effectively discards the transverse modes of the vector field and retains only the longitudinal ones.

V. SUMMARY AND CONCLUSIONS

We have shown that dynamical theory of currents of the kind suggested by Sugawara is equivalent to a canonical theory of massless scalar particles, provided that the currents are associated with nonlinear transformations of the field. These nonlinear transformations arise naturally in the context of spontaneous breakdown of symmetry, and it is interesting to note that in Lagrangian models of broken symmetry the Goldstone bosons fulfill the Sugawara criteria. This allows for a general procedure for obtaining canonical representations of Sugawara models for any group, and we have exhibited this mechanism specifically for a $SU(2)$ model. Finally, we have shown that if the Sugawara currents are coupled to gauge fields, the resulting theory is a massive Yang-Mills one.

We conclude that Sugawara models as formulated contain massless scalar particles and as such are unrealistic. We also conclude that any attempt to resolve the contradiction involved in associating a massive scalar particle with a Goldstone boson must fail, since this procedure removes the scalar particle altogether.[19]

ACKNOWLEDGMENT

We would like to thank Professor D. Lurié for many interesting and stimulating discussions.

[19] This phenomenon appears also in theories in which the symmetry-breaking scalar fields are not elementary dynamic variables but bilinear combinations of Fermi fields. Y. Freundlich and D. Lurié, Nucl. Phys. (to be published).

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Momentum-Space Behavior of Integrals in Nonpolynomial Lagrangian Theories

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Methods are developed for constructing momentum-space amplitudes corresponding to nonpolynomial nonderivative interactions of a real scalar field. The methods give rise to a supergraph technique and rules for writing down matrix elements very similar to Feynman techniques. The methods are not established rigorously; at several points the argument requires certain analytic properties of Feynman integrands which, though plausible, can only be demonstrated rigorously for the zero-mass case. Asymptotic behavior, both in spacelike and timelike directions, is discussed. Rough arguments are given that indicate that the singularity structure of the amplitudes is likely to be consistent with unitarity.

I. INTRODUCTION

IF it is to have any future, Lagrangian field theory must learn to cope with nonrenormalizable interactions. This becomes apparent when one examines what we currently believe are Lagrangians of physical interest.

1. These Lagrangians include the following.

(*a*) *Chiral $SU(2)\times SU(2)$ Lagrangians for strong interactions.* A typical example is Weinberg's Lagrangian for π mesons:

$$\mathfrak{L}=(\partial_\mu\phi)^2/(1+f\phi^2)^2.$$

(*b*) *Intermediate-boson-mediated weak Lagrangian.* An example is an intermediate neutral vector meson U_μ interacting with quarks Q. As is well known in Stückelberg's representation ($U_\mu=A_\mu+\kappa^{-1}\partial_\mu B$), $\mathfrak{L}_{\text{int}}$ can be written in the typical form

$$\mathfrak{L}_{\text{int}}=f\bar{Q}\gamma_\mu(1+\gamma_5)QA_\mu+m\bar{Q}(e^{i\gamma_5(f/\kappa)B}-1)Q.$$

(*c*) *The gravitational Lagrangian of Einstein* expressed in terms of the contravariant tensor $g^{\mu\nu}$

$$L=\kappa^{-2}(\sqrt{-g})g^{\mu\nu}(\Gamma_{\mu\rho}{}^{\lambda}\Gamma_{\nu\lambda}{}^{\rho}-\Gamma_{\mu\nu}{}^{\lambda}\Gamma_{\lambda\rho}{}^{\rho}),$$

where

$$\Gamma_{\mu\nu}{}^{\lambda}=\tfrac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}).$$

The covariant components $g_{\mu\nu}$ which enter the expression for $g=\det g_{\alpha\beta}$ are expressed as a ratio of two polynomials in $g^{\mu\nu}$.

The interaction Lagrangians in all these theories are typically of a nonpolynomial form in field variables. These Lagrangians can be expanded in power series of the type

$$\mathfrak{L}_{\text{int}}(\phi)=G\sum_n\frac{v(n)}{n!}(-\phi)^n. \tag{1.1}$$

(Here ϕ is a scalar field, and for simplicity we are ignoring derivatives.) The coefficients $v(n)$ are proportional to f^n, where f is a coupling constant.[1] All terms in such

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[1] In this paper we distinguish between the coupling constants G and f: G will be called the *major* coupling constant and f the *minor*. We shall be considering fixed order in G and all orders in f.

expansions with $n>4$ are nonrenormalizable. Thus, barring quantum electrodynamics, the rest of Lagrangian particle physics apparently needs a closer study with respect to nonrenormalizability.

2. Right from the very early days it was emphasized, particularly by Heisenberg, that the perturbation expansions of the S matrix in powers of $G^N f^r$ for the case of unrenormalizable theories suffer from two distinct (though related) difficulties.

(*a*) *Infinities*. The integrals in the theory become more and more infinite in each increasing order of perturbation. In each order one needs new counterterms containing *higher and higher derivatives of fields* if the conventional subtraction philosophy of renormalizable interactions is to be extended to these theories. (These infinite-order higher derivatives are likely to produce a *nonlocal* counter-term Lagrangian.)

(*b*) *Unacceptable high-energy behavior*. Even after a successful subtraction scheme has been carried out, the high-energy behavior (of the finite parts) of integrals is physically unacceptable. As external momenta become large, the dependence of these integrals on external momenta increases polynomially with the order of the approximation, unlike the case for renormalizable theories.

3. Of these two types of difficulties, the first—concerning the infinities of the integrals—has begun to be seriously investigated recently. Three types of approaches have been considered.

(*a*) *The conventional approach*, where the Feynman momentum-space integrals in each order $G^N f^r$ are considered as they stand and a consistent subtraction procedure defined.[2] It appears[3] that all three rigorous subtraction procedures used for renormalizable theories, i.e., (i) the Dyson-Salam method,[4] (ii) the Bogolubov-Parasiuk-Hepp method,[5] and (iii) the analytic renormalization method,[6] can be extended to nonrenormalizable theories. To our knowledge, the second technical problem of a systematic organization of the counterterms has not yet been examined for any of the theories, nor has the problem of physical interest posed by *finite* changes in the definition of renormalization constants. (Since there are an infinity of renormalization constants, such changes could reduce the predictive power of the theory to naught.)

[2] The subtractions do not affect whatever causality and unitarity properties the perturbation expansion may have.

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[4] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); Abdus Salam, *ibid.* **84**, 426 (1951). That this method carries through for nonrenormalizable theories has been shown in a set of basic papers by W. Zimmerman (unpublished).

[5] N. N. Bogolubov and O. S. Parasiuk, Acta Math. **97**, 227 (1957); K. Hepp, Commun. Math. Phys. **6**, 161 (1967).

[6] T. Gustafson, Arkiv. Mat. Astron. Fysik **34A**, No. 2 (1947); C. G. Bollini, J. J. Giambiagi, and A. Gonzalez Dominguez, Nuovo Cimento **31**, 550 (1964); E. R. Speer, J. Math. Phys. **9**, 1404 (1968).

(*b*) *The x-space approach of Efimov and Fradkin*[7,8] for theories with rational nonpolynomial Lagrangians. Formally one can write the Nth order approximation in the major coupling constant G in x space to a typical amplitude in the form of a divergent series:

$$F^N(x_1,x_2,\ldots)=G^N \sum_{p,q\cdots} (f)^{p+q+\cdots} a_{pq\cdots} \times \Delta_F{}^p(x_1-x_2)\Delta_F{}^q(x_2-x_3)\cdots. \quad (1.2)$$

Efimov and Fradkin have described an elegant technique of carrying out Borel sums of such series *in the minor coupling parameter f*. These sums can be examined in the ultraviolet limit $(x_1-x_2)^2\to 0$, $(x_3-x_4)^2\to 0$, ... The important result of their investigation (extended in Ref. 8) is that if the Dyson index D of the rational Lagrangians is less than or equal to *four*—i.e., the same as that for renormalizable theories—[the Dyson index D is defined by the limit $L(\phi)_{\phi\to\infty}=\phi^D$], only a few types of Borel sums exhibit any ultraviolet infinities—again like the case for renormalizable theories. In particular, if the Dyson index D is less than two, none of the Borel sums (including those representing vacuum-to-vacuum transitions) is ultraviolet infinite. Thus if in some sense the Borel sums represent the physical amplitudes, all theories with $D<2$ (and for these theories the Lagrangian *must* be nonpolynomial) are *super-renormalizable*.[9]

This is a beautiful result. The important question to decide is to what extent the Borel sums represent the physical amplitudes. Do the p-space Fourier transforms of these x-space functions possess the requisite analyticity and unitarity properties? And, finally, is the high-energy behavior of these p-space Fourier transforms polynomially bounded, as it should be if physical amplitudes are being represented?

A detailed study of the Fourier transform of the Efimov-Fradkin two-point function in *second order of the major coupling constant G* has been made by Lee and Zumino,[10] who have concluded (with Efimov) that (i) the corresponding Borel sum does possess the requisite analyticity and unitarity properties, (ii) it is not polynomially bounded, and (iii) the well-known lack of uniqueness of Borel sums of divergent series is reflected in an arbitrariness of the amplitudes up to an entire function.

[7] G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **44**, 2107 (1963) [Soviet Phys. JETP **17**, 1417 (1963)]; E. S. Fradkin, Nucl. Phys. **49**, 624 (1963); H. M. Fried, Nuovo Cimento **52A**, 1333 (1967); S. Okubo, Progr. Theoret. Phys. (Kyoto) **11**, 80 (1954) (this elegant paper was unknown to the authors until its existence was pointed out to them by Professor H. Lehmann); R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955).

[8] R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969). This paper will be referred to as I.

[9] As shown in Ref. 8, for the chiral π-meson Lagrangian the Dyson index is zero; for Einstein's gravity theory it equals *unity* [see R. Delbourgo, Abdus Salam, and J. Strathdee, Nuovo Cimento Letters **2**, 354 (1969)].

[10] B. W. Lee and B. Zumino Nucl. Phys. **B13**, 671 (1969).

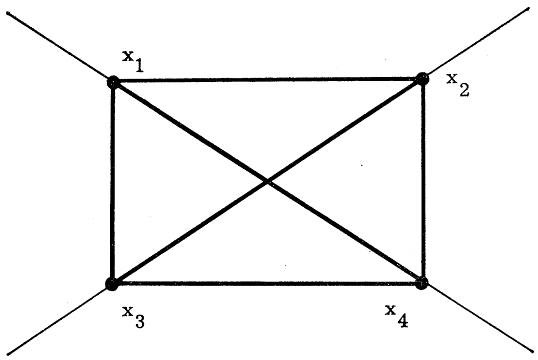

FIG. 1. Four-point supergraph. Heavy lines represent superpropagators each of which corresponds to the collection of functions $\Delta_F{}^n$, $n=1, 2, 3\ldots.$

(c) *The p-space method.* Since it is the momentum-space Fourier transforms of the amplitude (1.2) which are the quantities of primary physical interest, it is valuable to have a summation method which works directly within p space. The present paper is devoted to the development of such a method, following a procedure first discussed in this context by Volkov[11] and which in its essentials goes back to a discussion (in the appropriate region of x and n) of the Fourier transform of $(-1/x^2)^n$ by Gel'fand and Shilov.[12] In particular we show the following.

(i) The amplitudes appear to possess the analyticity structure associated with the unitarity requirements.

(ii) The method immediately gives the asymptotic behavior for large values of external momenta; and, in particular, for the two-point amplitude in the second order in G^2 studied by Lee and Zumino, we reproduce their result very simply.

(iii) The discussion of ultraviolet infinities of Borel sums in x space is closely parallelled by a similar one in p space.

