Analysis of the Experimental Meaning of Coherent Superposition and the Nonexistence of Superselection Rules*

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The experimental meaning of coherent superposition of state vectors is analyzed. It is shown that the involvement of the measuring instrument is of basic importance and that it is erroneous to study superposition without it. From this analysis it is found that there are no such things as superselection rules. The meaning of the sign change of spinors under 2π rotation is considered, and it is found to have experimental consequences. The nonmeasurability of the relative parity of states with different z components of angular momentum is shown. An appendix discusses the application of some of the considerations of the paper to decaying states and elementary particles.

I. INTRODUCTION

THE concept of a superselection rule was first introduced by Wick, Wightman, and Wigner[1] (WWW) through an analysis of the behavior of wave functions of different angular momentum under time inversion, and hypothesized by them to extend to the charge quantum number also. Recently, their conclusions have been challenged by Aharonov and Susskind,[2,3] on the (implied) grounds that the experimental meaning of superposition was not considered, and that when it is considered the bases upon which WWW rest their conclusions turn out to be irrelevant. Some further discussion of this question has also been given in other papers.[4,5]

Although all the arguments have been given by AS1, many of them have only been implied. It seems therefore worthwhile to discuss the entire subject in complete and explicit detail and to present a full analysis of the meaning of the concept of superposition, in this context, and to show why there are no superselection rules as the term was used by WWW. We shall attempt to do this here.

The basic conclusion of this paper is that, in principle, from the point of view of superposition, there is no difference between familiarly superposed quantities like momentum, or angular momentum, and others such as charge, baryon or lepton number, or univalence [that is $(-1)^F$, where F is an even integer for states of integer spin, and an odd integer for states of half-odd-integer spin].[6]

From this result, we find that there are no superselection rules, and that in this context all quantum numbers have analogous properties.

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[1] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952), referred to as WWW. See also G. C. Hegerfeldt, K. Kraus, and E. P. Wigner, J. Math. Phys. **9**, 2029 (1968).

[2] Y. Aharonov and L. Susskind, Phys. Rev. **155**, 1428 (1967), referred to as AS1.

[3] Y. Aharonov and L. Susskind, Phys. Rev. **158**, 1237 (1967), referred to as AS2.

[4] W. Rolnick, Phys. Rev. Letters **19**, 717 (1967).

[5] R. Mirman, Phys. Rev. **186**, 1380 (1969).

[6] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964), p. 5.

In Sec. II the Aharonov-Susskind experiment is analyzed in detail, the meaning of superposition is discussed, and the nonexistence of superselection rules is considered. In Sec. III we consider a closely related topic, which illustrates many of our points, the sign change of spinors under 2π rotation. In Sec. IV the concept of the phase conjugate to particle number is analyzed, in Sec. V we consider the error in the proofs of the superselection rules, and in Sec. VI it is shown how to construct the correlated containers needed for the Aharonov-Susskind experiment. Section VII considers the possibility of measuring the relative parity of states with different charge. The paper concludes with an appendix which studies another example of one of the basic questions considered in the paper, the factorization of state vectors.

II. AHARONOV-SUSSKIND EXPERIMENT AND ITS IMPLICATIONS

The basic instrument in studying the meaning of superposition is the AS1 experiment. At first sight it seems to have many odd features, and it may appear that the ability to superpose coherently states of different charge, or different univalence, for example, is dependent on these oddities. In fact, it may appear that these oddities destroy the conclusion that such superposition is possible, as they seem to require different definitions of such terms as superposition and state in the charge case than in the angular momentum case. And that would mean that we are really discussing different things when we talk about superposition in the different cases.

Actually, what we wish to show is that the AS1 experiment is in fact exactly analogous to the corresponding experiments for the other quantum numbers, and that the apparent new features are present there also. In other cases they were only implied and so perhaps not fully noticed; AS1 simply made them explicit.

The idea of AS1 is to construct a system consisting of two containers filled with mesons, such that the ratio of the numbers in the two containers is undetermined, although the total number may be fixed. Thus we can describe each container by a quantum-number operator whose eigenvalue is the number of mesons in that box. Conjugate to this quantum-number operator is another which gives the phase (a concept which we shall discuss in detail in Sec. IV). The containers are in a minimum uncertainty state, so the relative phase is accurately known. The total phase of the system can be unknown since the total particle number is determinable.

If a proton is now sent into one box it emerges as a combination of a proton and a neutron. AS1 claim that the wave functions describing these two particles combine coherently to describe the state of the beam emerging from the first container. The states are coherently superposed if there is a definite measurable phase between them, and AS1 provide an experimental prescription for measuring that phase.

The prescription is to duplicate the setup many times, and in some of the setups to measure the relative numbers of protons and neutrons emerging from the first container, and in others to allow the beam to pass first through the second container and then measure the ratio. From these two numbers AS give a formula for finding the phase while the beam is in the region between the two containers.

It is perhaps clear why coherent superposition of charge states is not as obvious as coherent superposition of angular momentum states. For it requires the transformation of one particle into another, and not the transformation of one spin state into another. Hence it requires considerably higher energy, and so experimentally it can occur, at best, only under very restricted conditions. Unlike angular momentum superposition, it does not occur in "everyday" laboratory experience.

The two basic concepts of the experiment are that the measuring apparatus is an integral part of the definition of a coherent state, and that the number of particles in each box must be undertermined. Let us see the reason that these requirements must hold.

The coherent state is defined relative to the containers, so that if, while the particle is in transit between the two, the second container is removed, then it would be impossible to measure the relative phase, which would become meaningless. Thus, what is being described is not the state of the particle, but the state of the system consisting of the particle and the containers.

Clearly, the number of particles in the containers must be uncertain. Let us assume that before the proton entered the first container the number of particles in it was measured. Then after the particle left, the number could be measured again without disturbing the particle. But then from conservation of charge, the charge of the particle could be determined. Thus it could not be in a coherent state of two charges. Likewise if the number of mesons in the second chamber were measured, then the number in the first could be computed and the charge on the particle determined without disturbing it. The superposition would be incoherent because the measurements make uncertain the relative phase between the containers and, as can be seen from the analysis of AS1, prevent the measurement of the relative phase carried by the particle.

What we now wish to show is that these two principles, that the system must be defined to include the measuring apparatus and that the number of particles in each container is uncertain, hold for "normal" situations like the superposition of different states of momentum or angular momentum, or z components of angular momentum. We first consider the question of correlation starting with some general considerations and then study some examples.

Let us consider the inclusion of the measuring apparatus in the definition of the state vector. Given any experiment whatever, the proper state vector to study is that of the entire Universe. But, of course, in order to make the analysis feasible we factorize the state vector of the Universe into $|r)|s)$, where s refers to the system (that part of the Universe) that we wish to study, and r to the rest of the Universe. We now assume that we can study each term in the product separately, and that they have no effect on each other. This factorization process is really fundamental, but it is usually carried out without any thought. However, it turns out to be by no means as obvious as it is generally taken, and if not done carefully it can easily be done incorrectly. The superselection rules arose because the factorization was done incorrectly for coherent states.

We shall assume (and it seems wise to state this assumption explicitly, and perhaps not so wise not to analyze it) without further consideration that we can factor out into the rest of the Universe everything except the particle and the apparatus acting on it.

It seems "obvious" that we can consider the experiment done in a region of space far from all other matter so that the rest of the Universe will have no effect on the system. But what we have also factored out above is the experimenter, the person who interacts with the measuring apparatus to get the data, as well as the instruments that prepare the measuring apparatus. It is far less obvious that these can be factored out also.

To study how, and why, the instruments affect our description of the process, we consider, for simplicity, a system consisting of a particle, the coherence properties of whose states we wish to study, and two containers, which may be the ones of AS1 filled with mesons, or may be magnets, etc., depending on the experiment.

After the particle passes through one container and is on its way to the second, we want the system to be in a coherent superposition of states which, if we could write the state vector as the product of state vectors for the particle p and the two containers 1 and 2, would be described as

$$|1)'|2)'|p)'+|1)|2)|p). \tag{2.1}$$

Actually what is usually meant by a coherent superposition is the state

$$|1)|2)[|p)'+|p)]. \qquad (2.2)$$

It is at this point that the usually assumed factorization breaks down, for the expression (2.2) is not the same as (2.1). Further, (2.1) is not correct. Before discussing these statements, we give an example to show the former.

