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Equivalence of the Dirac Equation to a Subclass of Feynman Diagrams*

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The Dirac equation of an electron in a Coulomb field is shown to be equivalent to a certain subclass of Feynman diagrams taken within the eikonal approximation. The result is obtained via analysis of a two-body scattering amplitude, wherein the role of the heavier of the two particles involved is to generate the effective external potential in which the lighter particle propagates. The poles of the scattering amplitude are found to be the well-known relativistic binding energies.

I. INTRODUCTION

The present paper is concerned with a new derivation of the classical one-particle Dirac equation which appears in the present context as a sum of a certain subclass of Feynman diagrams taken within the eikonal approximation. We propose an approach based on the analysis of the scattering amplitude of a fermion and a heavy charged scalar particle. By way of performing certain appropriate approximations upon the correct Green's functions associated with those particles and equating the above scattering amplitude to its spectral representation, we can locate the corresponding energy poles which are shown to agree with the well-known energy spectrum of an electron in a Coulomb field. This result, which makes substantial use of the eikonal nature of the high-energy approximation, is the consequence of an infinite mass ($m'\to\infty$) limiting process.

II. SCATTERING AMPLITUDE—GENERALIZED LADDER GRAPHS

We are interested in the scattering amplitude of a two-particle reaction $p_1+p_2\to p_1'+p_2'$, where p_1 (p_1') and p_2 (p_2') are the initial (final) four-momenta of a fermion and a charged scalar particle. Subscript 1 refers to the fermion, mass m, while index 2 denotes the heavy charged scalar particle, mass m'. We limit our attention to incoming undressed particles which exchange virtual photons in all possible generalized ladder-type ways, i.e., we study the contribution to the scattering amplitude $T(p_2'p_2|p_1'p_1)$ arising from all diagrams shown in Fig. 1. Those Feynman graphs can be formally evaluated in closed form by an exponential operation that acts upon the Green's functions associated with the two particles[1,2]; i.e., if $G(x_2'x_2|x_1'x_1)$ denotes the four-point Green's function corresponding to the exchange of all photons between the charged scalar and fermion line, the quantity of interest is

$$G(x_2'x_2|x_1'x_1) =\exp\left(-i\int dudv\,\frac{\delta}{\delta A_{1\mu}(u)}D_c(u-v)\frac{\delta}{\delta A_{2\mu}(v)}\right)\times G_1(x_1'x_1|A_1)G_2(x_2'x_2|A_2)\Big|_{A_1=A_2=0}\,, \tag{2.1}$$

where G_1 refers to the fermion and G_2 to the charged scalar Green's function.

It is easy to see that the expansion of the exponential operator of (2.1) yields the sum of all ladder and crossed graphs.

Formula (2.1) is an exact statement as long as self-energy structure and vertex-type corrections are neglected. $D_c(x-y)$ represents the undressed photon propagator taken in the Feynman gauge. The external c-number field $A_\mu(x)$ is introduced to carry out the combinatorics and at the same time appears as a test source in the functional argument of the Green's functions whose response upon $A_\mu(x)$ will reveal the dynamics. Thus, we first write down the differential equations satisfied by the Green's functions, without approximation,

$$\left[m-\gamma\left(\frac{\partial}{\partial x}-ieA_\mu(x)\right)\right]G_1(x,y|A_\mu)=\delta(x-y)\,, \tag{2.2}$$

$$\left[m'^2-\left(\frac{\partial}{\partial x}-ieA_\mu(x)\right)^2\right]G_2(x,y|A_\mu)=\delta(x-y)\,. \tag{2.3}$$

In the following we are mainly interested in a closed-form solution of Eq. (2.3). In order to proceed, we first

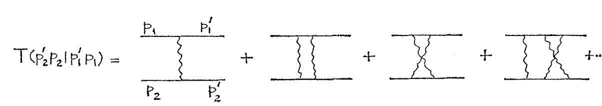

FIG. 1. Class of all ladder and crossed graphs summed by Eq. (2.1).