(iv) One can develop a graph technique of Feynman-like diagrams with *superlines* representing *superpropagators* $[\Delta_F(x)]^n$ replacing normal lines corresponding to Feynman propagators $\Delta_F(x)$. In p space, the closed-loop integrations for supergraphs can be performed with the help of Feynman's auxiliary parameters in exactly the same fashion as for conventional polynomial Lagrangians. Insofar as there is (essentially) just *one* supergraph in each order G^N, the topological analysis of supergraphs is simpler than Feynman diagrams for polynomial Lagrangians.

4. It would appear from the above that for nonpolynomial interaction Lagrangians with index $D<2$, one can construct amplitudes with no ultraviolet infinities and which (if one can extrapolate from the limited experience so far) are likely to satisfy the correct analyticity and unitarity requirements (unitarity verified to each order in the major coupling constant G). There are two remaining problems:

(*a*) *The arbitrariness* in the amplitudes which the Borel-summation method in x space allows or, as we shall see, its weaker analog, which still exists when the p-space method is used. The problem is analogous to the problem of arbitrariness of finite renormalization constants in renormalizable theories.

(*b*) *The more serious problem* on nonpolynomial bounded high-energy behavior of the amplitudes. We believe that any inference in this respect on the basis of order-by-order calculations in powers of G is likely to be misleading, and a final verdict on the true asymptotic behavior of these theories can only be given after a summation of the series in the major coupling constant G has been performed. For renormalizable theories, as is well known, this summation has been carried out for certain sequences of graphs for the four-point function and for a number of production amplitudes. The result is the emergence of Regge behavior for large values of energy, unsuspected if one had only considered individual terms of the perturbation expansion. We believe, on the basis of certain indicative considerations, that a similar (drastic) change in the high-energy behavior also occurs in the present theories when the summation in G is carried out. The p-space method is extremely convenient for summing the supergraphs insofar as the analytical expressions for the supergraphs resemble those for conventional polynomial Lagrangian theories.

5. To make the plan of the present paper clear and to bring out the parallel sets of ideas involved in the x-space and the p-space methods, we set down here a brief and nonrigorous summary of the paper.

(*A*) *Supergraphs.* Consider

$$\mathcal{L}_{\text{int}}(\phi)=G\sum_{n=0}^{\infty}\frac{v(n)}{n!}(-\phi)^n$$

[$v(n)$ contains the minor coupling parameter f^n. The factor $(-)^n$ is included for later convenience.]

It is easy to verify that the G^N contribution to an amplitude $F(x_1,\ldots,x_N)$ with E external line can be written as a sum of contributions from a set of *supergraphs* constructed as follows.

(a) Take N points $x_1, x_2, \ldots, x_N$.

(b) Join all points pair-wise with just one superline joining two distinct points (x_i,x_j); associate with this line a positive integer n_{ij}.

(c) For each line write the factor

$$(1/n_{ij}!)[\Delta_F(x_i-x_j)]^{n_{ij}}.$$

(d) For each point x_i write a vertex factor

$$v(\textstyle\sum_j n_{ij}+m_i).$$

Here m_i is the number of external lines impinging on the point x_i.

[11] M. K. Volkov, Ann. Phys. (N.Y.) 49, 202 (1968).

[12] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I.

(e) The contribution of the supergraph to the amplitude equals

$$F_{m_1 m_2\cdots}(x_1,\ldots,x_N)$$
$$=G^N \sum_{n_{ij}} \prod_i v(\sum_j n_{ij}+m_i) \prod_{i<j} \frac{[\Delta_F(x_i-x_j)]^{n_{ij}}}{n_{ij}!}. \quad (1.3)$$

(f) To get the total contribution in order G^N, sum over all partitions, $[m_1,\ldots,m_N]$, of the external lines with m_i lines at the ith vertex such that

$$\sum m_i = E.$$

(g) In the above set of rules we omit all tadpole contributions (lines joining a point x_i to itself). This is justified if we consider instead of (1.1) a suitably Wick-ordered interaction. Figures 1 and 2 show a typical supergraph and a superline.

(h) It is clear that the Green's function

$$F_{m_1,m_2,m_3\cdots}(x_1,\ldots,x_N),$$

with m_1, m_2 nonzero, is simply related to

$$F_{m_1-1,m_2-1,m_3,\cdots}.$$

For example,

$$\Delta(x_1-x_2)F_{1,1,0,0,\cdots} = \left.\frac{\partial F_{0,0,0,\cdots}(\lambda)}{\partial\lambda}\right|_{\lambda=1},$$

where $F(\lambda)$ on the right-hand side of (1.4) is obtained from (1.3) by replacing $\Delta(x_1-x_2)$ by $\lambda\Delta(x_1-x_2)$.

(*B*) *To illustrate* the x-space and p-space techniques, consider a simple example with $\mathcal{L}_{\text{int}}$:

$$V(\phi) = \frac{G}{1+f\phi} = G\sum_{n=0}^{\infty}(-f\phi)^n. \quad (1.4)$$

Here $v(n)=f^n n!$

(*a*) *The formal series expansion for amplitudes.* Formally an expectation value like

$$F(\Delta) = F_{00}(x_1,x_2) = \langle V(\phi(x_1)), V(\phi(x_2))\rangle$$

equals the *divergent* series:

$$= G^2 \sum_{n=0}^{\infty} n! f^{2n} \Delta_F{}^n(x_1-x_2). \quad (1.5)$$

We are interested in giving a meaning to this divergent series such that the Fourier transform

$$\tilde{F}(p^2) = \int F(\Delta) e^{ipx} d^4x \quad (1.6)$$

possesses correct analyticity and unitarity properties.

(*b*) *The Euclidicity postulate.* To do this consider the Symanzik region in p space ($p^2<0$). (When more than one external momentum p_i is involved, the Symanzik region is the region for which $p_i{}^2 \leqslant 0$, $p_i p_j \leqslant 0$.) Following Efimov, we can define the integral (1.6) by making a Wick rotation $x_0 \to ix_4$. For this region in p space, one therefore needs to consider $\Delta(x)$ for Euclidean x space only. [For a zero-mass field $\Delta(x) = -1/4\pi^2 x^2$ is real and positive.] For p-space regions outside the Symanzik region, we must appropriately analytically continue (1.6). [It cannot be emphasized strongly enough that for divergent series of the type (1.5), one is not starting by "proving" the validity of the Wick rotation. Rather, Euclidicity is a basic postulate—part of the process of definining the theory. One *accepts* it for the Symanzik region; outside this region one makes an analytic continuation.]

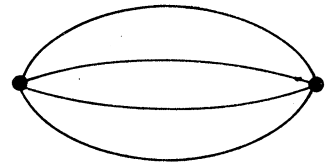

FIG. 2. Typical graph from the collection which is represented by a heavy line.

(*c*) *Borel summation.* To give meaning to the divergent sum $F(\Delta)$, use Borel transforms and write

$$F(\Delta) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\zeta}(f^2\zeta\Delta)^n. \quad (1.7)$$

Here we have used

$$n! = \int_0^{\infty} \zeta^n e^{-\zeta} d\zeta.$$

(*d*) *The x-space method.* The x-space method consists of inverting integration and summation in (1.7) and writing it as

$$F(\Delta) = \int_0^{\infty} d\zeta\, e^{-\zeta}(1-\zeta f^2\Delta)^{-1}. \quad (1.8)$$

The expression (1.8) defines the amplitude $F(\Delta)$. At this stage we encounter our first problem in the x-space method; the integrand has a pole on the integration path at

$$\zeta = \frac{1}{f^2\Delta} = \frac{4\pi^2 r^2}{f^2} \quad \text{for the case } m=0,\ r^2=\mathbf{x}^2+x_4{}^2.$$

We must define how to go around this singularity.

One obvious answer is: Take the principal value (P.V.). This is because $F(\Delta)$ in the Symanzik region is a sum of real terms. The P.V. prescription for the integral representation (1.8) of $F(\Delta)$ will guarantee this. Lee and Zumino show that this is essentially the correct prescription, barring an arbitrariness (to be specified later) associated with functions like $\exp[1/(f^2\Delta)]$

which possess an identically vanishing asymptotic expansion about the point $\Delta=0$, and which can be added to $f(\Delta)$ without affecting its representation in the form $\sum_{n=0}^{\infty}n!(f^2\Delta)^n$.

(*e*) One may now compute the Fourier transform, using $r=(-x^2)^{1/2}$,

$$\tilde{F}(p^2)=\int dx\, e^{ipx}\text{P.V.}\int_0^\infty d\zeta\, e^{-\zeta}(1-\zeta f^2\Delta)^{-1}$$

$$=\frac{4\pi^2}{(-p^2)^{1/2}}\int_0^\infty dr\, r^2 J_1((-p^2)^{1/2}r)$$

$$\times\text{P.V.}\int_0^\infty d\zeta\, e^{-\zeta}(1-f^2\zeta\Delta)^{-1} \quad (1.9)$$

for $p^2<0$ and continue direct to $0<p^2<m^2$. Inverting the order of integrations, we have

$$\tilde{F}(p^2)=\frac{4\pi^2}{(-p^2)^{1/2}}\int_0^\infty d\zeta\, e^{-\zeta}\int_0^\infty dt\, r^4 J_1((-p^2)^{1/2}r)$$
$$\times[r^2-\zeta(f/2\pi)^2]^{-1}.$$

This integral can be explicitly evaluated, and a continuation to timelike values of p^2 carried out, to demonstrate that $\tilde{F}(p^2)$ possesses the correct analyticity structure in the p^2 plane. The asymptotic behavior of $\tilde{F}(p^2)$ is

$$\tilde{F}(s)\to 1/(f^2s)^3,\quad s\to-\infty$$
$$\to\pm i\pi\exp(f^2s),\quad s\to+\infty\pm i0, \quad (1.10)$$

where $s=p^2$.

(*f*) *The p-space method.* Our procedure is different. It depends on Volkov's observation of the power of the Gel'fand-Shilov investigation of the Fourier transform of the generalized function $[\Delta(m=0)]^z=r^{-2z}$ in the range $0<\text{Re}z<2$. [In Sec. II we consider the case $m\neq 0$. We have no exact expression for this case similar to (1.11) below, but the general considerations are parallel to those treated here.]

The crucial formula is

$$\Delta^z(x)=\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{(-p^2)^{z-2}\pi(4\pi)^{2-2z}}{\sin\pi z\,\Gamma(z)\Gamma(z-1)},$$
$$0<\text{Re}z<2. \quad (1.11)$$

To use this formula go back to the Borel sum (1.7) and employ a *Sommerfeld-Watson transformation* to convert the series into an integral of the form

$$F(\Delta)=\tfrac{1}{2}i\int_\Gamma\frac{dz}{\sin\pi z}\int d\zeta\, e^{-\zeta}(-\zeta f^2\Delta)^z, \quad (1.12)$$

with the contour Γ enclosing the positive real axis in the z plane. (The conditions for validity of employing this transformation are discussed in Sec. III.) Straighten the contour to lie along the imaginary axis with Rez constrained to lie in the range $0<\text{Re}z<2$. Using the Gel'fand-Shilov formula to take the Fourier transform, we obtain

$$\tilde{F}(p^2)=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{(-f^2)^z(-p^2)^{z-2}\Gamma(z+1)}{\sin\pi z\,\Gamma(z)\Gamma(z-1)}$$
$$+(2\pi)^4\delta(p), \quad (1.13)$$

where $0<\alpha<2$. [The term $\delta(p)$ corresponds to a graph which contains no internal line.]

In the above passage from (1.5) to (1.13), we have changed orders of summation and integration repeatedly. The justification is provided in Sec. III. Here we are only concerned with a rapid exposition of the p-space method whose chief ingredients are the Sommerfeld-Watson transformation (1.12) and the Fourier transformation (1.11).

(*g*) Formula (1.13) is our master formula. By closing the contour along the left, one can immediately obtain the asymptotic behavior of $\tilde{F}(p^2)$ for $p^2\to-\infty$ [and, in particular, the Lee-Zumino result (1.10)]. As in Regge-pole theory, the rightmost pole of the integrand gives the leading contribution to the asymptotic behavior.

