the sum making up the total H, would not commute. It seems unlikely that such a scheme would be very useful.

This result leads to the mass of the single-particle state being the expectation value (properly defined, as discussed in Ref. 11) of the total Hamiltonian, and not the eigenvalue of a single-particle Hamiltonian.

In general, one must be careful in the factorization process when unstable states are involved.

These considerations do not seem to have been always throughly understood, and as a result errors have occasionally arisen.

A more detailed discussion of the whole question is given in Ref. 11.

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Plane-Wave Packets and Their Limitation in Nonlinear Compton Scattering*

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We show that scattering boundary conditions are incompatible with a monochromatic radiation field for the case of nonlinear Compton scattering. We demonstrate this by showing that (a) in the monochromatic limit, gauge invariance in a given order of the expansion of the S matrix is destroyed, and (b) the physical (scattering) boundary condition, that of a pulse of radiation incident on a target electron, cannot be reconstructed from the monochromatic limit of the S matrix. We then proceed to show by an example that the frequency profile of the scattered radiation is a function of both the intensity and line shape of the incident field. Another interesting feature of this calculation is that the profile of the photon scattered at a fixed angle is significantly broadened in comparison with the incident line shape. The worked-out example is a simple model, that of a neutral, scalar "electron" interacting with a bilinear scalar, massless external field, which contains all the important features of nonlinear Compton scattering. While from the point of view of gauge invariance it is sufficient to treat the external radiation field as a one-dimensional wave packet, for a complete description of the problem it is necessary to describe the incident radiation (quanta) in terms of normalizable states. An estimate of the breakdown of the plane-wave approximation is included.

I. INTRODUCTION

THE generalization of the Klein-Nishina formula[1] to include the effect of an intense light beam has been the subject matter of numerous articles.[2,3] These computations fall into two categories. The first group of authors[2] obtains the scattering amplitude via the Volkov[4] solutions for incident and outgoing electron states. Recall that the Volkov[4] wave function is a solution to the Dirac (Klein-Gordon) equation for a charged particle in the presense of an external, transverse electromagnetic field. The electromagnetic field is restricted to be a plane wave. This excludes the use of a three-dimensional wave packet in the description of the external field. The second approach[3] makes use of the adiabatic switching-on-and-off technique and covariant perturbation theory. The two methods yield diverging results. Briefly, the disagreements between the two methods are twofold: (a) In the kinematics, the scattering amplitude based on the Volkov[4] solutions yields an intensity-dependent frequency shift (IDFS), while the other method[3] gives no IDFS, and (b) the two amplitudes differ in their functional form. The merits of one approach versus the other have also been discussed in equally numerous articles.[5]

Recall that it was demonstrated[6] that the imposition of scattering boundary conditions on the Volkov[4] solutions is incompatible with unitary time evolution of the state vector. There is also something wrong, however, with the second approach,[3] viz., the scattering

* Work supported in part by U. S. Army Research Office (Durham) and administered by Lowell Technological Institute Research Foundation.

[1] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961), pp. 488–490.

[2] N. D. Sengupta, Bull. Math. Soc. (Calcutta) **39**, 147 (1947); A. I. Nikishov and V. I. Ritus, Zh. Eksperim. i Teor. Fiz. **46**, 776 (1963) [Soviet Phys. JETP **19**, 529 (1964)]; I. I. Goldman, Phys. Letters **8**, 103 (1964); L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964). A more complete set of references is given by J. H. Eberly, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1968), Vol. VII.

[3] Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).

[4] D. M. Volkov, Z. Physik **94**, 250 (1935).

[5] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965); L. M. Frantz, *ibid.* **139**, B1326 (1965); O. von Roos, *ibid.* **150**, 1112 (1966); Z. Fried, A. Baker, and D. Korff, *ibid.* **151**, 1040 (1966); H. Reiss and J. H. Eberly, *ibid.* **151**, 1058 (1966); P. Stehle and P. G. de Baryshe, *ibid.* **152**, 1135 (1966).

[6] Z. Fried, A. Baker, and D. Korff, Phys. Rev. **151**, 1040 (1966).

amplitude is not gauge invariant. In particular, the amplitude is not invariant under the gauge transformation

$$\epsilon_\mu'(k') \to \epsilon_\mu'(k') + \lambda k_\mu',$$

where ϵ_μ' is the polarization vector of the scattered photon. (Incidentally, the scattering amplitude based on the Volkov solutions[7] in the monochromatic limit is not gauge invariant either in ϵ_μ', in each order of the fine-structure constant. It is gauge invariant, however, if the amplitude as a whole is considered. From the point of view of covariant perturbation theory, the amplitude should be gauge invariant in every external photon line in all orders of the expansion if we are to obtain sensible results not only for the case when the laser beam is in a "coherent" state, but also in the case of an arbitrary multiphoton incident state. In the absence of compelling reasons for the presumed failure of perturbation theory, we shall look for the cause of this difficulty somewhere else.)