Let us take container 1 to be a magnet and the particle going through to have its spin flipped. Clearly, by conservation of angular momentum, the spin of the magnet is also changed so that states $|1)$ and $|1)'$ are not identical. Of course, for a magnet it is a good approximation to regard them as the same and go from (2.1) to (2.2). But one cannot correctly get general rules about quantum mechanics by ignoring the fact that this is an approximation.

The only way in which we can get (2.2), therefore, is if $|2)'|1)'$ and $|2)|1)$ are the same. But container 2 cannot be affected by the passage of the particle through container 1. Hence we conclude that it is impossible to get (2.2) rigorously. There is no state describing a coherent superposition of states of the particle, with the measuring instruments factored out. But (2.2) is, and must be, used in practice, so that the real question is how to construct a state given by the correct form of (2.1), such that (2.2) is a very good approximation to it.

The requirements for this are that the state of container 1 must be such that the passage of the particle through it produces a change so small that it can be ignored, and that either or both the initial or the final state of container 1 not be determinable.

The reason for this last requirement, as discussed above for the charge case, is that if they were both determinable then we could determine the state of the particle after it leaves container 1, and so it could not be in a coherent superposition of states. To show that this is where the indeterminacy comes in, we presume that the relevant quantum numbers of the state of the Universe (or, more particularly, that part which comprises the experiment) can be determined exactly before the experiment begins. Thus the total charge or the total angular momentum, for example, is known. Further we assume that the particle is initially so far away from the rest of the system that its quantum numbers can be determined. (We shall not concern ourselves here with the fact that this is a self-contradictory statement, for if the particle is so far away from the rest of the system, then it is not near enough to any piece of apparatus to have its quantum numbers measured.) Thus, its initial state as well as the quantum numbers of the system consisting of the Universe minus the particle, that is, the two containers, can be taken as known. We shall not consider here what other conditions must be satisfied in order to measure the particle's quantum numbers.

The final quantum numbers of the container cannot be uncertain, because they can always be measured without affecting the particle. Thus, we conclude that the initial quantum numbers of container 1 must be uncertain while the quantum numbers of the system of the two containers can be known (in principle). Thus, it is the relative values of the quantum numbers of the two containers which must be unknown.

Returning now to (2.1), we see that it should really be a sum over different states, which are eigenstates of the quantum-number operators with different eigenvalues:

$$\sum_i |1)_i'|2)_i'|p)'+\sum_i |1)_i|2)_i|p). \qquad (2.3)$$

The expression (2.2) now must be replaced by

$$\sum_i |1)_i|2)_i[|p)'+|p)], \qquad (2.4)$$

which means that, for the commonly accepted definition of a coherent state to be reasonable, the requirement is that to a good approximation,

$$\sum |1)_i'|2)_i'=\sum |1)_i|2)_i. \qquad (2.5)$$

Let us now consider the implications of these results. First we see that there must be some degree of correlation between the two containers, since the sum of the quantum numbers is fixed (whether the total value is known or not is irrelevant, since it can presumably be determined in principle without disturbing the system by studying the rest of the Universe). Since one container has a distribution of values of this quantum number, so must the other, and the distributions are related.

The state of the particle while in transit between the two containers is also correlated with the state of the containers, for it is the first container which determines its properties (for a given initial state). Thus, when we talk about a coherent state we are not discussing the state of a single particle, but rather that of a system consisting of the particle and the various instruments used in the experiment. This last part can often be ignored, but the fact that it is reasonable to ignore it must be demonstrated in each case and not just assumed.

The state of the containers must meet certain requirements, because it is the state of container 1 which determines the state of the particle, and on this state we have put a very strong requirement, that it be a coherent superposition. In addition, Eq. (2.5) must be satisfied, to a good approximation. We shall not consider here what class of states meets these requirements, but for the charge and similar cases there is one such set, the coherent states[7] used by AS1, as can be seen directly from this latter paper.

[7] P. Carruthers and M. M. Nieto, Am. J. Phys. **33**, 537 (1965); R. Jackiw, J. Math. Phys. **9**, 339 (1968), has shown that the coherent states are not always the minimum uncertainty states. There seems to be little doubt, however, that they are sufficient for our purposes.

Note that these states have a very well-defined phase, which is necessary if the state of the particle is to be a superposition of states with well-defined phase.

This, then, is a consideration of the correlation between the different instruments used in the experiment from the theoretical point of view. We wish now to consider some particular experiments to see whether the correlation is actually present.

First we shall study angular momentum, essentially the Stern-Gerlach experiment. Here we send a beam through one magnetic field, causing the spin of the particle to be aligned by the field; that is, the direction of spin of the particle leaving the first magnet is along the line defined by the first magnet. (Remember we are dealing with a large number of particles, so their initial phase averages to zero and can be ignored.) Then the particle passes through the field of the second magnet, and the direction of its spin is determined; that is, the direction of the first magnetic field is determined. Now if the state of the particle in transit between the two magnets is coherent, then we must be able to show this experimentally. This we do by repeating the above experiment, measuring the relative intensities of the up and down states, as the second magnet is rotated; that is, we must repeat the experiment many times. Coherence is a property that cannot be shown by one experiment, but requires a large number of repeated experiments.

What properties are required of the two magnets? These are that there be a rather well-defined angle between them, and that this angle change in time in a fairly well-defined way. These are necessary because the experiment, carried out over a period in time, measures the angle between the two magnets.

These requirements mean that the two magnets are correlated, and correlated in such a way that the knowledge of their relative angle and angular momentum is close to a maximum. A reasonable approximation to such a state is a coherent state.[7] This is exactly the same as in the charge case.[2]

The question of the correlation between the two magnets has been discussed somewhat further elsewhere.[5]

The situation is the same when we consider coherent superposition of momentum states (that is, measurement of position). Consider two infinite screens aligned parallel to each other with holes in them. A particle passes through the first screen and an eigenstate of position is created, with the position being defined by the hole in the first screen. The position is then measured by the second screen. In order to determine if the particle is in a coherent (and not incoherent) superposition of momentum states, the experiment must be repeated many times to show the presence of interference terms. But this clearly would not be possible if the relative position of the two holes, or the relative momenta of the two screens, were poorly known. Once again we see that we are led to correlation between the two boxes, which here are the two screens.

There is a difference between the spin case and the orbital angular momentum or momentum cases, that is, between internal and external variables. Since the state vectors for the latter are functions of a measurable variable (angle or position), there is a possibility that more types of experiments to measure coherence can be developed for the latter than the former. Hence some of our preceding statements may not impose requirements for all the experiments in this group and must be used with care. This does not affect the fundamental conclusion of the paper, which is that the class of quantum numbers for which superposition can be applied is larger than previously believed.

We turn now to the requirement that the number of particles in the container be uncertain and consider how this applies to the situations we are familiar with.

The first case is that of angular momentum, and we consider the magnet as consisting of a collection of atomic-sized bar magnets, each of which have spin $\frac{1}{2}$. If the spin of the traversing particle is flipped, one of the bar magnets must be flipped also. Clearly, if we had only one bar magnet, then we could tell the direction of the spin of the particle after it left the first magnet, so it could not be in a coherent superposition of states. We would, however, have a coherent superposition of states describing the particle and the bar magnet, but this sum could not be factored into a product of sums. We could not go from (2.1) to (2.2).

There cannot be an uncertainty in the number of bar magnets, since the total mass of the magnet is not altered by the particle passing through. The requirement is, in this case, that there be some bar magnets pointing up and some down, and that the excess of the number up over the number down be uncertain. If the excess were known, then the reasoning of the preceding paragraph would apply and we would not have coherence.

To understand the reason for this uncertainty in a more quantum-mechanical manner, let us again consider the case in which the magnet consists of a single spinning particle. Here one component of the angular momentum is determined but the other two must be completely uncertain. Thus, the direction of the magnetic field is uncertain. Suppose that we try to use this single particle as the first magnet in a Stern-Gerlach experiment. If we could show that the particle were in a coherent superposition of up and down states (with respect to the second magnet) and find its phase, we could find the direction of the magnetic field of the first magnet and so the direction of its angular momentum. In other words, we would be able to measure all three components of the angular momentum simultaneously.