(*h*) The principal value ambiguity of the x-space method noted in (d), which does not seem to arise explicitly in this treatment, has a weaker counterpart when we take into account the appearance of the negative sign in front of Δ in $(-\Delta)^z$ in the Sommerfeld-Watson transform. To see this more explicitly, introduce a multiplier λ in front of Δ; thus,

$$F(\lambda\Delta)=\tfrac{1}{2}i\int\frac{dz}{\sin\pi z}\int d\zeta\, e^{-\zeta}(-\zeta\lambda f^2\Delta)^z. \quad (1.14)$$

We must interpret the limit $\lambda\to+1$ by a *real* average of the values $(-\lambda)^z=e^{i\pi z}$ and $(-\lambda)^z=e^{i\pi z}$, obtaining finally

$$F(\Delta)=\int dz\left(\frac{1}{\tan\pi z}+b\right)\Gamma(z+1)(f^2\Delta)^z \quad (1.15)$$

with b an arbitrary real constant. The ambiguity introduced by $(-1)^z=e^{i\pi z}=e^{-i\pi z}$ we shall call the *signature ambiguity* of superpropagators. (Note that this method admits of an arbitrariness only up to a constant, and not up to an entire function of p^2, as in the x-space method.)

(*j*) In Paper I we showed that a superficial count indicates that the number of distinct types of ultraviolet infinities depends on the Dyson index D $[\lim_{\phi\to\infty}V(\phi)=\phi^D]$ of the Lagrangian. This number is finite if $D\leqslant 4$. The result was proved by considering $x\to 0$ (x spacelike) behavior of Borel sums (1.7). For example, explicitly when $V(\phi)=G/(1+f\phi)$, with

$D=-1$, $F(\Delta)$ is given by

$$F(\Delta)=G^2\,\mathrm{P.V.}\int_0^\infty d\zeta\, e^{-\zeta}x^2\left[x^2-\zeta\left(\frac{f}{2\pi}\right)^2\right]^{-1}$$

for zero-mass fields ϕ. This expression is finite in the limit $x\to 0$. For the interaction Lagrangian $V(\phi)=\phi^5/(1+f\phi)$ with $D=4$, however, we recover the ultraviolet infinities since the corresponding expression for $F(\Delta)$ is

$$F(\Delta)=\Delta^5\int d\zeta\, e^{-z}(1-f^2\zeta\Delta)^{-1}.$$

The question arises: Where in the p-space method is there an indication of a Dyson index? The answer, as we shall see in Sec. III, is that it is the Gel'fand requirement $0<\mathrm{Re}z<2$ for the unique definition of the Fourier transform of $\Delta^z(x)$ which forces us to distinguish between Lagrangians like $V(\phi)=1/(1+f\phi)$ and $V(\phi)=\phi^5/(1+f\phi)$. In order to give a precise meaning to $F(\Delta)$ for the latter, we are constrained to write it in the form

$$F(\Delta)=\sum_{n=2}^{4} n!\Delta^n+\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty} dz\,\frac{(-f^2\zeta\Delta)^z}{\sin\pi z}e^{-z}d\zeta,\ (1<\alpha<2).$$

The terms which appear in the sum $\sum_{n=2}{}^4 n!\Delta^n$ are just the ones which give rise to ultraviolet infinities.

(*k*) *Higher orders*. The great beauty of the p-space method lies in the similarity of the p-space expressions for supergraphs and normal Feynman diagrams.

One can introduce Feynman's auxiliary parameters and carry out the loop integrations. As we shall see below, the result is an elegant expression for the supergraph contribution as a weighted average integral of contributions of conventional graphs. The utility of such an expression is twofold. (i) The sums of supergraphs in different orders of G closely resemble the sums for conventional graphs, and the methods previously discussed by Polkinghorne, Federbush,[13] and others for carrying through the summation can be taken over. (ii) The discontinuity formulas of Cutkosky—and the proof of the unitarity relations using such formulas—follow the conventional lines.

For the zero-mass case, the integral expression for the Nth-order supergraph is the following.

Associate with each superline a four-momentum vector q_{ij}. The Sommerfeld-Watson transform of (1.3) in p space equals

$$\tilde{F}(p)=G^N\left(\prod_{i<j}\int dz_{ij}\rho(z_{ij})\int d^4q_{ij}(-q_{ij}^2)^{z_{ij}-2}\right)\times\prod_k\delta^4(p_k+\sum_{l\neq k}q_{kl}).\quad(1.16)$$

Here $\rho(z_{ij})$ is the product of the vertex factors $v(\sum_{j\neq i}z_{ij}+m_i)$, the factors $1/\sin\pi z_{ij}$, and the factors $1/[\sin\pi z_{ij}\Gamma(z_{ij})\Gamma(z_{ij}-1)]$ for each superline. The p_i's are the momenta carried by the external lines at the ith vertex, and the δ functions express conservation of energy and momentum.

Introduce Feynman's auxiliary parameters, using the integral representation[14]

$$(-q^2)^{z-2}=\frac{1}{\pi\Gamma(2-z)}\int_0^\infty d\alpha\,\alpha^{1-z}e^{\alpha q^2}.\quad(1.17)$$

One may now carry through the d^4q integrations in

$$I(p,\alpha_{ij})=\int(\exp\sum\alpha_{ij}q_{ij}^2)\prod_k\delta^4(p_k+\sum_{l\neq k}q_{kl})]^N\prod_{i<j}d^4q_{ij}.\quad(1.18)$$

The result is identical to the case as if we were dealing with *normal* Feynman graphs with $F=\frac{1}{2}N(N-1)$ internal lines rather than supergraphs. [This is because $I(p_i,\alpha_{ij})$ is not z_{ij} dependent.] Such normal graphs we shall call *skeleton graphs*. The evaluation of the functions $I(p_i,\alpha_{ij})$ for the skeleton graphs can easily be carried through using the methods of Chisholm[15]; the final expression for the amplitude $\tilde{F}(p_i)$ reads

$$\tilde{F}(p)=\prod_{i<j}\int dz_{ij}\rho'(z_{ij})\int d\alpha_{ij}\alpha_{ij}{}^{1-z_{ij}}I(p,\alpha_{ij}),\quad(1.19)$$

where ρ' differs from ρ by the factors

$$\prod_{i<j}\frac{1}{\pi\Gamma(2-z_{ij})}.$$

The result for the N-point function evaluated in order G^N can therefore be stated thus: Draw a normal Feynman graph with internal lines joining all the N points pair wise. We shall call such graphs skeleton graphs. Introduce Feynman parameters; the result of performing loop integrations in skeleton diagrams is the standard Chisholm expression $I(p,\alpha_{ij})$. Multiply this by the factors $(\alpha_{ij})^{1-z_{ij}}$ and the weight function $\rho'(z_{ij})$; integrate over Feynman parameters α_{ij} and the Sommerfeld-Watson parameters z_{ij}. One obtains the supergraph contribution.

(*m*) *Finiteness of the supergraphs*. One may examine the supergraph integrals for ultraviolet infinities. It is easy to see that a superficial power count would indicate that for an N-point function, with $F=\frac{1}{2}N(N-1)$ internal lines and $l=F-N+1$ loop momenta, the supergraphs have no ultraviolet infinities provided each superpropagator contributes a factor falling like $(q^2)^{-2}$. It is crucial to remember that with our Euclidicity

[13] J. C. Polkinghorne, J. Math. Phys. 4, 503 (1963); P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) 22, 263 (1963); 22, 299 (1963).

[14] For $z=1$ we recover Feynman's formula for normal propagators.

[15] J. R. S. Chisholm, Proc. Cambridge Phil. Soc. 48, 300 (1952).

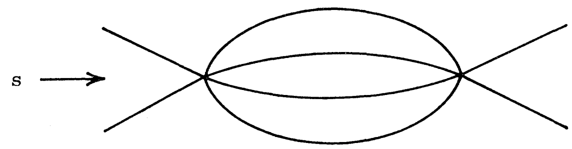

FIG. 3. Contribution to the s-wave scattering amplitude, $\tilde{F}(s) \sim \exp[(\text{const})s]$, $s>0$.

postulate it is the asymptotic behavior for spacelike q^2 for the superpropagators which is relevant here. This also leads us to stress once again that it would be a mistake to evaluate supergraphs in each order in the major coupling parameter G^N for the Symanzik region, to continue the external momenta to the physical region and *then* sum the series in G^N. *One must sum in G^N first* (as indeed has been done for the series in the minor coupling constant f) *and then continue the sum to the physical region.*

(n) As stated earlier, the G^2 approximation to the two-point amplitude in a typical nonpolynomial theory behaves asymptotically, as illustrated in Figs. 3 and 4. Thus, to order G^2, a simple evaluation for form factors (Fig. 4) immediately yields physically sensible results[16] $[\tilde{F}(t) \approx G^2/t^3]$, while the same approximation in the timelike region $s>0$ gives, for the four-particle scattering amplitude, a physically unacceptable behavior $[\tilde{F}(s) \propto G^2 e^{\alpha s}]$. It is clear that before rejecting nonrenormalizable theories on the grounds of unacceptability of their predictions in the lowest-order calculation in the timelike region of external momenta, one must first carry out a summation of a chain of diagrams. This crucial problem is being studied.

II. SUPERPROPAGATOR

As we have been stressing in Sec. I, the basic generalized function, in terms of which all the Green's functions are ultimately to be expressed, is Δ^z, where z denotes a complex number in the strip $0<\text{Re}z<2$, and Δ is the usual propagator for a free scalar field of mass m,

$$\langle T(\phi(x)\phi(0))\rangle = \Delta(x^2 - i0, m^2). \tag{2.1}$$

The function $\Delta(x^2,m^2)$ is analytic in the x^2 plane cut from 0 to $+\infty$. Explicitly,

$$\Delta(x^2,m^2) = mK_1(m\sqrt{(-x^2)})/4\pi^2\sqrt{(-x^2)}, \tag{2.2}$$

where K_1 denotes the modified Hankel (or Macdonald) function. The function Δ is real and positive on the negative real axis. It has no zeros in the finite x^2 plane, and its behavior near $x^2=0$ and $x^2=\infty$ is given by

$$\begin{aligned}\Delta(x^2,m^2) &\sim -1/x^2, \quad x^2 \to 0 \\ &\sim (-x^2)^{-3/4} e^{-m\sqrt{(-x^2)}}, \quad |x^2| \to \infty.\end{aligned} \tag{2.3}$$

The generalized function Δ^z, which we shall call the superpropagator, is well defined provided one can find a space of test functions $f(x)$ over which the integral

$$(\Delta^z, f) = \int d^4x\, \Delta^z f(x) \tag{2.4}$$

is convergent and satisfies the appropriate continuity conditions. This integral can certainly be defined for $f(x) \in \mathfrak{D}$, the space of infinitely differentiable functions with bounded support, provided z lies in the strip $0<\text{Re}z<2$. (Presumably it can be extended to larger spaces but we have not examined this problem.) Following the standard procedure[12] for defining a generalized function that corresponds to an ordinary function with an algebraic singularity, we can define (Δ^z,f) outside the strip $0<\text{Re}z<2$ by means of analytic continuation. The result is an analytic function of z with simple poles at the integers $z=2, 3, 4, \ldots$.

The Fourier transform $D(p^2,z)$ of the superpropagator must, like the latter, be an analytic function of z. It is defined on the segment $\text{Im}z=0$, $0<\text{Re}z<2$ by the classical integral

$$D(p^2,z) = \frac{1}{i}\int d^4x\, e^{ipx}\Delta^z, \tag{2.5}$$

which converges absolutely for $p^2<0$. In fact, one can perform a Wick rotation of the x_0 contour and replace the Minkowskian integral (2.5) by an equivalent Euclidean one which reduces to the form

$$D(p^2,z) = \frac{4\pi^2}{(-p^2)^{1/2}}\int_0^\infty dr\, r^2 J_1((-p^2)^{1/2}r)\Delta(-r^2, m^2)^z \tag{2.6}$$

after performing the angular integrations. It is clear that (2.6) converges for a wider range of z than does (2.5): $0<\text{Re}z<2$, $-\infty<\text{Im}z<\infty$. The analytic continuation of $D(p^2,z)$ outside this strip will be considered below. It will be shown that like the functional (Δ^z,f), it has poles at the integers $z=2, 3, 4, \ldots$.