It is our aim to show that all these difficulties are due to the fact that the intensity-dependent (nonlinear) scattering amplitude *does not exist* in the monochromatic limit. Needless to say, scattering problems should always be formulated in terms of wave packets which do not overlap initially. What distinguishes the linear (two-particle incident state) from the nonlinear problem is that in the former it is legitimate to discuss the scattering amplitude as a function of sharply defined four-momenta. In the linear case, one can always reconstruct the physical (wave-packet) amplitude by superposition. This cannot be achieved in the nonlinear case.[8]

In Sec. II we discuss the problem of gauge invariance and some of the additional reasons why the monochromatic limit does not exist for the nonlinear scattering amplitude. We also indicate that most (but not all) of these objections can be overcome by describing the external field as a plane wave, i.e., a one-dimensional wave packet.

In Sec. III we develop and present results based on a model interaction Hamiltonian, that of a neutral scalar "electron" interacting with a bilinear scalar massless field. The zero-mass field represents a scalar, plane-wave solution of the Maxwell wave equation. We believe, and so far there is no indication to the contrary anywhere in the literature, that all the important and troublesome features of nonlinear Compton scattering are included in this model.

In Sec. IV we point out that a plane-wave description of a laser beam is a poor approximation. Specifically, we estimate the breakdown of the plane-wave approximation by inspecting the most singular nonlinear amplitude. We find that the criteria for the breakdown of the plane-wave approximation involve Planck's constant. The implication of this result for the case of a focused laser beam is pointed out.

In Sec. V we summarize our results. Some additional features of nonlinear scattering are also discussed in a qualitative fashion.

II. VOLKOV SOLUTIONS AND MONOCHROMATIC LIMIT

To date, all semiclassical calculations of the intensity-dependent Compton scattering amplitude have made use of the Volkov[4] solutions. The "classical" intensity-dependent frequency shift (IDFS) follows, but only under either of the following two conditions: (i) in the monochromatic limit, or (ii) when the plane-wave describing the external vector potential has the shape of a square pulse. Besides the criticism raised in Ref. 6, there are other reasons why scattering boundary conditions are incompatible with the monochromatic (infinite extent) nature of the external field. We enumerate a few of them.

(a) For obvious physical reasons, if the external field is of infinite extent, the notion of scattering in the sense that the electron is at some time decoupled from the external field is untenable. Also, we already remarked in the Introduction that, unlike the case of linear scattering amplitude, the wave-packet (in the external field) scattering amplitude cannot be obtained from the monochromatic nonlinear amplitude.

(b) While this objection can be bypassed by the use of a square pulse,[9] this approach has its own difficulty. The use of a square pulse for the vector potential yields singular electromagnetic field intensities.

(c) The monochromatic "scattering" amplitude is not gauge invariant in a given order of α^n, α being the fine-structure constant. (Similar remarks pertain to the square-pulse case.) Specifically, given the monochromatic Volkov scattering amplitude[10] as a power series in α, the coefficients A_n are functions of the photon density ρ, the various four-momenta entering the problem, the polarization of the external field ϵ, and the polarization of the scattered photon ϵ'. These coefficients are not gauge invariant in ϵ'. Symbolically,

$$A_n(\rho,p,k,\epsilon;p',k',\epsilon') \neq A_n(\rho,p,k,\epsilon;p',k',\epsilon'+\lambda k'). \quad (2.1)$$

It is well known[11] that, at least in perturbation theory, the scattering amplitude is gauge invariant in all orders

[7] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964), especially Eqs. (3.12) and (3.22) and their discussion pertaining to these.

[8] The reader may wonder why any quantum system should be nonlinear. While it is correct to say that S-matrix elements are linear functions of the various particles in and out, the situation here is different. Here we are describing a large (and sometimes undetermined) number of incident quanta by a few parameters. The problem becomes nonlinear in these parameters.

[9] R. A. Neville, dissertation, Syracuse University (unpublished), and Sec. III of this paper. We thank Dr. Neville for sending us a copy of his thesis.

[10] See Ref. 7.

[11] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961), p. 493.

of the coupling constant, provided that the sums of the four-momenta of all external photon legs in the Feynman graphs add up to zero. (Thus, one could hope to be able to use this information to see whether the IDFS is a genuine effect.) The results of our calculation, which we omit here for the sake of brevity, show that the coefficients

$$A_n(\rho,p,k,\epsilon;\,p',k',\epsilon')$$

are not gauge invariant in ϵ', whether we use the standard Compton energy-momentum balance formula

$$p+k=p'+k' \qquad (2.2)$$

or whether we use the intensity-dependent expression for the "four-momentum" balance

$$p+(\alpha\rho/\omega 2p\cdot k)k+k=p'+(\alpha\rho/\omega 2p'\cdot k)k+k'\,, \qquad (2.3)$$

where p and p' denote the incident and outgoing electron momenta, k denotes the four-momentum of an absorbed laser photon, and k' denotes the four-momentum of the scattered photon. $\omega=k_0$. All other symbols have been defined previously. Equation (2.3) leads to the IDFS.