This is, of course, impossible; what is wrong with the above argument is that in order to measure the phase it is necessary to repeat the experiment many times. But if the direction of the angular momentum, and thus the magnetic field, is uncertain, the experiment is not reproducible. The phase difference between the up and down states will be different for each repetition, and thus not measurable.

Thus, the requirements that the coherence be a property of a set of experiments, and not just one, and that there be an uncertainty in a property of the measuring apparatus (the number of particles) are closely related to an uncertainty principle, here that of angular momentum. They are required in order that quantum mechanics be consistent. Ignoring the measuring apparatus leads to the danger of a basic error.

In order to get a well-determined magnetic field, it is necessary to have more than one particle and to have the direction of the total angular momentum of the system well determined.

There is another way of considering the interaction of the particle with the magnet, which brings out somewhat better the analogy with the AS1 experiment. The particle does not interact with the magnet directly; it interacts with the magnetic field which can be considered to be a box of photons. If the number of photons having spins up and down were measured before and after the particle passed through, then the direction of the spin of the particle would be determined, so once again an uncertainty in number is required.

Actually, the analysis of this situation is not quite as straightforward as in the box of mesons, because the magnetic field is a static field which cannot be analyzed into photons carrying angular momentum. When the particle passes through, it changes the magnetic field and in this way angular momentum is transferred. Thus, the analogy is somewhat spoiled, but the basic ideas are the same. It hardly seems necessary to present here a detailed analysis of this way of looking at the situation.

The next example is that of the coherent superposition of states of different momentum. Once again we must do a large number of experiments to show that the state of the system before the position was measured was a coherent superposition of different momentum states and not merely a single-momentum state which was changed to an eigenstate of position by the measurement of this quantity.

In order to see where the number uncertainty comes in, we consider an apparatus consisting of two screens parallel to each other, which have holes in them. The particle is put into a coherent superposition of momentum states by passing through the first hole and then it passes through the hole in the second screen and its position is measured. The screen is made up of particles each of which consist of a wave packet of momentum. Each particle's wave function can now be Fourier analyzed, and we arrive at a set of occupation numbers for the different momentum states. These occupation numbers must be uncertain, by the above arguments. But this resulting uncertainty in the momentum of the screen is, of course, what is expected, since something is known about the screen's position.

Now that we have shown generally the necessity for our requirements, we note one point about our standard example of coherent superposition: the addition of states with different z components of angular momentum. While this example is useful because of its familiarity, one should be aware of a difficulty in using it as a model: The magnetic field of the particle depends on the direction of the spin, and hence on the phase angle between up and down states. It may be possible to use this fact to construct experiments which have no analog in other cases, and hence are misleading. It is therefore often worthwhile to test analogies with other cases besides this one.

One such case would be a particle with zero angular momentum and spin, and positive parity, which enters a box filled with particles A which have spin 1, orbital angular momentum 0, and positive parity. A reaction may take place which produces a particle of spin 0, and positive intrinsic parity, which leaves the box, and a particle B with angular momentum 0, spin 1, and negative parity, which remains. In order to get a coherent superposition of orbital angular momentum 0 and 1 in the exiting beam, it is clear that the relative number of particles A and B in the box must be uncertain.

All of the above considerations are based on the assumption that if the initial state of a system is known and if for the final state all the quantum numbers but one are determined, then the value of that quantum number must be fixed. Thus, if we knew the initial and final number of mesons in a box and the initial charge of a particle, then the final charge of the particle is determined. In other words, we assume that the quantum numbers are additive.

However, this addivity of quantum numbers is not always true. As an example, consider a spin-1 particle which decays to give two particles of spins 0 and 1. Then the orbital angular momentum of the two particles is either 0 or 2. (The parities of all the particles are taken as positive.) The wave function describing the final state of the system is a coherent superposition of orbital angular momentum 0 and 2. The state function factors exactly, rather than just approximately as in all the above cases, into functions describing the spins of the particles, etc., and a function describing the orbital angular momentum, and the latter is a coherent sum. Because this factorization is exact, the different orbital angular momentum states are not correlated with other states and so cannot be determined by measuring them.

Having shown the nonexistence of a superselection rule for charge, we can ask whether there are, or can be, any quantum numbers for which a superselection rule exists. In the above discussion, and in the rest of the paper, we are not using any property of charge; it was just a label. Thus, we see that the states describing and two particles can be coherently superposed, provided it is possible through some reaction to convert the particles into each other.

If the particles have greatly different masses, then the phase difference between them will vary very rapidly. There are other possible experimental difficulties of this type, but none in principle.

III. SIGN CHANGE OF SPINORS UNDER 2π ROTATION

The behavior of spinors under rotation affords another opportunity for studying the importance of including the experimental apparatus in the analysis of experiments. We shall do this here.

In particular, we shall reply to the criticism by Hegerfeldt and Kraus[8,9] of the study of spinor rotation through 2π by AS2.[3]

The basic statement by HK is: "If, by some suitable apparatus, a quantum system is prepared in a certain state, then the rotated state is prepared by the rotated apparatus."

To clarify the error in the statement, we wish to see where it leads, so we consider as an example the statement that an inverted state has different properties from the original state (when weak interactions are considered). By the definition of HK, the inverted state is the state produced by the inverted apparatus. Thus, we can take a collection of spinning nuclei, apply a magnetic field in the up direction (by definition), and align the nuclei along the field. This is how we prepare the state. We next measure the ratio of the number of electrons emitted by the nuclei along the field to the number emitted opposite to it. The whole laboratory (which is the preparing apparatus) is now inverted so that the magnetic field is in the down direction and we prepare the inverted state and perform the experiment again. We find that the relative number of electrons is unchanged and so parity is conserved.

The thing wrong with the above discussion is that by using the inverted apparatus to produce an inverted set of nuclei, what we get is not the inverted state but the same state. The state is (pictorially) a linear combination of a spinning nucleus plus a nucleus with electrons being emitted preferentially along the direction of the spin of the first nucleus. The inverted state is a spinning nucleus plus a nucleus with electrons emitted preferentially opposite to the spin direction. Because of parity nonconservation, the inverted state does not exist.

Note that there is a difference between the initial and the inverted, or rotated, states. If this were not so, then the inversion or rotation process would be meaningless.

Returning now to the phrase "prepared by the rotated apparatus," we see that what is wrong is that only part of the apparatus can be rotated, for the instrument that performed the rotation must remain fixed. The assumption that this instrument can be disregarded is incorrect, for the statement that two instruments are rotated through some definite angle with respect to each other means that they are correlated. And it is the rotating apparatus which provided the reference to compare their correlation. Therefore, the phase of the two instruments with respect to the rotating apparatus, and thus with respect to the particle being studied, is different. We shall study in detail below an experiment which can in some ways serve as an example of this.

First we briefly discuss the "counterexample" given by HK. They state that they show, by what purports to be the argument of AS2, that a rotation through 0° produces an observable effect on spinors. What they do, however, is generate different mathematical state vectors by applying rotation operators to the original state vector. But this is not a rotation of the particle, which is a physical process. The difficulty is that one cannot draw physical conclusions about the set of vectors they generate until these vectors are correlated with physical objects. HK's result, that there are different vectors which can be correlated with the same object (an unrotated spinor), is different from AS2's result, that two different vectors must be correlated with two different physical objects, a rotated and an unrotated spinor. The latter result has physical implications and the former is not a counterexample to it.

HK also object to the "nonkinematical nature of the AS experiment." The problem here is that in quantum mechanics we tend to use abstract states without being clear about their experimental meaning. Thus, we can talk about two states which are rotated with respect to each other. But experimentally, how do we get two such states? We could say, as do HK, that the "rotated state is the state prepared by the rotated apparatus." But this statement only postpones the question—for how is the rotated apparatus prepared? Clearly, at some stage we must either take two instruments and prepare them in states which are rotated, and coherent, with respect to each other, or we must take one instrument and rotate it. This one rotated instrument, or the two correlated instruments, then either produce the rotated states or the devices to produce the states. Why do the two instruments have to be correlated? The answer is by definition, since they have an angle between them which is both definite and fixed, to a good approximation, and this is what we mean by correlation.