For the zero-mass case we can express the integral (2.6) in terms of elementary functions and so perform the analytic continuation explicitly. The result is

$$D_0(p^2,z) = \pi(4\pi)^{2-2z}\frac{(-p^2)^{z-2}}{\sin\pi z\,\Gamma(z)\Gamma(z-1)}, \tag{2.7}$$

which clearly exhibits the poles at $z=2, 3, 4, \ldots$. (It shows in addition the rather unexpected feature of *zeros* at $z=0, -1, -2, \ldots$. We have not yet been able

[16] That a typical form factor falls so fast is a welcome result. It is perhaps not a surprising result, since it has already been pointed out in the literature that falling form factors are most likely to be consequences of the existence of multiparticle intermediate states. See, for example, J. S. Ball and F. Zachariasen, Phys. Rev. **170**, 1541 (1968); D. Amati, L. Caneschi, and R. Jengo, Nuovo Cimento **98**, 783 (1968); A. O. Barut and H. Kleinert, Phys. Rev. **161**, 1464 (1967); C. Fronsdal, *ibid.* **171**, 1811 (1968).

to prove that this is true also of the massive superpropagator.)

The asymptotic behavior of the superpropagator both in p^2 and in z is of particular importance. For $|p^2|\to\infty$ with z fixed, we shall assume that $D(p^2,z)$ can be approximated by $D_0(p^2,z)$. For $|z|\to\infty$ with p^2 fixed, the situation is less clear. The behavior of the zero-mass superpropagator

$$D_0\sim(-p^2)^{z-2}|-z|^{-2\mathrm{Re}z-2},\quad |\arg z|<\pi \tag{2.8}$$

may or may not provide a useful guide to the massive case.

Consider the structure of $D(p^2,z)$ in the finite z plane with $p^2<0$. The integral representation (2.6) is valid only in the strip $0<\mathrm{Re}z<2$. In order to continue to the right of $\mathrm{Re}z=2$, we must modify the behavior of the integrand of (2.6) at $r=0$. We propose to subtract and add the first N terms of a Maclaurin expansion of J_1. Indeed, if we write

$$\frac{J_1((-p^2)^{1/2}r)}{(-p^2)^{1/2}r}=\frac{1}{2}\sum_{k=0}^{N-1}\frac{(\frac{1}{4}p^2r^2)^k}{k!(k+1)!}+R_N(p^2r^2), \tag{2.9}$$

then (2.6) takes the form

$$D(p^2,z)=2\pi^2\sum_{k=0}^{N-1}\frac{(\frac{1}{4}p^2)^k}{k!(k+1)!}\int_0^\infty dr\, r^{2k+3}\Delta(-r^2,m^2)^z$$
$$+4\pi^2\int_0^\infty dr\, r^3R_N(p^2r^2)\Delta(-r^2,m^2)^z. \tag{2.10}$$

The term involving R_N is easily shown to converge in the extended strip

$$0<\mathrm{Re}z<N+2 \tag{2.11}$$

since near $r=0$ we have $R_N(p^2r^2)\sim r^{2N}$. The other terms in (2.10) will of course exhibit poles when continued out of the strip $0<\mathrm{Re}z<2$. (This can be shown by the method of Gel'fand and Shilov.) The important point here is the fact that the residues of the poles at $z=2,3,4$ are *polynomials* in p^2. (This is simply a reflection of the well-known fact that the ultraviolet divergences manifest themselves in the coefficients of a polynomial.) By increasing N indefinitely, we can thus prove that $D(p^2,z)$ with $p^2<0$ is analytic in the half-plane $\mathrm{Re}z>0$ except at the points $z=2,3,4,\ldots$ where it has simple poles, the residues of which are polynomials in p^2 (of order $z-2$). The structure in the half-plane $\mathrm{Re}z<0$ is more difficult to unravel and we have not attempted this.

Consider now the structure of $D(p^2,z)$ in the p^2 plane. It is trivial but helpful to continue into the strip $0<\mathrm{Re}(p^2)^{1/2}<m\,\mathrm{Re}z$ by expressing the integral (2.6) in the modified form

$$D(p^2,z)=\frac{4\pi^2}{(p^2)^{1/2}}\int_0^\infty dr\, r^2I_1((p^2)^{1/2}r)\Delta(-r^2,m^2)^z. \tag{2.12}$$

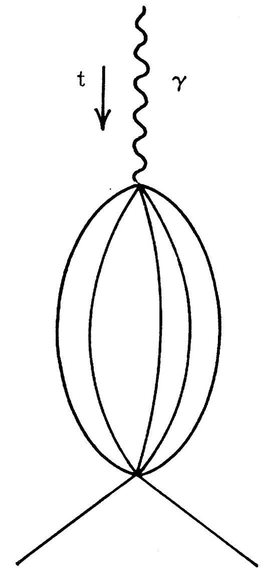

FIG. 4. Contribution to the electromagnetic form factors $\tilde{F}(t)\sim 1/t^3$, $t<0$.

It is the convergence or lack of it at the upper limit which controls the analyticity in p^2. We shall be able to make analytic continuations by displacing the r contour in (2.12). The integrand has no singularities in the finite plane, and for large values of $|r|$ it can be approximated by

$$r^{\frac{3}{2}(1-z)}e^{-(mz-\sqrt{p^2})r},$$

assuming $\mathrm{Re}(p^2)^{1/2}>0$. Let us keep z fixed in the strip $1<\mathrm{Re}z<2$. It is clear that we can rotate the r contour through the angle θ without affecting the value of the integral, provided we maintain the condition

$$\mathrm{Re}[(mz-\sqrt{p^2})e^{i\theta}]>0,$$

or, otherwise expressed,

$$|\arg(mz-\sqrt{p^2})+\theta|<\tfrac{1}{2}\pi. \tag{2.13}$$

This follows from the absence of singularities of Δ^z in the r plane apart from the branch point at $r=0$ where the convergence does not depend on $\arg(r)$. Starting with the original contour ,$\theta=0$, we have analyticity in the half-plane

(I) $-\frac{1}{2}\pi<\arg(mz-\sqrt{p^2})<\frac{1}{2}\pi$.

Increasing θ continuously to $+\frac{1}{2}\pi$ rotates the convergence domain into

(II) $-\pi<\arg(mz-\sqrt{p^2})<0$,

while decreasing θ to $-\frac{1}{2}\pi$ rotates the domain into

(III) $0<\arg(mz-\sqrt{p^2})<\pi$.

The regions (I)–(III) so obtained can be pictured in the plane of $\sqrt{p^2}$ as in Fig. 5. The dashed lines indicate boundaries between the regions. Thus it appears that the integral (2.12) defines a function which is analytic in the half-plane $\mathrm{Re}(\sqrt{p^2})>0$ except at the point $\sqrt{p^2}$

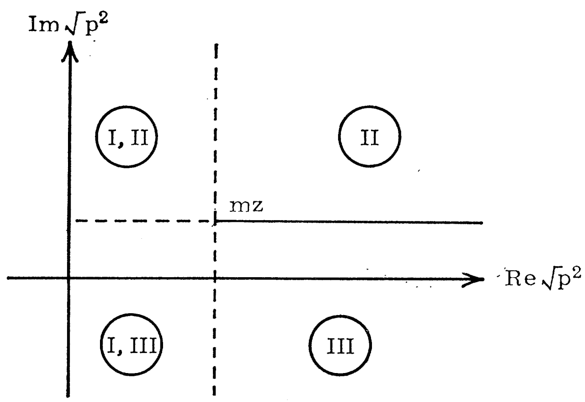

FIG. 5. Structure of the superpropagator in the complex $\sqrt{p^2}$ plane. The solid line represents a branch cut and the dashed lines mark out regions in which the superpropagator is represented by distinct contour integrals.

$=mz$, where it presumably has a branch point. The discontinuity across the branch cut is given by $D_{\rm II}-D_{\rm III}$, where $D_{\rm II}$ and $D_{\rm III}$ are defined by the contours with $\theta=\frac{1}{2}\pi-0$ and $\theta=-\frac{1}{2}\pi+0$, respectively. Equivalently,

$$D_{\rm II}(p^2,z)=\frac{4\pi^2}{(p^2)^{1/2}}\int_0^\infty du\,u^2J_1((p^2)^{1/2}u)\times\left(\frac{im}{8\pi}\frac{H_1^{(2)}(mu)}{u}\right)^z,$$

$$D_{\rm III}(p^2,z)=\frac{4\pi^2}{(p^2)^{1/2}}\int_0^\infty du\,u^2J_1((p^2)^{1/2}u)\times\left(-\frac{im}{8\pi}\frac{H_1^{(1)}(mu)}{u}\right)^z. \tag{2.14}$$

These formulas are valid for $1<{\rm Re}z<2$. If we add and subtract the Maclaurin terms as in (2.10), they can be extended in an obvious way into the half-plane ${\rm Re}z>2$. Presumably their range of validity can be extended down to ${\rm Re}z=0$ without difficulty (at $z=1$ the branch point should reduce to a simple pole).

To summarize, from the integral representation (2.6) and its modifications (2.10), (2.12), and (2.14) we deduce that the superpropagator $D(p^2,z)$ viewed as a function of two complex variables has the singularities: (a) fixed poles at $z=2, 3, \ldots$, the residues of which are polynomials of order $z-2$ in p^2; (b) a singular surface $p^2=z^2m^2$ which manifests itself as a branch point in the p^2 plane except when $z=1$, in which case it becomes a simple pole. The discontinuity across the branch cut is a regular function of z—at least for ${\rm Re}z>0$. Whenever z is an integer the singularity surface $p^2=z^2m^2$ corresponds to normal thresholds as implied by unitarity.

We have no clear idea of the behavior of the superpropagator for ${\rm Re}z\leqslant 0$, except, of course, in the zero-mass case where it is analytic with zeros at $z=0, -1, -2, \ldots$.

For large values of $|p^2|$ it is presumably adequate to approximate $D(p^2,z)$ by the zero-mass form (2.7). For large values of $|z|$ it may or may not be possible to use (2.8).

Finally, let us consider an alternative regularization of the superpropagator Δ^z which can be used in the zero-mass case. (This regularization will be referred to in Sec. III.) Introduce the regularizing parameter a to define

$$\Delta_{\rm reg}(-r^2,0)=\frac{1}{4\pi^2}\frac{1}{r^2+a^2},\quad a>0 \tag{2.15}$$

which has no singularity at $r=0$. Substituting this form into (2.6), we find the corresponding momentum-space superpropagator

$$D_{\rm reg}(p^2,z)=\frac{(2\pi)^{2-2z}}{(-p^2)^{1/2}}\int_0^\infty dr\frac{r^2J_1((-p^2)^{1/2}r)}{(r^2+a^2)^z}=\frac{2(4\pi^2)^{1-z}}{\Gamma(z)}\left(\frac{(-p^2)^{1/2}}{2a}\right)^{z-2}K_{z-2}(a(-p^2)^{1/2}), \tag{2.16}$$

and, as was to be expected, the fixed poles at $z=2, 3, \ldots$ have disappeared. In order to see what happens in the limit $\alpha\to 0$ let us assume that z is not an integer and replace K_{z-2} by its series expansion. The result is

$$D_{\rm reg}(p^2,z)=\frac{\pi}{\sin\pi z}(4\pi)^{2-2z}\frac{(-p^2)^{z-2}}{\Gamma(z)\Gamma(z-1)}\left[1-\frac{1}{z-1}\frac{a^2p^2}{4}+\frac{1}{2z(z-1)}\left(\frac{a^2p^2}{4}\right)^2-\cdots-a^{4-2z}\frac{\Gamma(z-1)}{\Gamma(3-z)}\left(-\frac{p^2}{4}\right)^{2-z}-a^{6-2z}\frac{\Gamma(z-1)}{\Gamma(4-z)}\left(-\frac{p^2}{4}\right)^{3-z}-\cdots\right]. \tag{2.17}$$

In the limit $a\to 0$ every term except the first vanishes if ${\rm Re}z<2$. In this way the superpropagator (2.7) can be obtained as the limit of a regularized function.