This lack of gauge invariance in a given order of α^n is even more disturbing if one tries to understand the diagrammatic basis of the intensity-dependent scattering amplitude. From Eq. (3.2) of Ref. 10, we can obtain the coefficient of α^2. It is

$$A_2=i\bar{u}_{p'\lambda'}(D_1-D_2+D_3)u_{p\lambda}\,, \qquad (2.4)$$

where

$$D_1=\gamma\cdot\epsilon'(\tfrac{1}{16}z_1)(4z_2-z_1^2)\,,$$

$$D_2=3\gamma\cdot\epsilon\boldsymbol{k}\gamma\cdot\epsilon'\boldsymbol{k}\gamma\cdot\epsilon(1/2p\cdot k)(1/2p'\cdot k)(\rho/4\omega)z_1\,,$$

$$D_3=[(\gamma\cdot\epsilon\boldsymbol{k}\gamma\cdot\epsilon'/p'\cdot k)+(\gamma\cdot\epsilon'\boldsymbol{k}\gamma\cdot\epsilon/p\cdot k)](\rho/2\omega)^{1/2}\times\tfrac{1}{8}(z_1^2+4z_2)\,,$$

and

$$z_1=(\rho/2\omega)^{1/2}[(2\epsilon\cdot p'/p'\cdot k)-(2\epsilon\cdot p/p\cdot k)]\,,$$

while

$$z_2=(\rho/4\omega)[(1/p'\cdot k)-(1/p\cdot k)]\,.$$

$\bar{u}_{p'\lambda'}$ and $u_{p\lambda}$ are Dirac spinors characterizing the final and initial electron states. It should be possible to obtain (2.4) from the set of Feynman graphs represented in Fig. 1, in the limit that k_1, $k_2\rightarrow k$, where k is the four-vector of the monochromatic laser photon. It is clear that before we take the limit, i.e., $k_1\neq k_2$, the set of Feynman graphs of Fig. 1 are gauge invariant in *all* photon legs. In particular, if we replace

$$\gamma\cdot\epsilon' \quad\text{by}\quad \gamma\cdot\epsilon'+\lambda\boldsymbol{k}'$$

the matrix element remains invariant. After all, this set of graphs also corresponds to a physical process in which two photons are incident on a target electron and two arbitrary photons are reradiated. Furthermore, for this problem gauge invariance will be maintained if and only if

$$p+2k_1=p'+k_2+k'\,, \qquad (2.5)$$

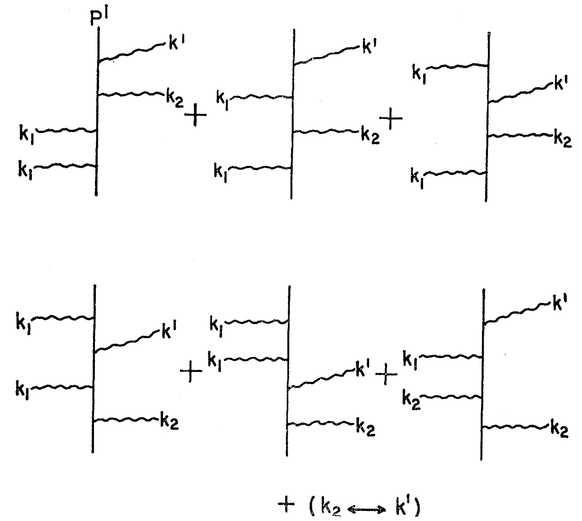

FIG. 1. Graphs representing two photons in and two photons out. The target is an electron.

i.e., the standard kinematic formula. What happens in the limit as $k_1\rightarrow k_2$? Some of the amplitudes diverge. The intermediate electron propagator approaches its mass-shell value. Finite values for the principal-value part of the propagator can be obtained if one uses some limiting procedure.[12] Whatever limiting procedure one chooses, however, gauge invariance is destroyed. Physical considerations suggest that as long as we do not change the nature of the problem (e.g., from scattering to bound state), all properties of the matrix element should be continuous in the monochromatic limit. We therefore conclude that the monochromatic limit for the nonlinear amplitude is *incompatible with scattering boundary conditions*.

Even if one adopts a purely semiclassical point of view and considers the external field as a potential, the scattering amplitude will depend on the spatial extent of this potential. One may very well wonder why the monochromatic limit (even if not exact) should not be a good approximation to the exact result—say, of the order of the ratio of the radiation wavelength to the length of the wave train. The answer can be stated in two ways. First, the effective potential consists of two parts: the sinusoidal part and a constant part. Both are of finite range. Since both contribute to the scattering amplitude, it is obvious that we cannot expect

[12] Fried and Eberly in their paper (Ref. 3) used a procedure based on Schrödinger-type perturbation theory for discrete energy levels. Another procedure, suggested by T. W. B. Kibble [Phys. Rev. **138**, B740 (1965)], is to integrate over the principal-value part of the propagator singularity with one-dimensional wave packets and take the monochromatic limit at the very end. We performed such a calculation to first order in the nonlinear parameter and obtained a result identical with Eq. (3.4).