Is the AS2 experiment dynamical? Clearly, producing two correlated instruments, or rotating one instrument, involves dynamics. An instrument cannot be rotated by applying an operator of the rotation group to it. There is no such thing as a "kinematical experiment." There are only discussions of experiments in which the dynamics is well hidden. AS2 were simply more explicit in indicating the dynamics than is usual. But there is not more dynamics in this experiment than in others.

As an example, consider an experiment in which it

[8] G. C. Hegerfeldt and K. Kraus, Phys. Rev. **170**, 1185 (1968), referred to as HK.

[9] J. S. Dowker, J. Phys. A **2**, 267 (1969), has considered this question but his considerations are not relevant to our discussion.

is the observer who is rotated. Suppose that we have a large number of observers arranged in a circle observing a spinor. As we interrogate each observer around the circle in turn, we will get reports of an increasingly rotated particle. Now consider two observers, one who is the first whom we interrogated and who is at an angle infinitesimally greater than zero, and the other, the last one interrogated, who is an angle infinitesimally less than zero. Will they report the same wave function except for a difference in sign? It seems that this question is regarded as the one that has to be answered affirmatively in order to say that a spinor changes under a 2π rotation. Despite the fact that the answer seems to be obviously no, it is actually yes.

The observer cannot report on the properties of the wave function unless he first interacts with the particle. This means that all the observers must be correlated with the particle, and hence with each other. The observers should be fermions, or else they will not get complete information about the wave function. Thus, we have a set of correlated fermions strung out along a circle. By this we mean that the fermion at position ϕ has a wave function $|\text{up}\rangle+e^{i\phi/2}|\text{down}\rangle$. Now how are these observers to be produced? They are produced by taking a set of observers all at 0° and rotating them. But this means that the two overlapping observers at 0 and 2π have been rotated with respect to each other, and so their wave functions differ by a sign. But they do not compare their wave functions with each other, but with that of a third object, the observed particle, and so they disagree on the sign of its wave function.

All these points are illustrated in the experiment suggested by AS2. However, it would seem worthwhile to consider another experiment in which these considerations might perhaps be made somewhat more explicit. The experiment is similar to that suggested by Bernstein,[10] but there is no restriction to infinitesimal magnetic fields here.

We consider a beam of spin-$\frac{1}{2}$ particles, polarized in the up direction along some z axis, sent into a magnetic field along the x axis. When the beam emerges there are both up and down particles, which we separate into two beams, one all up and the other all down. We send both beams (so that their history is the same) through magnetic fields along the x direction, and then split each emerging beam, once again, into up and down beams. We now have two beams with particles polarized in the up direction, with identical histories, except that one has been rotated through 2π. The phases of the two beams are compared, which we shall do mathematically below, and which can be done experimentally by combining them. The phases, of course, differ by a minus sign.

In fact, we can split the two beams into two parts, then recombine first the two subbeams from the same beam, and then two subbeams coming from the two different beams. The resulting interference pattern will shift with the subbeams because of the minus sign.

The phase of the rotated particle is different after it leaves the apparatus from what it was when it entered because it traveled through a potential. However, because the phase of the unrotated particle changed by the same amount owing to travel through the potential, we can subtract this change off and we thus see that we have rotated a spin-$\frac{1}{2}$ particle through 2π and as a result have changed its phase, in a measurable way, relative to its original phase, by -1.

Note that the unrotated particle's presence is not really relevant to this change of phase. What it does is to provide a reference phase, fixed throughout the experiment, equal to the original phase of the rotated particle, except for the phase shift due to the potential which we have taken care of above, with which to compare the final phase of the rotated particle. Clearly, such a fixed reference must be present in any experiment in which we consider the effects of a rotation. In many cases such a fixed reference is a macroscopic body, like a magnet, and so the present situation appears to be different because we are using a quantum body. But if we treated the magnet quantum mechanically, then the analogy would be more evident, but not really more exact.

We now solve the Schrödinger equation and show that the statement about the change in phase made above is correct.

For a particle of mass m moving along the y axis through a magnetic field H along the x axis, Schrödinger's equation is

$$\left(\frac{1}{2m}\frac{d^2}{dy^2}+E\right)\psi+\tfrac{1}{2}\omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\psi=0\,, \tag{3.1}$$

where $\omega=eH/mc$. Writing the wave function as

$$\psi=\begin{pmatrix}\alpha(y)\\ \beta(y)\end{pmatrix}, \tag{3.2}$$

Schrödinger's equation becomes

$$\begin{aligned}(1/2m)\alpha''+E\alpha+\tfrac{1}{2}\omega\beta&=0\,,\\ (1/2m)\beta''+E\beta+\tfrac{1}{2}\omega\alpha&=0\,.\end{aligned} \tag{3.3}$$

Letting

$$\alpha=\sum\alpha_m e^{ik_my}\,,\quad \beta=\sum\beta_m e^{ik_my}\,, \tag{3.4}$$

we find the solution is (ignoring reflections at field discontinuities)

$$\psi=\begin{pmatrix}\alpha_+e^{ik_+y}+\alpha_-e^{ik_-y}\\ \alpha_+e^{ik_+y}-\alpha_-e^{ik_-y}\end{pmatrix}, \tag{3.5}$$

where

$$k_\pm=(2mE\pm m\omega)^{1/2}\,. \tag{3.6}$$

[10] H. J. Bernstein, Phys. Rev. Letters **24**, 1102 (1967); Sci. Res. (N. Y.) **4**, 32 (1969).

The boundary condition is

$$y=0, \quad \psi=\begin{pmatrix}1\\0\end{pmatrix},$$

giving the state after the particle has traveled through the magnetic field which has a length l to be

$$\psi=\frac{1}{2}\begin{pmatrix}e^{ik_+l}+e^{ik_-l}\\ e^{ik_+l}-e^{ik_-l}\end{pmatrix}. \tag{3.7}$$

The beams are now split and each enters a magnetic field H' covering a distance l' and with corresponding ω' and $k_\pm'$.

The initial conditions on the two beams are

$$\psi_0|_{\text{up}}=\frac{1}{2}\begin{pmatrix}e^{ik_+l}+e^{ik_-l}\\ 0\end{pmatrix},$$
$$\psi_0|_{\text{down}}=\frac{1}{2}\begin{pmatrix}0\\ e^{ik_+l}-e^{ik_-l}\end{pmatrix}, \tag{3.8}$$

where ψ_{up} and ψ_{down} are the beams resulting from the up and down components of the original beam.

The state vectors after the particles leave the field are

$$\psi_{\text{up}}=\begin{pmatrix}e^{ik_+'l'}+e^{ik_-'l'}\\ e^{ik_+'l'}-e^{ik_-'l'}\end{pmatrix}\times\tfrac{1}{4}(e^{ik_+l}+e^{ik_-l}), \tag{3.9}$$

$$\psi_{\text{down}}=\begin{pmatrix}e^{ik_+'l'}-e^{ik_-'l'}\\ e^{ik_+'l'}+e^{ik_-'l'}\end{pmatrix}\times\tfrac{1}{4}(e^{ik_+l}-e^{ik_-l}). \tag{3.10}$$

The up components are

$$\psi_{\text{up}}\to e^{i(k_+'l'+k_+l)}+e^{i(k_-'l'+k_-l)} +e^{i(k_+'l'+k_-l)}+e^{i(k_+l+k_-'l')}, \tag{3.11}$$

$$\psi_{\text{down}}\to e^{i(k_+'l'+k_+l)}+e^{i(k_-'l'+k_-l)} -e^{i(k_+'l'+k_-l)}-e^{i(k_+l+k_-'l')}. \tag{3.12}$$

Now

$$k_\pm=\sum_\mu\frac{(\frac{1}{2})!(2mE)^{1/2-\mu}(\pm m\omega)^\mu}{\mu!(\frac{1}{2}-\mu)!} =\sum_{\mu\text{ even}}\{\ \}\pm\sum_{\mu\text{ odd}}\{\ \}. \tag{3.13}$$