If, on the other hand, we were to take $2<{\rm Re}z<3$, then the limit would be singular:

$$D_{\rm reg}(p^2,z)\to D_0(p^2,z)-(4\pi^2)^{1-z}a^{4-2z}/(z-1)(z-2). \tag{2.18}$$

This means that contour integrals involving $D_{\rm reg}(p^2,z)$ must be moved to the left of ${\rm Re}z=2$ *before* the regularization is removed.

III. REGULARIZED PERTURBATION SERIES

Consider now the problem of developing perturbation expansions in the nonpolynomial interaction $V(\phi)$. As discussed in Sec. I, we could begin formally by expanding V in powers of ϕ and then, for each power ϕ^n, develop the usual series. Let us therefore suppose that the interaction Hamiltonian, considered as a function of the complex variable ϕ, is analytic in some neighborhood of $\phi=0$ so that we can write

$$V(\phi)=\sum_n (v_n/n!)(-\phi)^n, \quad |\phi|<R. \tag{3.1}$$

It is convenient for our purposes to consider, instead of the usual many-body Green's functions, the equivalent set of amplitudes

$$(-)^{\Sigma m}F_{m_1\cdots m_N}(\Delta)=\langle T(V^{(m_1)}(\phi_1)\cdots V^{(m_N)}(\phi_N))\rangle, \tag{3.2}$$

where $\phi_1=\phi(x_1)$, ... and $V^{(m)}(\phi)=\partial^m V(\phi)/\partial\phi^m$. If we substitute the power series (3.1) into (3.2) then, after some straightforward manipulations, we obtain the series

$$F_{m_1\cdots m_N}(\Delta)\simeq\sum_{n_{ij}} v(m_1+\sum_j n_{1j})\cdots v(m_N+\sum_j n_{Nj}) \times\prod_{i<j}[\Delta(x_i-x_j)]^{n_{ij}}/n_{ij}!, \tag{3.3}$$

where the indices i, j, run from 1 to N and the $n_{ij}=n_{ji}$ from 0 to ∞. We shall suppose that $n_{ii}=0$ since the factors $\Delta(0)^{n_{ii}}$ produce no more than a rather simple renormalization of coupling strengths which we can regard as having been done already.[17]

For the interactions we shall be considering, the series (3.3) generally diverge. Thus, for example, corresponding to the interaction (1.4) we have $v(n)=Gf^n n!$ and so in the simplest case $N=2$, $m_1=m_2=0$,

$$F_{00}(\Delta)\simeq G^2\sum_{n=0}^{\infty} f^{2n}n!\Delta^n,$$

which clearly diverges. However, we can easily regard such series as asymptotic expansions and direct our attention to the problem of defining the amplitudes which are so represented. As a first step towards such a definition, let us consider the Borel sum of (3.3),

$$F_{m_1\cdots m_N}{}^B(\Delta)=\int_0^\infty\left(\prod_{i<j}d\zeta_{ij}e^{-\zeta_{ij}}\sum_{n_{ij}=0}^{\infty}\right)v(m_1+\sum n_{ij})\cdots \prod_{i<j}\frac{[\zeta_{ij}\Delta(x_i-x_j)]^{n_{ij}}}{n_{ij}!}. \tag{3.4}$$

The summations over n_{ij} are by this method—or some generalization[18] of it—made to converge so that we are left with the problem of integrating over the ζ_{ij}. The ζ integrals may well be undefined since the analytic function in the integrand can develop singularities on the path of integration. These can always be avoided by distorting the ζ contours or by giving the Δ's suitable imaginary parts.

The precise details of the method used to define the functions $F^B(\Delta)$ are not important since we are using the Borel integrals only at an intermediate stage of our program. The essential step will be the replacing of the sums over n_{ij} in (3.4) by contour integrals over z_{ij}. To do this we need to interpolate the expansion coefficients v_n by an analytic function $v(z)$. The existence of such an interpolating function is assured by the condition[19]

$$\int_0^\infty d\phi|\phi^{-\alpha-1}V(\phi)|<\infty \tag{3.5}$$

for some range of α. Indeed, if (3.5) is satisfied then $V(\phi)$ may be represented by the Mellin integral

$$V(\phi)=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{v(z)}{\Gamma(z+1)}\phi^z, \tag{3.6}$$

where $v(z)$ is an analytic function defined by the integral

$$v(z)=\frac{1}{\Gamma(-z)}\int_0^\infty d\phi\,\phi^{-z-1}V(\phi), \quad \mathrm{Re}z=\alpha. \tag{3.7}$$

In order to compare the integral representation (3.6) with the power series (3.1) we have only to perform the inverse Watson-Sommerfeld transformation. It appears that α is constrained by the minimum value n_0 of n appearing in the sum (3.1), i.e.,

$$n_1<\alpha<n_0, \tag{3.8}$$

where n_0 and n_1 are determined by the limiting behavior

$$\begin{aligned} V(\phi)&\sim\phi^{n_0}, \quad \phi\to 0\\ &\sim\phi^{n_1}, \quad \phi\to\infty. \end{aligned} \tag{3.9}$$

[We shall be assuming later that $V(\phi)$ can be expanded in powers of $1/\phi$ for $|\phi|>R_1$. In this expansion the lowest power to appear is $-n_1$.]

To avoid the notational obscurities consequent upon the use of general formulas like (3.4), we devote the remainder of this section to a detailed treatment of the second-order amplitudes,

$$F_{m_1m_2}{}^B(\Delta) =\int_0^\infty d\zeta\, e^{-\zeta}\sum_{n=0}^{\infty} v(m_1+n)v(m_2+n)\frac{(\zeta\Delta)^n}{(n!)^2}. \tag{3.10}$$

Formal generalizations for the higher-order amplitudes can usually be made without difficulty. New features connected with the problem of defining products of superpropagators may appear in the higher orders; about these we can draw only tentative conclusions. These problems will be presented in Sec. IV.

[17] R. Delbourgo and K. Koller (unpublished).

[18] Examples of such generalizations are given in Refs. 7 and 8.

[19] E. C. Titchmarsh, *Theory of Fourier Integrals*, 2nd ed. (Oxford U. P., Oxford, England, 1967), p. 46.

It will prove convenient in the following to have an auxiliary complex variable λ at our disposal. Let us therefore consider the functions $F^B(\lambda\Delta)$, bearing in mind that we shall take a limit $\lambda\to 1$ at the end. Let us now replace the summation over n in (3.10) by a contour integral using the $v(z)$ of (3.7), analytically continued, to interpolate the coefficients v_n,

$$F_{m_1m_2}{}^B(\lambda\Delta)=\int_0^\infty d\zeta\, e^{-\zeta}(\tfrac{1}{2}i)\int_\Gamma \frac{dz}{\sin\pi z}\times\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)^2}(-\lambda)^z(\zeta\Delta)^z, \quad (3.11)$$

where Γ denotes a contour coming from $+\infty$ which encircles the integers $z=M$, $M+1$, $M+2$, ..., in the negative sense, and returns to ∞. The non-negative integer M is fixed as the lowest power of Δ appearing in the sum (3.10). It is defined by

$$M=\max(n_0-m_1,\, n_0-m_2,\, 0)\,. \quad (3.12)$$

Let us now open the contour Γ after the fashion of Watson and Sommerfeld. This is possible because of the postulated existence of the Mellin representation for $V(\phi)$. The factor $\Gamma(z+1)^{-2}$ produces a strong damping effect for $|z|\to\infty$, $|\arg z|<\frac{1}{2}\pi$. Once the contour has been opened, however, the factor

$$\Gamma(z+1)^{-2}\sim\exp(\pi|\mathrm{Im}z|)$$

acts to weaken convergence. Hence it is advisable at this stage to interchange the ζ and z integrals and perform the ζ integration to obtain

$$F_{m_1m_2}{}^B(\lambda\Delta)=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)}(-\lambda)^z\Delta^z, \quad (3.13)$$

where α lies in the range

$$M_1<\alpha<M; \quad (3.14)$$

here M_1 denotes the position of the singularity of the integrand nearest on the left to $z=M$. This will generally be a pole at $z=M-1$ due to the factor $(\sin\pi z)^{-1}$. However, if $M=1$ or 0, this will be a pole from one or both of the factors $v(z+m_1)$ and $v(z+m_2)$. Let us defer these questions until we come to consider the construction of asymptotic series for the momentum-space amplitudes.

In formula (3.13) we have thus obtained an integral representation of the Mellin type involving the superpropagator Δ^z. If $M_1<2$, then this integral defines unambiguously the generalized functions $F^B(\lambda\Delta)$. If, on the other hand, $M_1\geqslant 2$, then it will be necessary to translate the contour to the left of the line $\mathrm{Re}z=2$ and, in so doing, pick up the poles at $z=2, 3, \ldots [M_1]$. The part of the generalized function defined by the new contour is unambiguous but the separated terms, involving $\Delta^2, \Delta^3, \ldots, \Delta^{[M_1]}$, are not. They carry the usual ultraviolet divergences.

A simple illustration of this effect can be given in the zero-mass case using the regularized superpropagator $D_{\mathrm{reg}}(p^2,z)$ in place of Δ^z in (3.13). This propagator, given by (2.16), has no z poles and in the strip $0<\mathrm{Re}z<2$ it reduces to the correct form (2.7) in the limit when the regularization is removed. However, if $\mathrm{Re}z>2$ then singular terms appear when the regularization is removed. That is, the Fourier transform of $F^B(\lambda\Delta)$ is well defined when the regularization is removed only if the contour is contained in the strip $0<\mathrm{Re}z<2$.

Assuming now that $0<\alpha<2$, we can immediately write down the Fourier transform of $F^B(\lambda\Delta)$. It is given by

$$\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\times\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)}(-\lambda)^zD(p^2,z)\,. \quad (3.15)$$

At this point we must remark that the continuation to $z=0$ of $D(p^2,z)$ must be treated carefully. For the zero-mass case the explicit formula (2.7) gives

$$D(p^2,0)=0\,.$$

On the other hand, we should expect the Fourier transform of $\Delta^0=1$ to be $(2\pi)^4\delta^4(p)$. This means that if, in the integral representation (3.13), the contour lies to the left of $z=0$, i.e., if $M=0$, then before defining the Fourier transform we must translate the contour to the right of $z=0$ where we can use (3.15) and, in compensation, add the term

$$v(m_1)v(m_2)(2\pi)^4\delta^4(p)\,, \quad (3.16)$$

which is the Fourier transform of the contribution of the pole at $z=0$. Clearly (3.16) corresponds to a disconnected graph. The contour method apparently picks out the connected graphs only.