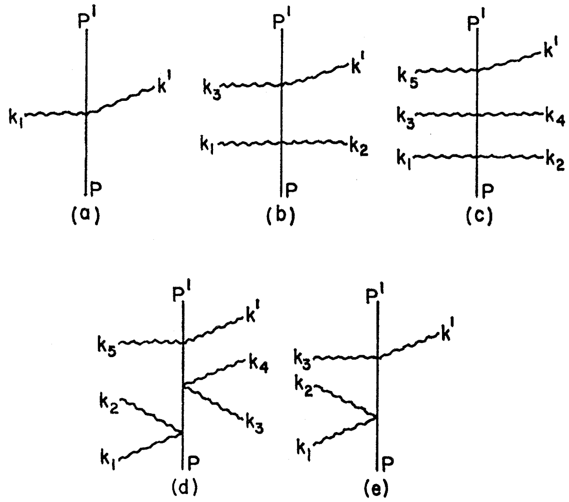

FIG. 2. Representative graphs to third order in coupling constant for semiclassical problem.

that the correction to the monochromatic scattering amplitude be of the order of the above-mentioned ratio. The characteristic length of the constant part of the potential is the length of the wave train. A second way of stating the same thing is: If all terms in the perturbation expansion were finite, then the correction to the monochromatic scattering amplitude would indeed be of the order λ/L. (λ is the radiation wavelength and L is the length of the wave train.) Some of the terms diverge, however, in the monochromatic limit. These terms are sensitive to the shape of the wave packet. The main part of Sec. III is devoted to illustrating this by a concrete example.

III. NONLINEAR SCATTERING AMPLITUDE WITH ONE-DIMENSIONAL WAVE PACKETS

Since all the difficulties encountered so far in nonlinear Compton scattering can be traced to the effective $A\cdot A$ (the square of the vector potential) term in the interaction, it is sufficient to illustrate the remedy on a simple model, that of the interaction of a charged scalar field (electron) with the square of a classical, one-dimensional zero-mass field:

$$\mathfrak{L}_I(x)=e^2\phi^*(x)\phi(x)[f(x)]^2. \tag{3.1}$$

Representative Feynman diagrams to third order in the interaction are shown in Fig. 2. As remarked earlier, if the external "photon" legs have all the same four-momenta (monochromatic limit), some of the second- and higher-order graphs contain infinities. These infinities must be avoided by making use of wave-packet description of the incident radiation. To make contact with Volkov-type solutions,[4] we take $f(x)$ to be of the following form:

$$f(x)=f(n\cdot x), \tag{3.2}$$

where $n=(1,0,0,1)$, $n\cdot n=0$.

Positive-energy solutions of the Klein-Gordon equation with Feynman boundary conditions for an $f^2(n\cdot x)$ interaction of this type are given by Volkov-type formulas,[4]

$$\psi_p{}^{(\pm)}(x)=\exp\left[-ip\cdot x-i\left(\frac{e^2}{2n\cdot p}\right)\int_{\mp\infty}^{n\cdot x}f^2(\tau)d\tau\right], \tag{3.3}$$

and the probability amplitude for scattering one quantum of momentum k' out of the packet is

$$T=N\int d^4x\,\psi_{p'}{}^{(-)*}(x)f(n\cdot x)\psi_p{}^{(+)}(x)e^{ik'\cdot x}, \tag{3.4}$$

with

$$N=ie^2[(2\pi)^9\times 2E_pE_{p'}\omega_{k'}]^{-1/2}.$$

Substitution of (3.3) into (3.4) gives

$$T=Ne^{-i\xi\delta}\int d^4x\,f(n\cdot x)e^{iq\cdot x+i\eta R(x)}, \tag{3.5}$$

where $q=p'+k'-p$ and

$$\xi=\tfrac{1}{4}e^2[(1/n\cdot p)+(1/n\cdot p')],$$
$$\eta=\tfrac{1}{4}e^2[(1/n\cdot p')-(1/n\cdot p)],$$
$$R(x)=\int_{-\infty}^{n\cdot x}f^2(\tau')d\tau'-\int_{n\cdot x}^{\infty}f^2(\tau')d\tau',$$

and

$$\delta=\int_{-\infty}^{+\infty}f^2(\tau')d\tau'.$$

Since the integration in Eq. (3.5) extends over the complete space-time volume, we may change the variables of integration to $t-z$ and $t+z$, obtaining

$$T=N\delta(q_x)\delta(q_r)\delta(q_t-q_0)I(q_0)e^{-i\xi\delta}, \tag{3.6}$$

where

$$I(q_0)=(2\pi)^{-1}\int_{-\infty}^{+\infty}d\tau\,f(\tau)e^{iq_0\tau+i\eta R(\tau)}.$$

It can be shown that the expansion of (3.6) in a power series in e^2 gives contributions to the scattering amplitude from Feynman diagrams such as that shown in Fig. 2. Note that Eq. (3.6) includes such processes as double absorption and double emission at a single vertex; furthermore, it includes processes where the number of quanta in does not equal the number of quanta out,[13] as, for example, the diagram shown in

[13] T. W. B. Kibble, in *Cargèse Lectures in Physics*, edited by M. Lévy (Gordon and Breach, New York, 1968), p. 299.