Let

$$\zeta=\sum_{\mu\text{ even}}\{\ \},\quad \eta=\sum_{\mu\text{ odd}}\{\ \}. \tag{3.14}$$

Then the up components are

$$\begin{aligned}\psi_{\text{up, down}}&=\exp i(\zeta'l'+\eta'l'+\zeta l+\eta l)\\&\quad+\exp i(\zeta'l'-\eta'l'+\zeta l-\eta l)\\&\quad\pm[\exp i(\zeta'l'+\eta'l'+\zeta l-\eta l)\\&\qquad+\exp i(\zeta l+\eta l+\zeta'l'-\eta'l')]\\&=\exp i(\zeta'l'+\zeta l)\{\exp i(\eta'l'+\eta l)\\&\quad+\exp(-i(\eta'l+\eta l))\\&\quad\pm[\exp i(\eta'l'-\eta l)+\exp(-i(\eta'l'-\eta l))]\}\\&=2\exp i(\zeta'l'+\zeta l)[\cos(\eta'l'+\eta l)\pm\cos(\eta'l'-\eta l)].\end{aligned}$$

So

$$\psi_{\text{up}}=4\exp i(\zeta'l'+\zeta l)\cos\eta'l'\cos\eta l,$$
$$\psi_{\text{down}}=-4\exp i(\zeta'l'+\zeta l)\sin\eta'l'\sin\eta l.$$

The phase of the wave functions is $\psi/|\psi|$ and differs by -1, as stated.

Here, once again, we have an uncertainty in the particle number of each beam. For suppose that there were only one particle going through the apparatus. Then if we determined which beam it was in, we would know that the wave function describing the other beam was zero and so there would be no interference. This is exactly the same as the elementary double-slit experiment, which emphasizes the generality of the concept of number indeterminacy.

It seems unnecessary to consider the case in which there is more than one particle. The point here is that a measurement of the particle number of a beam will require a force on the particles and thus will change their phase. The more accurate the determination of the particle number, the greater the uncertainty in phase. And if the relative phase of the two beams were indeterminate, the experiment would be impossible.

IV. PHASE

One of the basic concepts used in the AS1 experiment is the phase which is conjugate to the number of particles. It is this phase which determines the relative phase between the proton and neutron states when they are coherently superposed, and it is the value of this phase which must be reasonably well determined for the containers.

It would seem worthwhile, therefore, to study this quantity in somewhat further detail, to show that it is conjugate to the number of particles, and to attempt to get an intuitive understanding of its meaning and the rôle it plays.

If the phase is conjugate to the particle number, then it is indeterminate when there is a definite number of particles. It will help to understand the concept if an example of this is given. We shall do this next, first for the angular momentum case, then for the charge. The particle number is taken as 1. This one particle we shall call the magnet, and the particle whose coherence properties are being studied is called the particle. The subscripts m and p shall denote their states.

We shall use the interaction Hamiltonian

$$H=\omega\boldsymbol{\sigma}_m\cdot\boldsymbol{\sigma}_p, \tag{4.1}$$

where ω is a constant and the σ's are Pauli spinors.

The initial state is written as

$$\begin{aligned}\psi_0&=\begin{pmatrix}e^{i\varphi/2}\cos\frac{1}{2}\theta\\ ie^{-i\varphi/2}\sin\frac{1}{2}\theta\end{pmatrix}_m|\uparrow\rangle_p\\&=e^{i\varphi/2}\cos\tfrac{1}{2}\theta\,|\uparrow\rangle_m|\uparrow\rangle_p+ie^{-i\varphi/2}\sin\tfrac{1}{2}\theta\,|\downarrow\rangle_m|\uparrow\rangle_p,\end{aligned} \tag{4.2}$$

where φ is the magnet's azimuthal angle and θ is the angle between it and the z axis, which is defined as the direction of the spin of the initial particle. The arrows denote the direction of the spin, and the spin direction of the magnet can have any value, since the incoming particle and the first magnet of the Stern-Gerlach experiment are uncorrelated.

Solving the Schrödinger equation and assuming that the particle remains in interaction with the magnet for a time T, we find that after the particle has left the magnet the wave function is

$$\psi=[e^{i\varphi/2}\cos\tfrac{1}{2}\theta e^{-i\omega T}|\uparrow\rangle_m|\uparrow\rangle_p +\tfrac{1}{2}ie^{i\varphi/2}\sin\tfrac{1}{2}\theta(1-e^{2i\omega T})|\uparrow\rangle_m|\downarrow\rangle_p] +[\tfrac{1}{2}ie^{-i\varphi/2}\sin\tfrac{1}{2}\theta(1+e^{2i\omega T})|\downarrow\rangle_m|\uparrow\rangle_p]. \quad (4.3)$$

Notice that this result is different from the simple coherent wave function instinctively used in the Stern-Gerlach experiment, because the state of the magnet enters. Therefore, before we study the behavior of the particle let us first determine the state of the magnet, which we shall take to be determined as up. Then the wave function of the particle is given by the first term in the above equation, which is indeed a coherent superposition of spin up and spin down.

Now the phase difference between the up and down states of the particle depends on the phase difference between the up and down states of the magnet, as given in Eq. (4.2). This phase is, of course, the orientation of the angular momentum, and it is more usual to write it in such a way that this is implied, as in the first line of Eq. (4.2). But it is important to note that a state representing a particle whose spin is not along the z axis can of course be written as a linear combination of spin-up and spin-down states, as in the second line of Eq. (4.2). This makes the analogy with the charge case clearer, and it is helpful to write it in this form since we are studying such a coherent superposition.

Since the phase of the coherent superposition of the states of the particle depends on the direction of the spin of the magnet, which cannot be measured exactly, what can be said about the phase? Very simply, it cannot be measured at all. To show this we shall, for simplicity, quantize the magnet along the z axis so that the wave functions become

$$\begin{pmatrix} e^{i\varphi/2} \\ 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 \\ ie^{-i\varphi/2} \end{pmatrix}.$$

Note that they are not

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

because the component of the spin in the xy plane is unknown, and so the phase angle must be put in. This point is usually not important, but it is here. But since the direction of the spin is not determined, neither can the phase angle be.

So we immediately see that if the magnet consists of one particle, then its phase must be undetermined and it cannot produce a coherent superposition of up and down states of the particle passing through it.

Let us now consider the charge case. In the angular momentum situation, if we applied a torque the direction of the angular momentum changed. Or, to put it another way, if we put the particle in an angular-dependent potential then the relative phase between up and down states changes. For the charge case the situation is exactly the same. If we put the particle in a potential for a certain time, its phase changes. In fact, if we use isotopic spin notation, then we put the particle in a potential which depends on the (isotopic) angle, that is, in an electric field, so that the potential is different for up (positive charge) and down (negative charge) states; then the relative phase between up and down states changes. Note that the phase gives the direction of the particle's isotopic spin.

Consider now the AS1 Hamiltonian [Eq. (5) of AS1],

$$H=g(\sigma^+a^-+\sigma^-a^+), \quad (4.4)$$

and apply it as above, starting with a one-particle magnet (here container) initial state given by Eq. (4.2), with $\theta=\pi$, to get the state after the particle leaves the magnet:

$$\psi=e^{-i\varphi/2}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_c|\downarrow\rangle_p\right]\sin gt +ie^{-i\varphi/2}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_c|\uparrow\rangle_p\right]\cos gt, \quad (4.5)$$

where the subscript c stands for container.

Thus, we see that the state is an incoherent mixture of plus and minus charge because the state of the container enters. So we confirm our previous statements that a container containing a single particle cannot produce a coherent state.

