}}{\partial s^{2k}}\psi^n{}_{\Delta_5}(\mathbf{y},s)\bigg|_{s=0}\right),\qquad(52)$$

$$\hat{\mu}(y)=:\mu(y)-(-1)^{n'+v-m}\mu(-y).\qquad(53)$$

Here $O_{\mathbf{y}}{}^{|t|}$ and $\hat{O}_{\mathbf{y}}{}^{|t|}$ mean differential operators of (total) degree t in y^i ($i=1, 2, 3$). In the following we will also use the abbreviation

$$f^{|v|}(\mathbf{y})=:O_{\mathbf{y}}{}^{|v|}f(\mathbf{y}).$$

Let us first consider (52). The leading term in ϵ is given by

$$H_{v,m}{}^{n,n'}(\epsilon)=\int_{\epsilon a}^{b}dy^0\int_{\mathbf{y}^2\leq a^2}d^3y\,\hat{\mu}(y^0,\epsilon\mathbf{y})(y^0)^{n'+2N-v+m}$$

$$\times\frac{1}{(2N-v+m)!}\sum_{t=0}^{m}\epsilon^{n+1-t-2N}h^{|m-t|}(\epsilon\mathbf{y})\binom{m}{t}\hat{O}_{\mathbf{y}}{}^{|v|}\frac{\partial^{2N}}{\partial s^{2N}}\psi^n{}_{\Delta_5}(\mathbf{y},s)\bigg|_{s=0}.\qquad(54)$$

If $h(y)$ is from $P_3{}^{2N+m-1-n}$, then owing to Lemma 2 (ii), $h^{|m-t|}(\mathbf{y})$ belongs to $P_3{}^{2N+t-1-n}$. Since the whole integrand is continuous, we can take the limit and obtain

$$\lim_{\epsilon\to0}\tilde{\Pi}_{v,m}{}^{n,n'}(\epsilon)=\lim_{\epsilon\to0}H_{v,m}{}^{n,n'}(\epsilon)$$

$$=\int_0^b dy^0\int_{\mathbf{y}^2\leq a^2}d^3y\,\hat{\mu}(y^0,\mathbf{0})(y^0)^{n'+2N-v-m}\frac{1}{(2N-v+m)!}\sum_{t=0}^{m}\binom{m}{t}\hat{O}_{\mathbf{y}}{}^{|v|}\frac{\partial^{2k}}{\partial s^{2k}}\psi^n{}_{\Delta_5}(\mathbf{y},s)\bigg|_{s=0}$$

$$\times\left(\sum_{i=1}^{3}y^i\frac{\partial}{\partial Z^i}\right)^{2N-n-1+t}h^{|m-t|}(\mathbf{z})\bigg|_{\mathbf{z}=0}.\qquad(55)$$

If, moreover, $h(\mathbf{y})$ belongs to $P_3{}^{2N+m}\subset P_3{}^{2N+m-1}$, then $h^{|m-t|}(\mathbf{y})\in P_3{}^{2N+t}\subset P_3{}^{2N+t-1}$ and this expression vanishes owing to Lemma 2 (i). In the same way one shows that $\hat{\Pi}_{v,m}{}^{n,n'}(\epsilon)$ vanishes for $\epsilon\to0$ and all $h(\mathbf{y})\in P_3{}^{2N+m}$. If, in addition, the spectrum condition holds, then because of Borchers[19] m can be set equal to zero in (12) or (37) and therefore also everywhere in (40) and (46)–(55). This proves our theorem.

IV. CONCLUSIONS

We have proved our theorems only for the case of tempered distributions. The reason is that this is the simplest case from a technical point of view. However, the proofs can be extended by a slightly larger amount of technicalities to functionals on other spaces, like D_4, or more generally to any test function space M with the two properties: (i) M contains the space $K(b)_4$ for arbitrary small b as a subspace[25]; and (ii) there exists a one-to-one correspondence between the functionals on M vanishing in the open spacelike cone and the Fourier transforms of the solutions of the five-dimensional wave equation.

Property (i) guarantees that the functionals on M have a finite order.[14] The test function spaces introduced by Jaffe[26] for nonrenormalizable local theories, however, do not share this property.

Our results have some important consequences in the applications to current algebras. As we have shown, the *finite* set of relations

$$\lim_{x^0\to 0} \langle\Psi|\left[\int d^3x \prod_{i=1}^{3} (x^i)^{v_i} j^0{}_\alpha(x); j^\mu{}_\beta(0)\right]|\Phi\rangle^T=0 \quad (56)$$

for $v_i \geq 0$ integer and

$$0<\sum_{i=1}^{3} v_i \leq 2N-1$$

together with (3) is completely equivalent to the local current algebra (4). We can forget about the local algebra (4) and insert (3) and (56) for it.

Let us consider this finite set of relations for the special case of one-nucleon states. Because of the integration over x space, the contribution of the one-nucleon intermediate state (or any other state with the mass of the nucleon) is always a product of a coupling constant with a form factor or one of its derivatives with respect to the invariant momentum transfer.[27] The reason for this is that the space integration causes an identification of the (total) three-momentum of the intermediate states with the three-momentum of one of the external states.

Furthermore, one can apply Gauss's theorem to the charge moments in (3) and (56).[9,10] After this step, one can make either a local one-particle approximation[28,29] in the resulting expression or one can introduce dispersion relations first and make a naive one-particle approximation in the absorptive parts of the dispersion integrals.[9,10] In both cases,[9,29] owing to microcausality, the result is a frame-independent, finite set of linear differential relations between form factors.

This result has to be compared to the nonlinear algebraic relations of the Gell-Mann–Dashen program of saturation in the infinite-momentum frame[4,5,30] with all its serious difficulties.[7,8] Our results indicate where the source of these deceases has to be sought—in the destruction of microcausality even in the case of an infinite number of discrete intermediate states (for a finite number of states and nonconstant form factors it is bound to occur).

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[25] $K(b)_4$ is the space of all C^∞ functions with support contained in $|x^\alpha|<b^\alpha$ and equipped with a natural topology (Ref. 14).

[26] A. M. Jaffe, Phys. Rev. **158**, 1454 (1967).

[27] The moments $(x^i)^{m_i}$ in (56) lead to differential operators in momentum space.

[28] U. Völkel and A. H. Völkel, Commun. Math. Phys. **7**, 261 (1968).

[29] P. Stichel and A. H. Völkel (unpublished).

[30] H. Kleinert, *Ergebnisse der Exakten Naturwissenschaften* (Springer, Berlin, 1969), Vol. 49.