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[1] H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969), and further references cited there.
[2] G. W. Erickson and H. M. Fried, J. Math. Phys. 6, 414 (1965).

rewrite (2.3) using translational invariance of G, i.e.,

$$G(x,y|A(\xi)) \equiv G(x+h,y+h|A(\xi+h)) = G(x,y|A(\xi)) + h^\mu\left(\frac{\partial}{\partial x^\mu}+\frac{\partial}{\partial y^\mu}+\int d\xi\frac{\partial}{\partial \xi^\mu}[A(\xi)]\frac{\delta}{\delta A(\xi)}\right) \times G(x,y|A(\xi))+\cdots.$$

Therefore, as $h\to 0$,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\int d\xi\frac{\partial}{\partial \xi}[A(\xi)]\frac{\delta}{\delta A(\xi)}\right)G(x,y|A(\xi))=0.$$

This formula permits us to rewrite the gauge-covariant derivative on the left-hand side of Eq. (2.3). We obtain

$$\left[m'^2-\left(\frac{\partial}{\partial y}+\int d\xi\frac{\partial}{\partial \xi}[A(\xi)]\frac{\delta}{\delta A(\xi)}-ieA(x)\right)^2\right] \times G_2(x,y|A)=\delta(x-y).$$

Introducing

$$G(x,y|A(k))=\int d^4p\, e^{ip(x-y)}G(p|A(k)e^{ikx}),$$

$$A(k)=\frac{1}{(2\pi)^4}\int d^4\xi\, e^{-ik\xi}A(\xi)$$

then yields, in momentum space,

$$\left[m'^2+\left(p-\int dk\, kA(k)\frac{\delta}{\delta A(k)}+e\int dk\, e^{ikx}A(k)\right)^2\right] \times G_2(p|A(k)e^{ikx})=1. \quad (2.4)$$

This equation, however, can only be solved under certain assumptions. The approximation we are interested in corresponds to neglecting terms quadratic in the internal four-momenta k in functions of type $m'^2+(p-\sum_i k_i)^2$. Then (2.4) leads to

$$\left[m'^2+p^2-\int dk\, 2pkA(k)\frac{\delta}{\delta A(k)}-2ep\int dk\, e^{ikx}A(k)\right] \times G_2(p|A(k)e^{ikx})=1, \quad (2.5)$$

which also can be written as

$$\exp\left(-e\int dq\frac{p\cdot A(q)}{p\cdot q}e^{iqx}\right) \times\left(m'^2+p^2-\int dk\, 2pkA(k)\frac{\delta}{\delta A(k)}\right) \times\exp\left(e\int dq\frac{p\cdot A(q)}{p\cdot q}e^{iqx}\right)G_2(p|A(k)e^{ikx})=1.$$

This equation implies

$$\{\cdots\}\exp\left(e\int dq\cdots\right)G_2=\exp\left(e\int dq\cdots\right),$$

where we have used an obvious notation. Then,

$$\exp\left(e\int dq\cdots\right)G_2 =\{\cdots\}^{-1}\exp\left(e\int dq\cdots\right) =i\int_0^\infty d\alpha\exp(-i\alpha\cdots)\exp\left(e\int dq\cdots\right)$$
$$=i\int_0^\infty d\alpha\exp[-i\alpha(m'^2+p^2)] \times\exp\left[i\alpha\int dk\, 2p\cdot kA(k)\frac{\delta}{\delta A(k)}\right] \times\exp\left[e\int dq\frac{p\cdot A(q)}{p\cdot q}e^{iqx}\right]$$
$$=i\int_0^\infty d\alpha\exp[-i\alpha(m'^2+p^2)] \times\exp\left[e\int dk\frac{p\cdot A(k)}{p\cdot k}e^{i\alpha 2p\cdot k}\right],$$

where we used the formula

$$\exp\left[\int\Phi(\xi)A(\xi)\frac{\delta}{\delta A(\xi)}\right]\exp\left[\int\chi(\xi)A(\xi)\right] =\exp\left[\int\chi(\xi)e^{\Phi(\xi)}A(\xi)\right].$$

Thus, Eq. (2.5) can be solved, with the result

$$G_2(p|A(k)e^{ikx})=i\int_0^\infty d\alpha\exp[-i\alpha(p^2+m'^2)] \times\exp\left[e\int dk\frac{p\cdot A(k)}{p\cdot k}e^{ikx}[e^{i\alpha 2p\cdot k}-1]\right], \quad (2.6)$$

or in configuration space

$$G_2(x,y|A_\mu)=\int dp\, e^{ip(x-y)}i\int_0^\infty d\alpha\exp[-i\alpha(p^2+m'^2)] \times\exp\left[ie2p_\mu\int_0^\alpha d\alpha' A_\mu(x+2\alpha' p)\right]. \quad (2.7)$$