The multiparticle state can produce a coherent superposition of states, but it must be a mixture of states of different values of the number of particles (or excess of up over down). As remarked previously, this is essentially because the phases of the various states add to zero so that the average phase is determined (as zero). Clearly, in order to do this a sum of states of different phase is needed, and the form of this sum must not depend on the phase. Since the phase of the individual terms must vary over all the quadrants, no matter how small the individual phase is, an infinite number of terms is needed. Clearly, the coherent states meet the requirement, for the phase of the different states in the sum depend on n, and therefore take on the required range of values of all φ, since n goes to infinity. Next we show that the phase is conjugate to the number of particles. Consider an n-particle state whose phase we shall take as zero. Put it in a potential for a time T, such that the energy of each particle increases by E, so that of the entire state increases by nE. After it is removed, its phase will be nET. A state which is a sum of states of different n will have the phase of each of them changed by a different amount. Thus, an arbitrary function can be written $\sum_n f(n)e^{in\varphi}$, analogous to $\int f(p)e^{ipx}dp$, so φ and n are conjugate variables. Therefore, the uncertainty in the phase increases as that of the number of particles decreases.

At this point one can raise the question: What quantity is conjugate to the charge (and remember that the properties of charge here are irrelevant; charge simply labels the particles)? It is the question that Rolnick[4] has asked: What is the observable that does not commute with charge? The answer is that the phase does not commute with the charge, as we have seen above, and it is an observable since it determines the value of the cross terms in an interference experiment.

The same discussion as above applies to univalence also, where the phase is conjugate to the number of fermions in the container.

V. ANALYSIS OF PROOFS OF SUPERSELECTION RULES

The position that there is no such thing as superselection rules in the sense generally used, and that there is no reason to believe that it is impossible to add coherently states differing by any quantum number whatever, has been argued in detail above. However, there are proofs of superselection rules (as in WWW), and it is necessary to consider where we disagree with these proofs.

The proofs are basically as follows. Consider two state functions f_1 and f_2 which differ in the value of some quantum number (charge, univalence, etc.). Then it is claimed that the coherent superposition f_1+f_2 has no meaning because by some transformation under which the system remains invariant, time reversal, rotation, gauge transformation, etc., the superposition becomes $f_1+e^{i\varphi}f_2$. From this it follows that the phase between the two state vectors is physically meaningless.

We agree that the superposition f_1+f_2 is meaningless, but we do not agree that this is the state function of the system that we wish to describe. The correct state function can be coherently superposed and does not vary under the transformations which leave the system invariant.

The definition of the proper state vector of the system is discussed above. In this section we wish to consider the examples we are using to show that the transformations leave the correct state vector invariant. The first example we consider is the Stern-Gerlach experiment with the particle between the two magnets. What is the state function of the system? As we have discussed above it is not $|$particle spin up$)+|$particle spin down$)$, which is what WWW have used in their proof, but rather $\sum[|1)'|2)'|\mathrm{up})+|1)|2)|\mathrm{down})]_i$, where the numbers refer to the magnets, and this cannot be factored.

Of course, this can be factored to a very good approximation, but it is just this small difference between the approximation and the exact formula which nullifies the proof. As an example, let us assume that $|2)'$ and $|2)$ are identical and that the only difference for the states of 1 are that a single dipole has its spin down in one case and up in the other. Then, ignoring the factored parts, the state function becomes $|$magnet down$)|$particle up$)+|$magnet up$)|$particle down$)$. Now, applying time reversal to this state, we see that each term in the product changes sign so that the product does not, and as a result the sum remains the same.

When we have something of the order of 10^{23} atoms in a magnet, we can usually ignore the behavior of a single one. But this situation is an exception, for the single magnet represents the difference between the two states. Further, the use we put it to is not additive (as would be the case if we were computing the change in energy of the magnet, for example) but multiplicative, where its sign, and sign variation, is crucial and the sign depends not on the total number of atoms, but whether that number is even or odd. We cannot ignore the behavior of a single atom here, for it invalidates the proof.

What about the charge case; is the relative phase of the proton and neutron changed by a gauge transformation?

The answer is that the relative phase between the two charge states is determined by the relative phase of the two containers in the experiments described. This phase is determined by the phase of the mesons, which in turn is changed by a transformation by an amount which is the negative of the change in phase of the charge states due to the transformation. The result of the transformation is therefore not to change the relative phase of the particles at all.

To show this, we carry out a gauge transformation e^{isQ}, so that a state vector describing a state with charge nQ is multiplied by e^{insQ}. Thus, the phase term in the state vector of Eq. (4) of AS1 now becomes $e^{in(\theta-sQ)}$. From their Eq. (9), the relative phase of the two particles is $\theta-\theta'$, where θ and θ' are the phases of the two containers. Clearly, this remains unchanged when the two phases are changed equally.

Or consider the same result as expressed by their Eq. (7). We take Q of the mesons and the charge difference of the superposed particles as both 1. Then the phase difference between the two states is $\theta-\varphi$, where φ is the proton-neutron phase difference when the particle is between the containers. Now φ is increased by s because of the gauge transformation. Again the phase difference between the two states remains unchanged by this transformation.

Unlike several other quantum numbers, no superselection rules have been given for angular momentum. Yet we have argued that all the quantum numbers must be treated identically. To complete the analogy, we must therefore give a "proof" similar to the other quantum numbers for a superselection rule for angular momentum.

To "show" the impossibility of coherently superposing two states with different z components of angular momentum, we take a spin-$\frac{1}{2}$ system and use the wave function given in Eq. (4.2), ignoring the magnet, with $\theta=\frac{1}{2}\pi$, $\varphi=0$. We now rotate around the z axis by some angle χ, so the wave function becomes

$$\psi'=e^{iL_z\chi}\psi=e^{i\chi/2}|\uparrow)_p+ie^{-i\chi/2}|\downarrow)_p. \tag{5.1}$$

Notice that the wave function is varied in exactly the same way that a superposed wave function describing positive and negative charge changes under gauge transformations. Hence, if we were to follow the analogy, we would postulate that there is a superselection rule for states of different m values. (Remember that WWW did not "prove" the superselection rule for charge, they only hypothesized it. The only quantum number for which a "proof" was given was univalence.) We should be able to conclude that the phase difference between up and down states is arbitrary and so unmeasurable.

The difficulty here is that the phase difference between the up and down states gives the orientation of the spin with respect to the xy plane. But this orientation is changed by a rotation around the z axis. Thus, the variation in phase does not rule out a coherent superposition, it is required by it.

The xy plane is determined by the apparatus, and this result is, therefore, a way of restating our previous results.

There is a case noted above where we can factor the state vector exactly into terms, one of which is a coherent superposition of states of different angular momentum. Is it possible to do the same thing in the charge case, and then use gauge invariance to show that the resulting superposition is meaningless?

The two states in the superposition differed in total angular momentum, but they both had the same value of the z component of angular momentum. The analog in the charge case is a coherent superposition of two states with different total isotopic spin, but with both having the same charge. Under transformations both states change by the same phase, and the coherent superposition of different isotopic spin states is a perfectly reasonable and accepted procedure. Hence the analog is, in fact, exactly analogous.

Thus, again we can find no difference, in principle, between the angular momentum and the charge cases.

VI. CONSTRUCTION OF CORRELATED CONTAINERS

Although it should now be clear that given two correlated boxes of charged particles in coherent states we can create a coherent superposition of charge states, we have not shown that such a coherent superposition is possible until we prescribe a means of getting the correlated containers with coherent states. To this question we now turn.

First we consider how a coherent state of a simple harmonic oscillator is produced. As shown by Carruthers and Nieto,[7] this can be achieved simply by subjecting the oscillator to a constant force so that the Hamiltonian becomes

$$H=\hbar\omega(a^\dagger a+\tfrac{1}{2})-x_0F(a+a^\dagger), \tag{6.1}$$

where the a's are the creation and annihilation operators, x_0 is a constant, and F is the force. The ground state of this Hamiltonian is then

$$e^{x_0F/\omega}(a^\dagger-a)|0), \tag{6.2}$$

which is clearly an eigenstate of the annihilation operator a.

This shows the mathematics of obtaining a coherent state; we now have to consider the physics. As the first example, we again consider a magnet. How do we construct a magnet?