Fig. 2(e). All of these must be considered because, in the semiclassical wave-packet description of the external field, these amplitudes can add up coherently.

The kinematic constraints, $q_x=q_y=n\cdot q=0$, require q to be a multiple of n; $q=\gamma n$, and since p, p', and k' are on their respective mass shells and in the forward light cone, we find

$$\gamma=p\cdot k'(n\cdot p-n\cdot k')^{-1}\geqslant 0\,,$$

and $p'=\gamma n+p-k'$. Using these results, we obtain from Eq. (3.6),

$$\eta=\gamma\nu\,,$$

where

$$\nu=\tfrac{1}{4}e^2 n\cdot\bar{k}'[(n\cdot p)(p\cdot\bar{k}')]^{-1}\,,$$

with $\bar{k}'=(1,\hat{k}')$, independent of the energy of the scattered quantum.

The probability of detecting a "photon" scattered in the direction $\hat{k}'$, in the frequency range $d\omega'$ and solid angle $d\Omega'$ per pulse per incident "electron," is obtained from the amplitude, Eq. (3.6), in the usual way. We find

$$d^2P/(d\omega')(d\Omega')=4\pi e^2\gamma|I(\gamma)|^2\Theta(\gamma)[E_p(p\cdot\bar{k}')]^{-1}\,, \quad (3.7)$$

where Θ is the unit step function. Here

$$I(\gamma)=(2\pi)^{-1}\int_{-\infty}^{+\infty}f(\tau)d\tau\, e^{i\gamma\tau+i\gamma\nu R(\tau)} \quad (3.8)$$

and

$$\omega'=\gamma(n\cdot p)[(p\cdot\bar{k}')+\gamma(n\cdot\bar{k}')]^{-1}. \quad (3.9)$$

It is clear from Eq. (3.8) that the amplitude $I(\gamma)$, and hence the frequency shift, will depend on the pulse shape. In order to study this effect, we have computed $I(\gamma)$ for two different pulses: a square pulse modulation,

$$\begin{aligned} f_1(\tau)&=A_1\cos(\omega_0\tau-\alpha)\,, & -L<\tau<L\\ &=0\,, & \tau>|L| \end{aligned}$$

and a Lorentzian modulation,

$$f_2(\tau)=A_2e^{-\Delta|\tau|}\cos(\omega_0\tau-\alpha)\,.$$

For the first case, we find, for $-L<x<L$,

$$R_1(\tau)=A_1^2[\tau+(1/2\omega_0)\sin(2\omega_0\tau-2\alpha)+G]\,,$$

where G is some constant. Let us set $\kappa=\frac{1}{2}\nu A_1^2$. Then, from Eq. (3.8), we find

$$(2\pi)I(\gamma)e^{-i2\kappa\gamma G}=A_1\int_{-L}^{+L}d\tau\cos(\omega_0\tau-\alpha)e^{i\gamma F(\tau)}\,, \quad (3.10)$$

where

$$F(\tau)=[(1+2\kappa)\tau+(\kappa/\omega_0)\sin(2\omega_0\tau-2\alpha)].$$

Performing the integration, we get

$$2I(\gamma)e^{-i2\kappa\gamma G}=\sum_{n=-\infty}^{+\infty}[J_n(-\kappa\gamma/\omega_0)+J_{n+1}(-\kappa\gamma/\omega_0)]\times e^{i(2n+1)\alpha}\Delta_{L,n}\,, \quad (3.11)$$

where

$$\pi[\gamma(2\kappa+1)-\omega_0(2n+1)]\Delta_{L,n}=\sin\{[\gamma(2\kappa+1)-\omega_0(2n+1)]L\}\,,$$

and J_n are the cylindrical Bessel functions of order n. Thus, for $L\gg 1/\omega_0$, maxima occur for all values of the strength parameter κ. They occur for values of

$$\gamma=\omega_0(2n+1)/(2\kappa+1)\,.$$

Thus, from Eqs. (3.8)–(3.11), we see that the most probable (and also the average) frequency of the emitted quantum will show an intensity-dependent shift, the IDFS.[9] For the unphysical problem considered in our example, this fictitious square pulse may serve us well enough. To predict from this, however, that in scattering a real electron off a real laser beam one will also encounter the IDFS is quite another matter. For the real problem, it is the vector potential A_μ which enters into the Dirac equation. A vector potential with a square pulse shape yields singular electric and magnetic field intensities. Hence we are motivated to pursue this question one step further and consider a somewhat more physical arrangement, where the potential is of Lorentzian shape. For this case, we have approximately