Taking the mass-shell Fourier-transformed version of Eq. (2.7) and performing two amputations upon G_2

then yields

$$G_2^{(E)}(\bar{p}_2'\bar{p}_2|A_\mu)\lim_{p_2'^2+m'^2\to 0,\ p_2^2+m'^2\to 0}(p_2'^2+m'^2)\times(p_2^2+m'^2)G_2^{(E)}(p_2'p_2|A_\mu)$$

$$=(2\pi)^{-4}\int dx_2' e^{-i(p_2'-p_2)x_2'}e2p_\mu A_\mu(x_2')\times\exp\left(ie2p_\mu\int_0^\infty d\alpha\,A_\mu(x_2'-2p\alpha)\right)$$

$$=(2\pi)^{-4}\int dx_2' e^{-i(p_2'-p_2)x_2'}\times\left[i\frac{\partial}{\partial\alpha}\exp\left(ie2p_\mu\int_\alpha^\infty d\alpha' A_\mu(x_2'-2p\alpha')\right)\right]_{\alpha=0}, \quad (2.8)$$

where the index (E) is introduced to indicate that p is now to be taken in its eikonal-approximation average, i.e., $p:=\frac{1}{2}(p_2'+p_2)$.

At this stage it is in order to compare our approach to the semiclassical derivation given in Ref. 3. While in Ref. 3 as well as in Ref. 1 the eikonal approximation is performed on either line of the colliding particles, we can improve the calculation by keeping only the charged scalar particle on the mass shell and subjecting it to the eikonal approximation. The fermion Green's function will be retained in its complete form. Needless to say, we can reproduce the results of Refs. 1 and 3 by simultaneously subjecting both propagators involved to the eikonal approximation.

Under those assumptions, we obtain for the scattering amplitude

$$T(x_1'x_1|\bar{p}_2'\bar{p}_2)=(2\pi)^4\exp\left(-i\frac{\delta}{\delta A_1}D_c\frac{\delta}{\delta A_2}\right)\times G_2^{(E)}(\bar{p}_2'\bar{p}_2|A_2)G_1(x_1'x_1|A_1)|_{A_1=0=A_2}. \quad (2.9)$$

Inserting the expression for $G_2^{(E)}$ into Eq. (2.9), the exponential can be carried out immediately to result in

$$T(x_1'x_1|\bar{p}_2'\bar{p}_2)=i\int d^4z\,e^{iqz}\left[\frac{\partial}{\partial\alpha}G_1\left(x_1'x_1\middle|e2p_\mu\int_\alpha^\infty d\alpha'\times D_c(z-2p\alpha'-\omega)\right)\right]_{\alpha=0}, \quad (2.10)$$

where $q:=p_2'-p_2$. The structure of Eq. (2.10) makes it obvious that particle m can now be considered as moving in an effective external potential created by the incoming particle m'. Looking at the functional argument of G_1 in the rest frame of particle m', we learn that this turns out to be precisely the Coulomb field.

[3] R. Torgerson, Phys. Rev. **143**, 1194 (1966).

III. INTERMEDIATE MASS STATES

Our final task is to introduce a spectral representation for the scattering amplitude and set up a relation to the result as expressed in Eq. (2.10). For this reason we write

$$\langle p_2+q|(\psi(x)\bar{\psi}(y))_+|p_2\rangle=e^{iqx}\int d\Delta\,e^{i\Delta(x-y)}G(\Delta), \quad (3.1)$$

where $q:=p_2'-p_2$. In order to compute the masses of the intermediate states which are defined by the decomposition of the left-hand side of Eq. (3.1) into a complete set of energy-momentum eigenstates, we rewrite

$$\langle p_2+q|\psi(x)\bar{\psi}(y)|p_2\rangle=\sum_{\Delta'}\langle p_2+q|\psi(x)|\Delta'+p_2+q\rangle\langle\Delta'+p_2|\bar{\psi}(y)|p_2\rangle$$
$$=\sum_{\Delta'}\langle p_2+q|\psi(0)|\Delta'+p_2+q\rangle\times e^{i\Delta'(x-y)}\langle\Delta'+p_2|\bar{\psi}(0)|p_2\rangle,$$

where the sum is taken over all fixed states Δ' and the causal arrangement is taken such that $x_0>y_0$. Then the mass spectrum of those states is defined via

$$(\Delta'+p_2)^2=-m_B^2. \quad (3.2)$$

Applying this spectral decomposition to our problem, first notice that Eq. (3.1) can be continued to yield

$$e^{iqx}\int d\Delta\,e^{i\Delta(x-y)}G(\Delta)=\frac{1}{i}T(x,y|\bar{p}_2'\bar{p}_2)=\int d^3z'e^{-i\mathbf{q}\cdot\mathbf{z}'}G_1\left(x-z',y-z'\middle|-\frac{p_\mu}{m'}\int_{-\infty}^{+\infty}d\alpha\,D_c(\omega-(p^\mu/m')\alpha)\right), \quad (3.3)$$