The magnet we consider will be composed of spin-$\frac{1}{2}$ particles, and before we construct the magnet these are randomly oriented; there are as many up states as there are down ones. We shall divide the particles into pairs, each consisting of one up and one down particle. To construct the magnet, we must take some of the pairs and flip their down particles to up ones. Thus the ground state, the "vacuum," consists of zero flipped pairs, and the various excited energy levels are determined by the number of flipped pairs. If an external magnetic field is imposed so that the energy of the flipped pairs is lower than that of the unflipped, the Hamiltonian (after subtracting a constant due to the total mass of the system) is just the number of flipped pairs times their energy, exactly as for the simple harmonic oscillator.

Physically, to construct the magnet we take the unmagnetized specimen, impose an external field, and subject the system to a beam of impinging particles whose purpose it is to take up the excess energy and angular momentum. These "particles" could, of course, be the external motions of the spinning particles, in which case the excess energy appears as heat. The Hamiltonian can be represented schematically as that in Eq. (6.1), where x_0F is replaced by a coupling constant times the density of the impinging particles, and the a's represent the creation and annihilation operators for flipped pairs. They are linear rather than quadratic by definition, since when they create a flipped pair they also annihilate an unflipped one.

Thus the formalism is identical, and a coherent state, a magnet, is created. However, this is not quite what we want in our experiment.

The problem is that each state with different numbers of flipped pairs n is correlated with different states of the beam. Thus, if we measure the number of particles impinging on the sample and the number of particles leaving, then we can compute n from such conservation laws as those of energy and angular momentum. Likewise, if the magnet is produced by thermal means we can find n by measuring the initial and final temperatures.

Clearly, this is not satisfactory and what we want is a correlation between two magnets. Here we must be careful, for we can get correlation by allowing the magnets to interact through their magnetic fields. The direction of the magnetic field of the magnet depends on the phase, but this seems to be essentially an accidental property and we shall avoid this method so that our results can be extended to other situations.

What we shall do is send a beam of particles through two samples. Then by studying the properties of the beam we will be able to find the total number of pairs, but we shall not be able to tell how the pairs are distributed between the two magnets. Thus, we can produce coherent states which are correlated with each other.

Consider a beam of particles with average spin in some direction going along through the specimen. The Hamiltonian will be of the form of Eq. (6.1), with the x_0F replaced by a term depending on the coupling constant and the particle density of the beam. Let us assume that the interactions between the particles in the beam and those in the box have a very small effect on the beam. Then the beam will be essentially undisturbed after passing through the specimen. The specimen will be magnetized in the direction determined by the direction of spin of the particles in the beam.

When it reaches the second specimen the average spin of the beam will be in the same direction as it was when it passed the first specimen. Hence the effect on both specimens will be the same and the result is that two correlated magnets will be created.

Clearly, the spin of the beam can be rotated while it is in transit between specimens, which will result in two correlated magnets at an angle with each other.

To create two correlated boxes of charge, we shall consider them filled with neutrons (only for simplicity; stable nuclei can be used) and protons, and in the AS1 experiment the incoming proton may scatter off the neutrons and exchange charge, coming out either a proton or a neutron, and, if done properly, a coherent mixture.

The container will be in an electric field (only to make the discussion analogous to the case of the oscillator) and the ground state will consist of all neutrons. We now send through a beam of protons, some of which will undergo charge-exchange collisions so that the state of the container will be transformed to one describing a mixture of neutrons and protons: an excited state.

The situation here is then exactly analogous to the case of a magnet, and it is unnecessary to repeat the above discussion. Likewise, to get two correlated containers we let the beam enter the second container after it leaves the first, just as above.

In practice, the above experiment can be carried out by taking a crystal whose nuclei are isobars of some stable nucleus, placing it in a reactor to cause some conversion of one nucleus to the other, and then cutting it. This then supplies the correlated containers.

Of course, we do not discuss here the question whether it is feasible to carry this out in practice, but there are no reasons, in principle, why it cannot be done.

VII. MEASUREMENT OF PARITY

The major conclusion that WWW drew from their postulation of superselection rules is that it is impossible to measure the relative parity of two particles of different charge, or of different univalence. We have argued above that such superselection rules do not exist, and that in this regard charge is exactly similar to angular momentum. Since we know that we can measure the relative parity of different angular momentum states, we should therefore be able to measure that of different charge states, and in order to complete the analogy we must exhibit an experiment to measure it.

Actually, we can exhibit no such experiment, and the reason is that the above argument is incorrect; for we cannot measure the relative parity of different angular momentum states, although one of the possible conventions, the one which is actually used, is far more esthetic than any other. Such a convention can, of course, also be used in the charge case, and usually is. But there it is clearer that it is a convention.

When we say states of different angular momentum, we mean states which differ in their z components (m values), and we do not consider here states of different total angular momentum. The situation is the same as in the charge case, for the m value determines the charge and there seems never to have been a claim that it is impossible to measure the relative parity of two states only because they have different isotopic spin; for example, the p and the Σ^+.

Although the idea that states with different m values have arbitrary relative parity may not seem too strange for spin, it does seem to conflict with what is known of orbital angular momentum wave functions. And if the orbital angular momentum m values have their relative parity determinable, and not just definable, then since spin can be converted into orbital angular momentum, the wave functions with different m values for spin have theirs determinable also.

In order to prove the statement that parity is a convention, we define the wave function as $p_m Y_m$, where the Y's are spherical harmonics and the p's are functions of the m's whose only properties are a phase which may change under a parity transformation, $p_m{}^* p_m = 1$, and are otherwise arbitrary except that they are subject to the constraints arising from the relations $Y_{m_1} Y_{m_2} \sim Y_{m_1+m_2}$.

There are two questions about such a procedure: Does it violate inversion invariance, and does it violate rotational invariance?

As far as parity conservation is concerned, it should be noted that m is conserved in all reactions. Thus, given any system, the parity of a state of the entire system with any z component of the total angular momentum is determined by definition and, since that m state can never be converted into any other m state, it is not necessary that these definitions be related. Thus, there are no reactions which do not conserve parity because of this definition, which would conserve parity if all the p's are taken to be 1.

Parity conservation means that it is possible to define consistently a set of parities for different states and that once having been defined by means of one reaction, there will be no other reaction for which this definition will not result in the total parity after the reaction equaling the total parity before the reaction. Let us assume that this is possible for all the p's equal to 1 (so that parity is conserved using the standard convention for the p's). The requirement of conservation of parity then becomes that the product of the intrinsic and orbital parities times that of the p's remain constant in the reaction. By the above statement the product of intrinsic times orbital parities is constant, so that the requirement reduces to the requirement that the product of the p's remain constant. But since the product of the p's is defined to be equal to the p for the total m, and since m is conserved, this follows. Thus, using arbitrary p's does not cause a violation of parity conservation if it is not violated for all the p's equal to 1.

By inversion invariance we mean simply conservation of parity. Of course, we may speak of it as meaning that if we reflected the system in a mirror we would see the identical system. But it is impossible to reflect a wave function in a mirror. Thus, a definition with all the p's equal to 1 would give a simpler formalism, and nicer mirror reflection properties, but there would be no experimental distinction between this convention and any other, or any experimental way of determining the p's.

And it is likewise to an analysis of experiment that we must return to determine the rotational properties of a wave function with nonunit p's.

First let us consider how we determine the time dependence of a state of mixed m, since it does not appear to satisfy the Schrödinger equation, which is formally rotationally invariant, while the state vector is not. The answer is that the state function for each m value obeys the equation separately, and since all m values have the same energy and time dependence, so does the sum of terms. Note that the requirement that the state vector obey the rotationally invariant Schrödinger equation does not necessitate rotational invariance; it just allows it. Clearly, a universe consisting of one free particle with spin up is not rotationally invariant, for there is a universal z axis, along the particle's spin direction. Likewise, consider a universe consisting of the interior of a box containing two particles of different mass and opposite spins. Define the z axis at each point as pointing along the spin of the particle whose probability density is larger at that point. Thus, the direction of the universal z axis varies in space, and if we wish to consider traveling waves, we could have it varying in time. Thus, the fact that we use as the wave function pY rather than Y does not prevent it from satisfying the Schrödinger equation.

To determine whether this convention violates rotational invariance, we must ask two questions. First, is each component of angular momentum conserved in every reaction, and second, will two experimenters rotated with respect to each other get the same experimental result? The affirmative answer to the second clearly implies the affirmative answer to the first question.