$$R_2(\tau)\simeq A_2^2[(1-e^{-2\Delta|\tau|})\epsilon(\tau)/2\Delta+(1/2\omega_0)\sin(2\omega_0\tau-2\alpha)]\,, \quad (3.12)$$

where the damping factor in the oscillating term has been neglected. Substitution of (3.12) into (3.6) leads to the result

$$2I(\gamma)\Delta=A_\epsilon\sum_{n=-\infty}^{+\infty}[J_n(-\kappa\gamma/\omega_0)+J_{n+1}(-\kappa\gamma/\omega_0)]\times e^{i\alpha(2n+1)}\tilde{g}_n\,, \quad (3.13)$$

where

$$\tilde{g}_n=g(2\kappa\gamma/\Delta;[\gamma-(2n+1)\omega_0]/\Delta)=g(z;x)\,.$$

$g(z;x)$ is represented by the following integral:

$$\pi g(z;x)=\int_0^\infty ds\cos[xs+\tfrac{1}{2}z(1-e^{-2s})]e^{-s}\,, \quad (3.14)$$

and κ is defined as before. The function $g(z;x)$ is related to an incomplete gamma function of complex argument, and shares some of the properties of the δ function. For example,

$$\int_{-\infty}^{+\infty}g(z;x)dx=1\,,\quad \int_{-\infty}^{+\infty}xg(z;x)dx=-z\,.$$

Therefore, the average frequency of the emitted quantum will be fixed by a value of γ given by

$$(2\kappa+1)\gamma_{\text{av}}=(2n+1)\omega_0. \quad (3.15)$$

However, g is not symmetric about the average value, but instead is somewhat skewed. To get a better idea

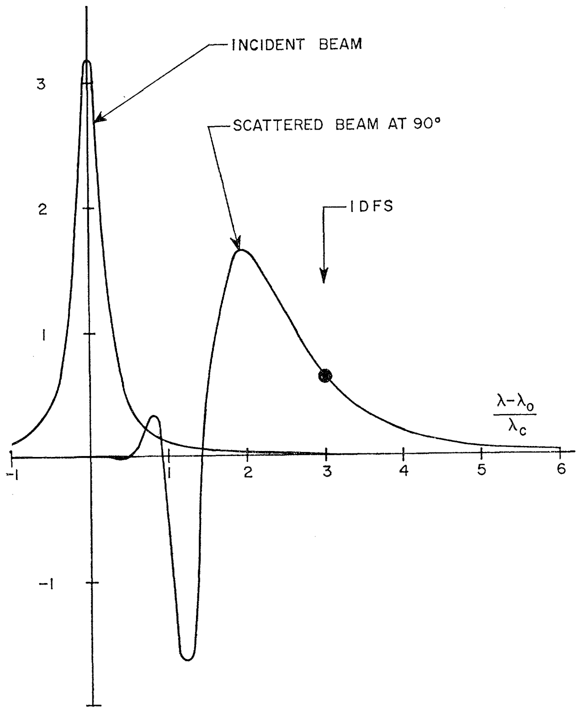

FIG. 3. A plot of $g(z;x)$ as a function of wavelength of the final photon, detected at right angle to the incident beam. A plot of the profile of the incident laser beam is also included.

of what this means, we show in Fig. 3 a detailed plot of $g(z;x)$ as a function of wavelength of the final photon, detected at right angle to the incident beam, for the parameters $e^2A^2/4m^2=10^{-5}$, $\Delta\omega/\omega_0=10^{-6}$, and $\omega_0/m=5\times10^{-6}$ appropriate to a high-intensity optical laser. The IDFS for this intensity is given by $\lambda'=\lambda_0+3\lambda_C$, where λ_0 is the central wavelength in the incident Lorentzian packet, and λ_C is the Compton wavelength of the electron. The predicted shift is, indeed, small here and could probably be described as "nonclassical," according to the criteria given by Kibble,[13] $\lambda_C\ll\lambda_0$. The IDFS is not small, however, when compared with the standard Compton shift; in fact, it is three times larger for the parameters we have chosen. Note that for these parameters the incident laser packet has a spread at half-maximum of only *one-fifth* the electron Compton wavelength. The integral (3.14) was computed numerically and the results are shown in Fig. 3. As a check on the numerical integration, asymptotic formulas were found which agreed with the computed one to about 10% in the region of interest. Note that the line shape has been *broadened* about ten times the incident beam, and that the wavelength of the *most probable* photon differs considerably from the wavelength of the *average* (the IDFS value) photon.