Let us now consider the problem of constructing an asymptotic series to represent the function $\tilde{F}^B(p^2,\lambda)$. To simplify the discussion let us suppose that $M=0$ or 1. (If $M\geqslant 2$ the necessary modifications can be made without difficulty.) For large spacelike p^2, we shall use the zero-mass approximation for $D(p^2,z)$ and write (3.15) in the form

$$\begin{aligned}\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)&\simeq\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)^2}\times\frac{z(-\lambda)^z(-p^2)^{z-2}}{\sin\pi z\,\Gamma(z-1)}(4\pi)^{2-2z}\\&=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)^2}\frac{z}{\pi}\times\Gamma(2-z)(-\lambda)^z(-p^2)^{z-2}(4\pi)^{2-2z}.\end{aligned} \quad (3.17)$$

Our aim, in order to get a series in inverse powers of p^2, is to collapse the contour onto the negative real axis—a mirror image of the conventional Watson-Sommerfeld contour on the positive real axis. By our assumption that $V(\phi)$ can be expanded in inverse powers of ϕ for sufficiently large $|\phi|$, it follows that $v(z)/\Gamma(z)$ is damped with sufficient strength for the inverse Watson-Sommerfeld transformation to go through in the half-plane $\mathrm{Re}z<0$. The only factor in (3.17) which hinders this operation is $\Gamma(2-z)$, which explodes. However, this can be removed by the Borel trick. That is, we can write

$$\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)\sim\int_0^\infty\frac{d\zeta}{\zeta}e^{-\zeta}(\tfrac{1}{2}i)\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\times\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)^2}\frac{z}{\pi}(-\lambda)^z\left(-\frac{p^2}{\zeta}\right)^{z-2}(4\pi)^{2-2z}. \quad (3.18)$$

The dominant term as $p^2\to\infty$ is going to come from the pole or dipole of $v(z+m_1)v(z+m_2)$ furthest to the right. For simplicity let us suppose that the poles of $v(z)$ occur at negative integer values of z. This happens in many cases of interest and it is a consequence of requiring that $V(\phi)$ has a Laurent expansion at $\phi=\infty$. Suppose

$$V(\phi)=\sum_n\frac{u(n)}{n!}(-\phi)^{-n}.$$

Comparing this with the integral representation (3.6), we find

$$v(z)=-\frac{\Gamma(z)}{\Gamma(-z)}u(-z),$$

so that, in particular,

$$\frac{v(z+m)}{\Gamma(z+1)}=-(z+m)(z+m-1)\cdots(z+1)\frac{u(-z-m)}{\Gamma(1-z-m)}, \quad (3.19)$$

which vanishes for $z=-1,-2,\ldots,-m$. It will vanish also for

$$z=-m-1,\ldots,-m-n_2+1,$$

where n_2 denotes the lowest power of ϕ occurring in the regular part of the Laurent expansion of $V(1/\phi)$, i.e., $u(n)=0$ for $n=0,1,\ldots,n_2-1$. The poles of the integrand of (3.18) therefore occur at

$$z=-N-1,\ -N-2,\ldots,$$

where N is given by

$$N=\max(m_1,m_2,m_1+n_2-1,m_2+n_2-1). \quad (3.20)$$

Hence we can write

$$\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)\sim\int_0^\infty\frac{d\zeta}{\zeta}e^{-\zeta}\sum_{n=N+1}^\infty(-)^n\frac{(n-1)!}{(n-m_1-1)!}\times\frac{u(n-m_1)}{(n-m_1)!}\frac{(n-1)!}{(n-m_2-1)!}\frac{u(n-m_2)}{(n-m_2)!}\times(-\lambda)^n\frac{n}{\pi}\left(-\frac{\zeta}{p^2}\right)^{n+2}(4\pi)^{2(n-1)}. \quad (3.21)$$

Finally, to obtain the asymptotic[20] series we interchange the ζ integration with the summation. We obtain the result

$$\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)\simeq\frac{1}{\pi}\sum_{n=N+1}^\infty\lambda^n\times\frac{[(n-1)!]^2u(n-m_1)u(n-m_2)}{(n-m_1-1)!(n-m_1)!(n-m_2-1)!(n-m_2)!}\times n(n+1)!\left(-\frac{4\pi^2}{p^2}\right)^{n+2}. \quad (3.22)$$

There are two important aspects of this formula. Firstly, the leading term in $\tilde{F}^B$ turns out to be $(-p^2)^{-N-3}$ where N is given by (3.20). [If $M\geqslant 2$ we must include additional terms corresponding to the poles at $z=1,2,\ldots,M_1$. These are $(1/p^2)$, $\ln p^2,\ldots(p^2)^{M_1-2}\ln p^2$.] Secondly, the asymptotic series (3.22) is *single valued* in λ.

The functions $\tilde{F}^B(p^2,\lambda)$ defined by the integral (3.15) are not single valued in λ. Generally they have a logarithmic singularity at $\lambda=0$. This can be seen if we collapse the contour of (3.15) onto the positive real axis and pick up the residues of the *dipoles* at $z=2,3,\ldots$. Let us use again the zero-mass expression for the superpropagator to write

$$\begin{aligned}\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)&=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{\sin\pi z}\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)}(-\lambda)^z\frac{(-p^2)^{z-2}}{\sin\pi z\,\Gamma(z)\Gamma(z-1)}\\&=\frac{v(1+m_1)v(1+m_2)}{p^2}\lambda+\frac{1}{\pi}\sum_{n=2}^\infty\frac{\partial}{\partial z}\left[\frac{v(z+m_1)v(z+m_2)(-\lambda)^z(-p^2)^{z-2}}{\Gamma(z+1)\Gamma(z)\Gamma(z-1)}\right]_{z=n}\\&=\frac{v(1+m_1)v(1+m_2)}{p^2}\lambda+\frac{1}{\pi}\sum_{n=2}^\infty\frac{v(n+m_1)v(n+m_2)(-\lambda)^n(-p^2)^{n-2}}{n!(n-1)!(n-2)!}\\&\quad\times\left[\frac{v'(n+m_1)}{v(n+m_1)}+\frac{v'(n+m_2)}{v(n+m_2)}-\psi(n+1)-\psi(n)-\psi(n-1)+\ln(-\lambda)+\ln(-p^2)\right], \quad (3.23)\end{aligned}$$

[20] We shall not attempt to prove that the series (3.22) is indeed asymptotic. Presumably one could ensure this by imposing appropriate conditions on $V(\phi)$.

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. That is, we can write

$$\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)=A_{m_1m_2}(p^2,\lambda)+\ln(-\lambda)B_{m_1m_2}(p^2,\lambda)\,, \quad (3.24)$$

where A and B are entire functions of λ. Moreover, B is an entire function of p^2 as well.

That $B(p^2,\lambda)$ is an entire function of p^2 can be seen from the fact that the discontinuity of $\tilde{F}^B$ is an entire function of λ and must therefore come from the function $A(p^2,\lambda)$. Considering the massive case, we can use the property that the absorptive part of the superpropagator, $D(p^2,z)$, vanishes for $p^2<(mz)^2$ to evaluate the absorptive part of $\tilde{F}^B$ by translating the contour in (3.15) to the right of $\mathrm{Re}z=(\sqrt{p^2})/m$ for given $\sqrt{p^2}>0$. Only the poles separated in this way can contribute to the discontinuity, which is therefore given by

$$\mathrm{disc}\tilde{F}_{m_1m_2}{}^B(p^2,\lambda)=\sum_{n=1}^{[\sqrt{p^2}/m]}\frac{v_{m1+n}v_{m2+n}}{n!}\lambda^n\,\mathrm{disc}D(p^2,n)\,, \quad (3.25)$$

which is just a polynomial in λ.

The form (3.24) which has been derived for the zero-mass case is probably true in general. Its validity depends only on the feasibility of the inverse Watson-Sommerfeld transformation together with the fact that $D(p^2,z)$ has simple poles at the integers $z=2,3,\ldots$. These combine with the zeros of $\sin\pi z$ to make dipoles.

The dispersive part of $\tilde{F}^B$ is certainly not an integral function of λ and we shall have to adopt some definition of the limit $\lambda\to 1$. It is at this point that a basic uncertainty enters the program. In the absence of guidelines we can interpret $\lim_{\lambda\to 1}(-\lambda)^z$ by an average of the terms

$$e^{i(2k+1)\pi z},\quad k=0,\pm1,\pm2,\ldots.$$

That is, we should write

$$\tilde{F}_{m_1m_2}(p^2)=\sum_k a_k\tilde{F}_{m_1m_2}{}^B(p^2,-e^{i(2k+1)\pi}) \quad (3.26)$$

with arbitrary complex parameters a_k. Substituting the form (3.24), this reads

$$\tilde{F}_{m_1m_2}(p^2)=(\sum_k a_k)A_{m_1m_2}(p^2,1)+i\pi(\sum_k(2k+1)a_k)B_{m_1m_2}(p^2,1)\,, \quad (3.27)$$

so that there are really only two arbitrary constants.

There is one very important constraint to be imposed. That is unitarity. The imaginary part of $\tilde{F}(p^2)$ should be given by (3.25) with $\lambda=1$, and it should vanish for $p^2<0$. This gives us the conditions

$$\sum a_k=1 \quad\text{and}\quad i\pi\sum(2k+1)a_k=b\,, \quad (3.28)$$

where b is real. Thus it appears that the Fourier transform of the generalized function $F(\Delta)$ may contain one arbitrary parameter b. Possibly we could take $b=0$. This corresponds to the choice advocated in earlier references (Efimov[7] and Paper I[8]) and it may receive justification when we are able to impose the unitarity requirement in higher orders.

The final choice of amplitude $\tilde{F}(p^2)$ represented by (3.27) and (3.28) can be expressed by the integral

$$\tilde{F}_{m_1m_2}(p^2)=\tfrac{1}{2}i\int_{\alpha-i\infty}^{\alpha+i\infty}dz\,\frac{v(z+m_1)v(z+m_2)}{\Gamma(z+1)}D(p^2,z)\times\left(\frac{1}{\tan\pi z}+b\right). \quad (3.29)$$

IV. HIGHER ORDERS AND FEYNMANIZATION OF SUPERGRAPHS

Corresponding to a diagram with N vertices which are pairwise connected by superpropagators there exists the momentum-space amplitude

$$\int dx_1\cdots dx_N e^{i\Sigma px}\prod_{i<j}\Delta(x_i-x_j)^{z_{ij}}=(2\pi)^4\delta(\sum p)D(p_1\cdots p_N;z_{12}\cdots) \quad (4.1)$$

depending upon $N-1$ independent momenta and $\frac{1}{2}N(N-1)$ independent complex parameters z_{ij}. The singularities of this integral occur on the various light cones $(x_i-x_j)^2=0$. An over-all convergence condition can be obtained by considering the behavior of the integrand when all components x_i approach 0 simultaneously. This gives the over-all singularity

$$x^{-2\Sigma\mathrm{Re}z_{ij}}=x^{-N(N-1)\mathrm{Re}z}$$

if we assume for simplicity that $\mathrm{Re}z_{ij}$ is the same for every z_{ij}. This singularity is compensated by $4(N-1)$ integrations. Hence we have superficial convergence if

$$N(N-1)\,\mathrm{Re}z<4(N-1)\,,$$

i.e.,

$$\mathrm{Re}z<4/N\,. \quad (4.2)$$

An equivalent representation of the amplitude (4.1) is given by the momentum-space integral

$$D(p_1\cdots p_N;z_{12}\cdots)=\int\prod_{\text{loops}}(dk)\prod_{i<j}D(q_{ij}{}^2,z_{ij})\,, \quad (4.3)$$

where $D(q^2,z)$ denotes the superpropagator of Sec. II. The $\frac{1}{2}(N(N-1)$ momenta q_{ij} associated with the superlines are expressed in the usual way by linear combinations of the loop momenta $k_1,\ldots,k_l$ and the external momenta $p_1\cdots p_N$. The convergence of (4.3) can be justified by the same power-counting arguments as were used above. Using the asymptotic form $D(q^2,z)\sim(-q^2)^{z-2}$, one arrives again at the condition (4.2).