where the reduction to a three-dimensional integral has been achieved by a change of variables in Eq. (2.10):

$$z=z'+(p/m')\beta,\quad z'\cdot p/m'=0=q\cdot p$$

and only the connected part of G_1 has been taken into account. Looking at the right-hand side of (3.3), we observe that in p^μ/m', for $m'\to\infty$, only the time component survives, leaving us with $p^\mu/m'\to n^\mu=(\mathbf{0},1)$. So, indeed, we obtain for the functional argument of G_1

the Coulomb potential, since

$$\int_{-\infty}^{+\infty} dt\, D_c(x-x') = \frac{1}{4\pi}\frac{1}{R}\,.$$

Therefore (3.3) reduces to

$$\frac{1}{i}T(x,y|\bar{p}_2'\bar{p}_2) = e^{iqx}\int d^4\Delta\, e^{i\Delta(x-y)}G(\Delta) = \int d^3z\, e^{-iq\cdot z}G_{1\,\mathrm{Coul}}(\mathbf{x}-\mathbf{z},\,\mathbf{y}-\mathbf{z};\,x_0y_0)\,, \quad (3.4)$$

and inverting with respect to $G(\Delta)$ then yields the result that the poles Δ_0 of $G(\Delta)$ coincide with the energy spectrum as described by

$$[m+\gamma(\partial-ieA_{\mathrm{Coul}})]G_{1\,\mathrm{Coul}}(\mathbf{x},\mathbf{y};\,x_0-y_0) = \delta^3(\mathbf{x}-\mathbf{y})\delta(x_0-y_0). \quad (3.5)$$

This is the desired Dirac equation of particle m moving in a Coulomb potential generated by an infinitely heavy particle m'.

A more formal proof leading to the same answer starts by introducing the Fourier transform of $G_1(x,y)$ and performing a Lorentz transformation which takes the four-vector n_μ into p^μ/m': $L^{\mu\nu}(p/m')n_\nu = p^\mu/m'$. Then we find

$$G\left(x-z',\,y-z'\,\middle|\,-e(p^\mu/m')\int_{-\infty}^{\infty} d\alpha'\, D_c(\omega-(p^\mu/m')\alpha')\right) = L(p/m')\int d^4k d^4k'\,\delta(k_0-k_0')e^{ik\cdot L^{-1}(x-z')}e^{-ik'\cdot L^{-1}(y-z')} \times G_{\mathrm{Coul}}(k,k')L^{-1}(p/m)$$

and all the information needed to extract the energy spectrum is contained in

$$e^{iqx}\int d^4\Delta\, e^{i\Delta(x-y)}G(\Delta) = \int d^3z\, e^{-i\mathbf{q}\cdot\mathbf{z}}L(p/m')\int d^4k d^4k' \times\delta(k_0-k_0')e^{ik\cdot L^{-1}(x-z)}e^{-ik'\cdot L^{-1}(y-z)} \times G_{\mathrm{Coul}}(k,k')L^{-1}(p/m')\,.$$

Solving with respect to $G(\Delta)$ then yields

$$G(\Delta) = L\left(\frac{p}{m'}\right)\frac{m'}{|p_0|}G_{\mathrm{Coul}}\big((L^{-1})_\nu{}^k(\Delta-q)^\nu,(L^{-1})_\nu{}^0(\Delta-q)^\nu;\ (L^{-1})_\nu{}^k\Delta^\nu,(L^{-1})_\nu{}^0(\Delta-q)^\nu\big)L^{-1}(p/m')\,.$$

The quantity of interest is the Coulomb energy

$$\omega = (L^{-1})_\nu{}^0(\Delta-q)^\nu = -(1/m')(\Delta\cdot p)\,, \quad (3.6)$$

which is equal to Δ_0, for $m'\to\infty$, as shown by

$$p\cdot\Delta = p\Delta - (p^2+m'^2)^{1/2}\Delta_0 \xrightarrow[m'\to\infty]{} -m'\Delta_0\,.$$

Inserting for $\Delta'\cdot p_2$ the value found in (3.6), Eq. (3.2) then leads to

$$m_B = (m'+\Delta_0')\left[1-\left(\frac{\boldsymbol{\Delta}}{m'+\Delta_0'}\right)^2\right]^{1/2} \underset{m'\to\infty}{\cong} (m'+\Delta_0') = m'-\epsilon_n\,,$$

where $\Delta_0' = -\epsilon_n$ is the binding energy given by the system electron in a Coulomb field as defined by the Dirac equation (3.5).

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