As a partial summary of our results so far, we list a few of the more pertinent ones. To begin with, we have shown that the frequency spread of the scattered photon depends not only on the intensity of the external field, but also on the spatial (temporal) shape of the incident radiation. Furthermore the correction to the monochromatic scattering amplitude is *not* merely of the order $(\Delta\omega/\omega_0)$. From a theoretical standpoint these results are gratifying, in view of the fact that all this can now be obtained from covariant perturbation theory. There are no more ambiguities concerning singular propagators. One simply integrates over the support prescribed by the extent of the wave packet. Hence (in the real problem with vector photons) gauge invariance would be maintained with standard kinematic formulas. Furthermore, the questions of translation invariance and Hermiticity raised in Refs. 3 and 6, respectively, are also squared away, inasmuch as all these requirements are satisfied by covariant perturbation theory. In passing we should also point out that the spread and shift in the frequency of a nonlinearly scattered photon has been observed in the interaction of pulsed laser light with liquids.[14] (The dynamical response of atomic electrons in a liquid is of course different from that for the free-electron case. The essential features common to both are the pulsed laser light and the nonlinear dynamics.) Thus, these results should not be too surprising.[15] For completeness, we should also mention that other attempts[16] have been made to estimate the effect of a one-dimensional wave packet on the propagation of an electron through an external field. These authors,[16] however, focus their attention on the "momentum shift" of the electron inside the external field, and not on the shape of the scattered radiation.

IV. PLANE-WAVE PACKETS AND THEIR LIMITATION

Although several of the difficulties, such as lack of gauge invariance and translation invariance (as manifested in the IDFS), plus the incompatibility of the scattering boundary condition with the monochromatic limit, are eliminated by the use of one-dimensional wave packets, for a complete description of the problem, three-dimensional, i.e., normalizable, wave packets are necessary.

To demonstrate the importance of normalizable wave packets, one should compute the scattering amplitude in the presence of an external field of limited spatial extent. The resulting integrals cannot be easily expressed in simple analytic terms. Hence they are not too instructive. We therefore resort to simple estimates, which, though not accurate, are at least transparent. Before we do this, let us digress a bit to put the develop-

[14] F. DeMartini, C. H. Townes, T. K. Gustafson, and P. L. Kelley, Phys. Rev. **164**, 312 (1967).

[15] J. D. Childress and C. G. Hambrick, Phys. Rev. **136**, A411 (1964). In this paper the nonlinear interaction of two acoustic wave packets is considered.

[16] H. Reiss, Bull. Am. Phys. Soc. **11**, 96 (1966); J. H. Eberly and A. Sleeper, Phys. Rev. **176**, 1570 (1968).

ments of the previous sections in their proper perspective.

The T-matrix element for an arbitrary (linear) scattering problem is a product of two parts: a four-dimensional δ function of the kinematic variables times an amplitude. If the amplitude function should vary considerably over a small energy range (e.g., near a resonance), a monochromatic description of the physical scattering process might be inadequate. In this case one integrates the product of wave packet times T-matrix element. The integral is over the momentum-energy variables which appear as arguments both of the δ function and of the amplitude. In contrast to this, in nonlinear scattering, as in the presence of an external "Maxwell field," in the monochromatic limit one obtains a δ function which depends on the intensity of the external field. (This gives rise to the IDFS.) The physical problem of colliding wave packets cannot be obtained from this monochromatic amplitude, however, because the problem is nonlinear.

Retaining the wave-packet description of the external field throughout the computation, we have shown that the integral (3.6) has a power-series expansion, the terms of which are in agreement with the corresponding terms obtained from covariant perturbation theory. These terms are integral representations over monochromatic multivertex Feynman graphs with standard kinematic constraints in their δ functions multiplied by wave packets. All this would have been mere "nit-picking" if the result of a wave-packet analysis had yielded an expression identical with the monochromatic result except for a correction of order $\Delta\omega/\omega_0$. ($\Delta\omega/\omega_0$ is small even in the case of a pulsed laser.) This, however, is not the case. The plot of the frequency profile for the scattered photon in Fig. 3 shows that the wave-packet calculation yields a result qualitatively different from the monochromatic result. (If the monochromatic description were to be accurate, except for a $\Delta\omega/\omega_0$ correction, the graph in Fig. 3 would be a Lorentzian centered around the IDFS value.)

From this vantage point, it is only natural to ask the following question: How good is a plane-wave description for a light pulse emanating from a laser rod? The answer depends, of course, on the problem under consideration. For any process which can be adequately described by the lowest-order (nonvanishing) terms, such as the linear Thomson amplitude, harmonic production, etc., the plane-wave approximation is excellent. If higher-order (in the intensity parameter) terms have to be included, however, for an adequate description of the problem, the answer is no longer obvious.

We now proceed to estimate the limit of validity of the plane-wave approximation. In Fig. 4 we depict the two singular Feynman graphs corresponding to an interaction quartic in the field amplitudes. The amplitude is given by

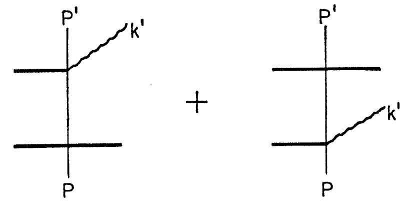

FIG. 4. Second-order singular Feynman graphs. The thick lines depict three-dimensional wave packets.

$$T=N\int \delta^{(4)}(m+q_1+q_2-q_3-p'-k')Fd^3q_1d^3q_2d^3q_3\,, \quad (4.1)$$

where

$$N=2^{1/2}[2\pi m p_0'\omega']^{-1/2} \quad (4.2)$$

and

$$F=B(q_1)B(q_2)B(q_3)\left[\frac{1}{2m(q_{10}-q_{30})-2q_1\cdot q_3+i\epsilon}+\frac{1}{2p'\cdot(q_3-q_1)-2q_1\cdot q_3+i\epsilon}\right]. \quad (4.3)$$

$B(q)$ represents a three-dimensional packet. The normalization is as follows:

$$\int 2q_0|B(q)|^2d^3q=M\,, \quad (4.4)$$

M being the number of photons.