The problem of analyzing higher-order contributions is a very standardized one. For each N there is one and only one skeleton graph. This graph is obtained by joining the vertices in pairs—one line to each pair. The resulting diagram has $\frac{1}{2}N(N-1)$ lines and $\frac{1}{2}(N-1)\times(N-2)$ loops. The amplitude which corresponds to this diagram would be highly divergent if ordinary bare propagators were associated with the lines. It is the possibility of taking $\mathrm{Re}z_{ij}$ sufficiently small which makes the amplitude converge when superpropagators are associated with the lines. The analytic function defined by this convergent integral can then be continued outside the original domain $\mathrm{Re}z_{ij}<4/N$ exactly as was done for the superpropagator itself.

In this way one is led to define the higher-order momentum-space amplitudes by the multiple contour integral

$$\tilde{F}_{m_1\cdots m_N}{}^B(p_1\cdots p_N;\lambda_{12}\cdots)=\int\prod_{\text{lines}}\left(\tfrac{1}{2}i\frac{dz_{ij}}{\sin\pi z_{ij}}\frac{(-\lambda_{ij})^{z_{ij}}}{\Gamma(z_{ij}+1)}\right)\times\prod_{\text{vertices}}v(m_k+\sum_l z_{kl})D(p_1\cdots p_N;z_{12}\cdots)\,,\quad(4.4)$$

where the z contours lie initially in the strip $0<\mathrm{Re}z_{ij}<4/N$. The auxiliary parameters λ_{ij} must then be set equal to $+1$, and it is at this stage that some ambiguity can enter the problem. The general procedure should be to define the true amplitude as a linear combination of the possible limits $\lambda_{ij}\to 1$, i.e.,

$$\tilde{F}_{m_1\cdots m_N}(p_1\cdots p_N)=\sum_{\nu_{12}\cdots}a_{\nu_{12}\cdots}{}^N\tilde{F}_{m_1\cdots m_N}{}^B(p_1\cdots p_N;\,-e^{i\nu_{12}\pi}\cdots)\,,\quad(4.5)$$

where each ν_{ij} takes the values $+1$ and -1. The coefficients $a_{\nu_{12}\cdots}{}^N$ are to be chosen consistently with unitarity but are otherwise arbitrary. Substituting the representation (4.4) into (4.5) one obtains the form

$$\tilde{F}_{m_1\cdots m_N}(p_1\cdots p_N)=\int\prod_{\text{lines}}\left(\tfrac{1}{2}i\frac{dz_{ij}}{\sin\pi z_{ij}}\frac{1}{\Gamma(z_{ij}+1)}\right)\times\prod_{\text{vertices}}v(m_k+\sum_l z_{kl})D(p_1\cdots p_N;z_{12}\cdots)\times a^N(z_{12}\cdots)\,,\quad(4.6)$$

where $a^N(z_{12}\cdots)$ denotes an entire function defined by the sum

$$a^N(z_{12}\cdots)=\sum_{\nu_{12}\cdots=\pm1}a_{\nu_{12}\cdots}{}^N\exp(i\pi\sum_{i<j}\nu_{ij}z_{ij})\,.\quad(4.7)$$

It is necessary to investigate in what way this "ambiguity function" (which resembles the signature ambiguity in Regge theory) is constrained by the requirements of unitarity.

The unitarity problem is of course an extremely intricate one and so we shall confine the discussion to a conjecture about normal thresholds. To this end let the N vertices be divided into two sets, $1, 2, \ldots, M$ and $1', 2', \ldots, M'$ $(M+M'=N)$. If the amplitude $D(p_1\cdots p_N;z_{ij})$, considered as a function of the variable $(p_1+\cdots+p_M)^2=(p_{1'}+\cdots+p_{M'})^2$ has a branch point at

$$(p_1+\cdots+p_M)^2=m^2(\sum_{ii'}z_{ii'})^2\,,\quad(4.8)$$

and if the discontinuity across the associated cut is given by the integral

$$\int(\prod dq_{ij'})\delta(p_1+\cdots+p_M-\sum q)\times D(p_1+\sum q_{1i'},\ldots;z_{ij})\prod_{ij'}\theta(q_{ij'})\,\mathrm{disc}D(q_{ij'}{}^2,z_{ij'})\times D(p_{1'}+\sum q_{1'i},\ldots;z_{i'j'})\,,\quad(4.9)$$

where the variable $q_{ij'}$ denotes the four-momentum carried from vertex i in the first set to vertex j' in the second, we would have a situation for supergraphs similar to the Landau-Cutkosky discontinuity formulas for normal Feynman graphs. The plausibility of (4.9) can be seen when we consider that the discontinuity (4.9) is a regular function of the $z_{ij'}$ which vanishes when the real part of $\sum z_{ij'}$ is taken sufficiently large. This follows from the properties, established in Sec. II, of the superpropagator $D(q^2,z)$.

If the expression (4.9) for the discontinuity of $D(p_1\cdots p_N;z_{ij})$ is used in conjunction with the representation (4.4), it is possible to give the discontinuity of $\tilde{F}^B$ in the form

$$\mathrm{disc}\tilde{F}_{m_1\cdots m_N}{}^B(p_1\cdots p_N;\lambda_{ij})=\int\prod_{\text{lines}}\left(\tfrac{1}{2}i\frac{dz_{ij}}{\sin\pi z_{ij}}\frac{(-\lambda_{ij})^{z_{ij}}}{\Gamma(z_{ij}+1)}\right)\prod_{\text{vertices}}v(m_k+\sum_{l\neq k}z_{kl})\,\mathrm{disc}D(p_1\cdots p_N;z_{ij})$$
$$=\int\prod_{\text{lines}}\left(\tfrac{1}{2}i\frac{dz_{ij}}{\sin\pi z_{ij}}\frac{(-\lambda_{ij})^{z_{ij}}}{\Gamma(z_{ij}+1)}\right)\prod_{\text{vertices}}v(m_k+\sum_{l\neq k}z_{kl})\int\prod(dq_{ij'})\delta(p_1+\cdots+p_M-\sum q_{ij'})\times D(p_1+\sum q_{ij},\ldots;z_{ij})\prod_{ij'}\theta(q_{ij'})\,\mathrm{disc}D(q_{ij'}{}^2;z_{ij'})D(p_{1'}+\sum q_{1'i},\ldots;z_{i'j'})\,.\quad(4.10)$$

The appearance of this formula can be simplified considerably by translating to the right the contours $z_{ij'}$ corresponding to particles exchanged between the two sets. In the course of this translation simple poles due to the zeros of $\sin\pi z_{ij'}$ will be crossed and their residues must be collected. Ultimately, when the contours have been pushed far enough, the contribution of these contours to the discontinuity will vanish. The discontinuity is thereby expressed—for given $(p_1+\cdots+p_M)^2$—by the residues of the *finite* set of poles at $z_{ij'}=n_{ij'}$ which have been crossed. The result is

$$\begin{aligned}
&\mathrm{disc}\tilde{F}_{m_1\cdots m_N}{}^B(p_1\cdots p_N;\lambda_{ij})\\
&=\sum_{n_{ij'}}\int\prod\left(dq_{ij'}\theta(q_{ij'})\right)\delta\left(\sum q_{ij'}-p_1-\cdots-p_M\right)\\
&\quad\times\tilde{F}_{m_1+\Sigma n_{1j'},\ldots}{}^B\left(p_1+\sum q_{1j'},\ldots,p_M+\sum q_{Mj'};\lambda_{ij}\right)\\
&\quad\times\prod_{ij'}\lambda_{ij'}{}^{n_{ij'}}\Omega_{n_{ij'}}(q_{ij'}{}^2)\\
&\qquad\times\tilde{F}_{m_{1'}+\Sigma n_{1'i},\ldots}{}^B\left(p_{1'}+\sum q_{1'i},\ldots,\lambda_{i'j'}\right),
\end{aligned}\tag{4.11}$$

where $\Omega_n(q^2)$ denotes the n-particle phase volume.

A similar factorization of the discontinuity into the products of lower-order amplitudes will be obtained for the true amplitude (4.6), provided the entire function a^N factorizes according to

$$\lim_{z_{ij'}\to n_{ij'}}a^N(z_{12}\cdots)=a^M(z_{ij})a^{M'}(z_{i'j'})(-)^{\Sigma n_{ij'}}.\tag{4.12}$$

(For the two-point function discussed in Secs. I and III, $a^2(z)=\cos\pi z+b\sin\pi z$.) One possible form for general a^N which satisfies (4.12) is

$$a^N(z_{ij})=\prod_{i,j=1}^{N}a^2(z_{ij}),$$

though this may not be the most general one. This is a strong result and would imply that there is just one arbitrary constant b in the whole theory. A result similar to this but not as strong has been claimed by Efimov, who shows on the basis of unitarity that, in his recent formulation[21] of the theory, there is just one arbitrary function $b(s)$ associated with superpropagators.

Another method of attack on the unitarity problem which may give more insight is to eliminate the loop momenta from (4.3) in favor of a set of Feynman parameters. As will be seen below, this method has the advantage of making a sharper separation between the factors which depend on the details of the interaction and the kinematical factors which are common to all interactions. In fact, the momentum-space amplitude in Nth order can be expressed in the form

$$\tilde{F}_{m_1\cdots m_N}(p_1\cdots p_N)=\int_0^\infty\prod_{\text{lines}}d\alpha_{ij}\mathbf{F}_{m_1\cdots m_N}(\alpha_{12}\cdots)\times\mathbf{C}(\alpha)^{-2}\exp\left(\frac{\mathbf{D}(\alpha,p)}{\mathbf{C}(\alpha)}\right),\tag{4.13}$$

where $\mathbf{C}(\alpha)$ and $\mathbf{D}(\alpha,p)$ are functions which are completely determined by the structure of the skeleton graph, which, for these considerations, will always be taken as the set of N vertices pairwise connected by $\frac{1}{2}N(N-1)$ lines. The functions $\mathbf{C}$ and $\mathbf{D}$ are those defined, for example, in Ref. 22 with the stipulation that zero-mass bare propagators be used in the definition. The function $\mathbf{F}_{m_1\cdots m_N}(\alpha_{12}\ldots)$ contains the dynamical information and also the ambiguities. It will be defined in the following.

The derivation of the integral representation (4.13) proceeds in the following way. Firstly, since the superpropagator $D(q^2,z)$ is analytic in the q^2 half-plane $\mathrm{Re}q^2<\mathrm{Re}(mz)^2$ and is bounded there by a power, it can be expressed as a Laplace transform

$$D(q^2,z)=\int_0^\infty d\alpha\,\tilde{D}(\alpha,z)e^{\alpha q^2},\quad \mathrm{Re}q^2<\mathrm{Re}(mz)^2.\tag{4.14}$$

The new amplitude $\tilde{D}$, the α representative of the superpropagator, is obtained by inverting this integral. For $\alpha>0$,

$$\begin{aligned}
\tilde{D}(\alpha,z)&=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}dq^2e^{-\alpha q^2}D(q^2,z),\quad \beta<\mathrm{Re}(mz)^2\\
&=\frac{1}{2\pi i}\int_{(mz)^2}^{\infty}dq^2e^{-\alpha q^2}\,\mathrm{disc}D(q^2,z),
\end{aligned}\tag{4.15}$$

where the latter form is obtained by collapsing the contour onto the cut which extends from $(mz)^2$ to $+\infty$ in the q^2 plane. The fixed poles at $z=2, 3, 4, \ldots$ contained in $D(q^2,z)$ are absent from its discontinuity and therefore also from the new amplitude $\tilde{D}$. It is clear that $\tilde{D}$, considered as a function of complex α, is analytic in the half-plane $\mathrm{Re}\alpha>0$. In general, there is a singularity at $\alpha=0$, where

$$\tilde{D}(\alpha,z)\sim\frac{\alpha^{1-z}}{\Gamma(1-z)},\quad \alpha\to0.\tag{4.16}$$

For $|\alpha|\to\infty$, $|\arg\alpha|<\frac{1}{2}\pi$, one finds

$$\tilde{D}(\alpha,z)\sim e^{-\alpha(mz)^2},\quad \mathrm{Re}z^2>0\tag{4.17}$$

[21] G. V. Efimov, Kiev Report Nos. ITF 68-52, ITF 68-54, and ITF 68-55 (unpublished).