To answer the first question, we consider some interaction Hamiltonian which consists of a product of creation and annihilation operators, times products of the p's, possibly. These operators annihilate and create various angular momentum states, and the products must be such that the various angular momentum components before and after application of the Hamiltonian are the same. But the types of operators that appear in the product are not going to be altered by the fact that the product is multiplied by the p's. So angular momentum will continue to be conserved in all reactions.

We are thus led to the question of whether two experimenters, rotated with respect to each other, will get the same results.

To answer this, we have to show that the matrix elements of any operator are rotationally invariant. Any state can be written as a sum of terms, each of which is the product of angular momentum eigenstates, and any operator can likewise be written as products of creation and annihilation operators on these states. We now redefine the operator by substituting for a creation or annihilation operator the same operator times the proper p. Thus, any matrix element is a sum of terms each of which contains an even number of p's and thus which contain no p's. Remember that for each creation operator for state m in the matrix element, there is an annihilation operator, and vice versa. Therefore, the matrix elements are rotationally invariant if they are when all the p's are defined as 1.

Therefore, no experimental results are altered if the different eigenstates of the z component of angular momentum have arbitrarily different parities, and so their parities are a matter of definition. Clearly, defining all the p's as 1 is the simplest possible choice, both for angular momentum and for charge.

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APPENDIX

One of the major points of this paper is that the factorization of the state vector of the Universe into a term describing the system under consideration and another term describing the rest of the Universe, which can be ignored, is of basic importance in analyzing the system. We should like here to describe another situation which will illustrate this point again. This has been discussed elsewhere,[11] but it seems worthwhile to discuss it from the present point of view.

Consider a Universe consisting of the decay products of some unstable state. Then, if we wish to study the subsystem consisting of one of these decay products, we have to factor the state vector of the Universe into a term describing the system under study and another term for the rest of the Universe. We should like to examine here the circumstances under which this factorization can meaningfully be carried out, and what restrictions it places on our description of the system.

The quantum-mechanical description of a system requires a complete set of operators whose eigenstates are the states of the system. The operators and the eigenstates are those of the whole Universe, and the questions of the preceding paragraph amount to the question of whether it is possible to carry out the factorization so as to define a complete set of diagonalizable operators for the subsystem, and for which operators is it possible to carry out this factorization.

The details of the analysis are given in Ref. 11. Here we shall only describe how the results of that paper can be viewed in the context of this one.

We consider first the hydrogen atom for which, unless we wish to limit our studies to the ground state, the Universe must consist, at a minimum, of the electron, the proton, and the photons making up the radiation field. Of course in many cases it is convenient to factor this into a term describing the electron-proton subsystem, and a term describing the photons and ignoring the latter. And it turns out that we can indeed find a complete set of diagonalizable single-particle operators (the single particle being the electron-proton subsystem) such as the spin of the single particle. But the important point is that the Hamiltonian is not one of these operators. There are no single-particle states which diagonalize it. When we talk about the mass or energy of the atom, we actually mean the expectation value of the Hamiltonian between single-particle states. The so-called "unperturbed Hamiltonian" is, of course, not a Hamiltonian but an operator whose eigenvalues are equal to these expectation values of the Hamiltonian. The single-particle states are not eigenstates of the Hamiltonian because they do not remain single-particle states. The Hamiltonian connects them with multiparticle states, that is, excited states of the atom decay.

Next we consider elementary-particle physics, where it is customary to regard the elementary particles as being eigenstates of the Hamiltonian, even if they decay as they all do [with, of course, the exception of the ground state(s)]. Further, the use of sets of operators to describe the particles is widespread, particularly when group and algebraic descriptions are used. Often, however, these operators are not clearly defined. For example, when we talk about the isotopic-spin operator, what operator do we mean? It may seem trivial to say that it is the operator operating on single-particle states, but if we consider the angular momentum operator and apply the same definition of it, we cannot then conclude that rotational invariance requires this operator to be conserved. Thus, consider a Universe consisting only of a single resonance. This then decays into a proton and a pion. If we now define the single-particle angular momentum operator for a pion, this is clearly not conserved, and if the pion were to decay while it was still in the vicinity of the proton, this nonconservation of angular momentum would have important consequences concerning the properties of the decay.

It turns out that, for elementary particles, it appears to be reasonable and useful to factor an operator into a single-particle operator and a background operator, provided that the factored operator is the generator of a semisimple Lie algebra. Thus, the single-particle spin or isospin operators seem both useful and meaningful, and the eigenstates of this (still incompletely defined) complete set of operators can be correlated with the elementary particles. This allows the elementary particles to be labeled.

However, while the semisimple operators can be diagonalized by the single-particle states, those operators forming Abelian subalgebras of the algebra describing the elementary particles (assuming that such an algebra exists), or at least of the Poincaré algebra, apparently cannot be.

In particular, there is no such thing as a single-particle Hamiltonian (disregarding situations where we ignore the entire system and just study one state, the ground state, in which case we would give up almost all of the information about the system). Consider the above example and write H_s as that Hamiltonian which acts on both the single-particle resonance state and the single-particle proton state. Then the factorization would be time dependent as the pion joins the background, changing it. Thus H_s and H_b, the two terms in

[11] R. Mirman, Progr. Theoret. Phys. (Kyoto) 41, 1578 (1969).

the sum making up the total H, would not commute. It seems unlikely that such a scheme would be very useful.

This result leads to the mass of the single-particle state being the expectation value (properly defined, as discussed in Ref. 11) of the total Hamiltonian, and not the eigenvalue of a single-particle Hamiltonian.

In general, one must be careful in the factorization process when unstable states are involved.

These considerations do not seem to have been always throughly understood, and as a result errors have occasionally arisen.

A more detailed discussion of the whole question is given in Ref. 11.

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Plane-Wave Packets and Their Limitation in Nonlinear Compton Scattering*

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We show that scattering boundary conditions are incompatible with a monochromatic radiation field for the case of nonlinear Compton scattering. We demonstrate this by showing that (a) in the monochromatic limit, gauge invariance in a given order of the expansion of the S matrix is destroyed, and (b) the physical (scattering) boundary condition, that of a pulse of radiation incident on a target electron, cannot be reconstructed from the monochromatic limit of the S matrix. We then proceed to show by an example that the frequency profile of the scattered radiation is a function of both the intensity and line shape of the incident field. Another interesting feature of this calculation is that the profile of the photon scattered at a fixed angle is significantly broadened in comparison with the incident line shape. The worked-out example is a simple model, that of a neutral, scalar "electron" interacting with a bilinear scalar, massless external field, which contains all the important features of nonlinear Compton scattering. While from the point of view of gauge invariance it is sufficient to treat the external radiation field as a one-dimensional wave packet, for a complete description of the problem it is necessary to describe the incident radiation (quanta) in terms of normalizable states. An estimate of the breakdown of the plane-wave approximation is included.

I. INTRODUCTION

THE generalization of the Klein-Nishina formula[1] to include the effect of an intense light beam has been the subject matter of numerous articles.[2,3] These computations fall into two categories. The first group of authors[2] obtains the scattering amplitude via the Volkov[4] solutions for incident and outgoing electron states. Recall that the Volkov[4] wave function is a solution to the Dirac (Klein-Gordon) equation for a charged particle in the presense of an external, transverse electromagnetic field. The electromagnetic field is restricted to be a plane wave. This excludes the use of a three-dimensional wave packet in the description of the external field. The second approach[3] makes use of the adiabatic switching-on-and-off technique and covariant perturbation theory. The two methods yield diverging results. Briefly, the disagreements between the two methods are twofold: (a) In the kinematics, the scattering amplitude based on the Volkov[4] solutions yields an intensity-dependent frequency shift (IDFS), while the other method[3] gives no IDFS, and (b) the two amplitudes differ in their functional form. The merits of one approach versus the other have also been discussed in equally numerous articles.[5]

Recall that it was demonstrated[6] that the imposition of scattering boundary conditions on the Volkov[4] solutions is incompatible with unitary time evolution of the state vector. There is also something wrong, however, with the second approach,[3] viz., the scattering

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