In contrast to this, the corresponding plane-wave packet amplitude is given by

$$T'=N\int \delta^{(4)}[m+n(\xi_1+\xi_2-\xi_3)-p'-k']\times G d\xi_1 d\xi_2 d\xi_3\,, \quad (4.5)$$

where

$$G=A(\xi_1)A(\xi_2)A(\xi_3)\left(\frac{1}{2n}-\frac{1}{2p'\cdot n}\right)\frac{1}{\xi_1-\xi_3+i\epsilon}. \quad (4.6)$$

A represents a one-dimensional (plane-wave) packet. The normalization is as follows:

$$\int 2\xi|A(\xi)|^2d\xi=(M/L^2)\,, \quad (4.7)$$

where (M/L^2) stands for the photon flux per unit area. (Recall that $c=1$.)

A comparison of Eqs. (4.3) with (4.6) immediately reveals the basic differences between the plane-wave and three-dimensional-packet cases. We list the differences in order of importance.

(a) In the plane-wave case the nonlinear part of the amplitude is zero for forward scattering (when $m=p'\cdot n$), whereas (4.3) does not vanish in this limit.

(b) In the plane-wave case, only the Cauchy principal part of the integral contributes to the differential scattering cross section; the $i\pi\delta$ terms sum up to an over-all phase factor [see Eq. (3.5)], whereas in the three-dimensional case the $i\pi\delta$ terms stand for multiple scattering in which the electron can suffer a momentum change. (Our estimates indicate that these terms do not modify appreciably the frequency profile of the scattered photon.) The most important difference stems from the following.

(c) If the frequency spread Δq_0 is small in comparison to the transverse momentum spread Δq_t, then (4.3) may differ significantly from (4.6). A simple estimate is the following: When

$$\Delta p \sim (\Delta q_t)^2/\Delta q_0 \tag{4.8}$$

or smaller, the plane-wave approximation is not valid. Δp stands for the magnitude of the momentum transfer to the electron. Such a situation arises in the case of almost forward scattering, or alternatively when Δq_t is large, as may be the case for a focused laser beam. We should mention here that classical arguments also indicate that for a focused beam the plane-wave approximation fails.[17] Our estimate, however, involves Planck's constant.

Another dividend of (4.3) comes from the recognition that for certain experimental configurations, such as crossed laser beams, $q_1\cdot q_3$ can be made larger than $m(q_{10}-q_{30})$, obtaining thus an effective nonlinear parameter which is approximately $e^2\rho/\omega^3$. (ρ stands for the photon density.) In contrast, the plane-wave nonlinear parameter is $e^2\rho/m^2\omega$. Such an experiment has been proposed recently.[18]

V. CONCLUSION

We have obtained the following results. In the case of nonlinear scattering, the shape and extent of the external field are fundamental aspects of the problem. The monochromatic limit, while it may exist as a mathematical exercise, is not germane to the physical scattering problem. Whether or not an IDFS is obtained in the monochromatic limit is thus irrelevant. The relevant thing is the frequency distribution (center and profile) of the scattered photon at a given scattering angle. As we have shown in Sec. III, the frequency distribution is a sensitive function of the intensity and line shape of the incident radiation. Furthermore, in the wave-packet case it is difficult to disentangle the classical from the quantum aspects of the problem. Thus, the marked broadening displayed in Fig. 3 may not show up at all when the electron is treated classically. We have also explored the limit of validity of the plane-wave approximation. Here we were forced to conclude that when the transverse extent of the wave packet has to be considered in the computation, the problem *has* to be done semiclassically. A classical calculation based on the Lorentz-force equation will give erroneous results.

Finally, we would like to mention at least one important omission in our discussion of this problem. We have assumed that the radiation was in a "coherent" state. For only then is it permissible to use the semiclassical equations. When the radiation is not in a coherent state, there is no substitute for the fully quantized theory. Since the frequency profile of the scattered photon depends on the coefficients of the higher-order terms, and these coefficients in turn depend on the (photon) density matrix, the ordering of the photon operators is an important aspect of the problem. In principle, one could carry out the calculation to a given order as a function of the multiple correlation values. These unknown parameters could then be determined from empirical data. This additional work would be premature, however, for the obvious reason that the nonlinear terms in Compton scattering are still extremely small for unfocused lasers. For the focused case the computations are even more involved since the plane-wave approximation is then no longer valid.

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