[22] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., Cambridge, England, 1966), Chap. II.

provided $m>0$. For the zero-mass case,

$$\tilde{D}(\alpha,z)=\frac{(4\pi)^{2-2z}}{\Gamma(z)}\alpha^{1-z} \tag{4.18}$$

exactly.

The Laplace representation (4.14) is easily generalized to higher forms like (4.3). One can do this formally by substituting an integral like (4.14) for each factor in (4.3) and then exchanging the loop integrals with the α integrals. One obtains

$$D(p_1\cdots p_N;z_{12}\cdots)=\int_0^\infty \prod_{\text{lines}}[d\alpha_{ij}\tilde{D}(\alpha_{ij},z_{ij})]\int\prod_{\text{loops}}(dk)e^{\Sigma\alpha q^2}. \tag{4.19}$$

This exchanging of the integrals can of course be justified only in the Symanzik region of the external momenta and only after a Wick rotation has been applied so as to make all the loop integrals Euclidean, $q_{ij}^2\leqslant 0$. Under these circumstances the loop integrals in (4.19) can be performed explicitly to give

$$\int\prod_{\text{loops}}(dk)e^{\Sigma\alpha_{ij}q_{ij}^2}=\mathbf{C}(\alpha)^{-2}\exp\left(\frac{\mathbf{D}(\alpha,p)}{\mathbf{C}(\alpha)}\right), \tag{4.20}$$

where $\mathbf{C}(\alpha)$ denotes a homogeneous polynomial of degree l while $\mathbf{D}(\alpha,p)$ is linear in the invariants p_ip_j and homogeneous of degree $l+1$ in $\alpha_{12}\cdots$. The degree l is equal to the number of loops in the skeleton graph which in this case is given by $l=\frac{1}{2}(N-1)(N-2)$.

To compute an amplitude, one must multiply (4.19) by the appropriate vertex factors and integrate over the z_{ij}, i.e.,

$$\begin{aligned}\tilde{F}_{m_1\cdots m_N}{}^B(p_1\cdots p_N;\lambda_{12}\cdots)&=\int\prod_{\text{lines}}\left(\frac{dz_{ij}}{2\pi i}\Gamma(-z_{ij})(-\lambda_{ij})^{z_{ij}}\right)\\&\quad\times\prod_{\text{vertices}}v(m_k+\sum_l z_{kl})D(p_1\cdots p_N;z_{12}\cdots)\\&=\int_0^\infty\prod_{\text{lines}}(d\alpha_{ij})\mathbf{F}_{m_1\cdots m_N}{}^B(\alpha_{12}\cdots;\lambda_{12}\cdots)\\&\quad\times\mathbf{C}(\alpha)^{-2}\exp\left(\frac{\mathbf{D}(\alpha,p)}{\mathbf{C}(\alpha)}\right),\end{aligned} \tag{4.21}$$

where the new amplitude $\mathbf{F}^B$ is defined by

$$\begin{aligned}\mathbf{F}_{m_1\cdots m_N}{}^B(\alpha_{12}\cdots;\lambda_{12}\cdots)&=\int\prod_{\text{lines}}\left(\frac{dz_{ij}}{2\pi i}\Gamma(-z_{ij})(-\lambda_{ij})^{z_{ij}}\tilde{D}(\alpha_{ij},z_{ij})\right)\\&\quad\times\prod_{\text{vertices}}v(m_k+\sum_l z_{kl}).\end{aligned} \tag{4.22}$$

There is no need to leave (4.22) in the form of a contour integral. One could collapse all of the z contours onto the positive axes and collect the residues of the poles there. The resulting sum, which represents the amplitude within some hypercircle of convergence, is given by

$$\begin{aligned}\mathbf{F}_{m_1\cdots m_N}{}^B(\alpha_{12}\cdots;\lambda_{12}\cdots)&=\sum_{n_{ij}}\prod_{\text{lines}}(\lambda_{ij}{}^{n_{ij}}\tilde{D}(\alpha_{ij},n_{ij})1/n_{ij}!)\\&\quad\times\prod_{\text{vertices}}v(m_k+\sum_l n_{kl}).\end{aligned} \tag{4.23}$$

For the class of theories considered in this paper, the series (4.23) converges for sufficiently large $|\alpha|$ and defines an analytic function of the Feynman parameters. In general, this function has singularities, some of which move onto the positive real axes when $\lambda_{ij}\to+1$. This phenomenon necessitates a distortion of the α contours in the integral (4.9). In order to define a sensible momentum-space amplitude, it will be necessary to take an average of the limits $-\lambda_{ij}\to\exp(i\pi\nu_{ij})$ with $\nu_{ij}=\pm1$ exactly as was done above.

To illustrate this, consider once more the second-order vacuum graphs corresponding to the interaction

$$v_n=Gf^nn!$$

in the zero-mass approximation. For this case, (4.11) reads

$$\begin{aligned}\mathbf{F}_{00}{}^B(\alpha,\lambda)&=G^2\sum_n\lambda^n\frac{\alpha^{1-n}}{(n-1)!}\frac{1}{n!}f^{2n}(n!)^2\\&=G^2\frac{\lambda f^2\alpha^2}{(\alpha-\lambda f^2)^2}.\end{aligned}$$

Corresponding to the definitions adopted in Sec. III, one can define the "true" amplitude by the limit

$$\begin{aligned}\mathbf{F}_{00}(\alpha)&=\tfrac{1}{2}(1+ib)\mathbf{F}_{00}{}^B(\alpha,-e^{i\pi})+\tfrac{1}{2}(1-ib)\mathbf{F}_{00}{}^B(\alpha,-e^{i\pi})\\&=G^2f^2\alpha^2[\text{P.V.}(\alpha-f^2)^{-2}-\pi b\delta'(\alpha-f^2)],\end{aligned}$$

where P.V. ( ) denotes the principal value and b is real. The momentum-space amplitude which corresponds to this is given by (4.13) with $\mathbf{C}(\alpha)=1$ and $\mathbf{D}(\alpha,p)=\alpha p^2$:

$$\begin{aligned}\tilde{F}_{00}(p^2)&=G^2f^2\left[\text{P.V.}\int_0^\infty d\alpha\frac{\alpha^2e^{\alpha p^2}}{(\alpha-f^2)^2}+\pi bf^2(f^2p^2+2)e^{f^2p^2}\right]\\&=G^2\left[p^2\left(\frac{\partial}{\partial p^2}\right)^3+2\left(\frac{\partial}{\partial p^2}\right)^2\right]e^{f^2p^2}E^*(-f^2p^2)\\&\quad+\pi bG^2f^4(f^2p^2+2)e^{f^2p^2},\end{aligned}$$

where $E^*(z)$ denotes the exponential integral function[23] cut from 0 to $-\infty$. The term containing $E^*(-f^2p^2)$

[23] Erdélyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. II, p. 147.

behaves asymptotically like $(1/p^2)^3$ provided $|\arg p^2|>0$. The term containing b, an entire function of p^2, dominates the asymptotic behavior if $|\arg(p^2)|\leqslant\frac{1}{2}\pi$. This may be a good reason for taking $b=0$.

In general, the result of this averaging of limits will be an integral like (4.13), with $\mathbf{F}$ given by

$$\mathbf{F}_{m_1\cdots m_N}(\alpha_{12}\cdots)=\sum_{\nu_{12}\cdots=\pm1}a_{\nu_{12}\cdots}\mathbf{F}_{m_1\cdots m_N}{}^{B}(\alpha_{12}\cdots;-e^{i\nu_{12}\pi}\cdots),\quad(4.24)$$

a limit which must be interpreted in the sense of generalized functions. That is, $\mathbf{F}(\alpha)$ will have prescribed singularities such as P.V.$(\alpha-\alpha_0)^{-n}$ or $\delta(\alpha-\alpha_0)$ on the integration contours.

Before discussing the unitarity problem it will be useful to have yet another representation at one's disposal. In the power series (4.23), one can substitute for the α representative of the superpropagator the form

$$\tilde{D}(\alpha,n)=\int_0^\infty d\kappa^2\theta(\kappa^2-(nm)^2)e^{-\alpha\kappa^2}\Omega_n(\kappa^2),$$

where $\Omega_n(\kappa^2)$ denotes the n-particle phase volume. One obtains in this way, after making an interchange of the $\kappa_{ij}{}^2$ integrals with the n_{ij} sums,

$$\mathbf{F}_{m_1\cdots m_N}(\alpha_{12}\cdots)=\int_0^\infty\prod_{\text{lines}}(d\kappa_{ij}{}^2)\sigma_{m_1\cdots m_N}(\kappa_{12}{}^2,\cdots)e^{-\Sigma\alpha\kappa^2},\quad(4.25)$$

where the *spectral function* σ is given by the finite sum

$$\sigma_{m_1\cdots m_N}(\kappa_{12}{}^2\cdots)=\sum_{n_{12}\cdots}\prod_{\text{lines}}\frac{\theta(\kappa^2-m^2n_{ij}{}^2)\Omega_{n_{ij}}(\kappa_{ij}{}^2)}{n_{ij}!}\times\prod_{\text{vertices}}v(m_k+\sum_{l\neq k}n_{kl}).\quad(4.26)$$

It must be emphasized, however, that the spectral integral (4.25) converges only for *sufficiently large* α_{ij} where it defines an analytic function. If the α's are decreased until a singularity of the function $\mathbf{F}$ is reached, then (4.25) will of course diverge. This happens for the class of theories considered in this paper because the spectral function defined by the sum (4.26) tends to increase like $\exp\kappa^2$.

Although the integral representation (4.25) is not valid for the entire range of the α's, it provides a very useful tool for the analysis of singularities in the momentum-space amplitudes. This is because, according to the integral representation (4.13), only the behavior of $\mathbf{F}(\alpha)$ for *large* α is relevant to the finite p-space structure. Thus if the α-space integrals are divided into two pieces $0<\alpha_{ij}<R$ and $R<\alpha_{ij}<\infty$, then the former yields an *entire* function of the momentum variables while the latter yields the expression

$$\tilde{F}_{m_1\cdots m_N}{}^{R}(p_1\cdots p_N)=\int_R^\infty\prod(d\alpha_{ij})\mathbf{F}_{m_1\cdots m_N}(\alpha_{12}\cdots)\mathbf{C}(\alpha)^{-2}\exp\left(\frac{\mathbf{D}(\alpha,p)}{\mathbf{C}(\alpha)}\right)$$
$$=\int_0^\infty\prod(d\kappa_{ij}{}^2)\sigma_{m_1\cdots m_N}(\kappa_{12}{}^2\cdots)\times\int_R^\infty\prod(d\alpha_{ij})\mathbf{C}(\alpha)^{-2}\exp\left(\frac{\mathbf{D}(\alpha,p)}{\mathbf{C}(\alpha)}-\sum_{i<j}\alpha_{ij}{}^2\kappa_{ij}{}^2\right).\quad(4.27)$$

The integral over α_{ij} contained here approximates to the simple Feynman amplitude corresponding to a diagram with N vertices joined pairwise by lines which correspond to the propagation of bare particles with mass $\kappa_{ij}{}^2$. It certainly has all the usual singularity structure that is proved for perturbation amplitudes. Since the amplitude (4.27) is just a summation of these simple processes weighted by a spectral function which is itself given by a sum of simple phase-space integrals, it seems at least plausible that the requirements of unitarity are met.

It must be remarked, however, that the formal interchange of κ^2 and α integrations employed in arriving at the result (4.27) is not usually permissible. The κ^2 integrals as written are divergent. This difficulty can be met simply by cutting off these integrals at some large mass M^2. Such a cutoff will not affect the singularity structure in any given range of the external momenta if M^2 is taken sufficiently large. One of course takes the limit $M^2\rightarrow\infty$ in the end.