for $\gamma=1$, which is of interest in the analysis of chiral Lagrangians, we have a finite expression. In the limit $\gamma\to 0$ the integrals in (A4) diverge at the lower end point. The Y_0 terms are then found to compensate for the $[\Gamma(\gamma)]^2$ in the denominators and we obtain the expressions

$$Q^{(1)}(z^2)\to\theta_1 \quad \text{for } \gamma\to 0\,,$$
$$Q^{(2)}(z^2)\to\theta_2 \quad \text{for } \gamma\to 0\,, \tag{A5}$$

of the IVB theory.

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Sugawara Model, Broken Symmetries, and Massless-Boson Fields

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It is asserted that the Sugawara theory of currents is equivalent to a canonical Lagranigan theory of massless scalar fields with a Yang-Mills type of interaction. The Sugawara currents are identified with the canonical currents associated with nonlinear transformations of the massless fields. These transformations arise naturally in the context of spontaneous symmetry breakdown. In models of spontaneously broken symmetry in which the symmetry-breaking scalar fields are elementary dynamical variables, that part of the stress-energy tensor containing the Goldstone bosons terms goes over into the Sugawara form when reexpressed in terms of the currents. However, additional terms relating to a massive field appear in the stress-energy tensor, and the massive field operator replaces the usual Sugawara constant. A Sugawara current theory is obtained by eliminating the massive field and retaining the Goldstone boson field only. The reverse is also true in that any Sugawara theory is necessarily equivalent to this canonical representation. Finally, it is shown that if the currents are coupled to gauge fields, a massive Yang-Mills field is obtained.

I. INTRODUCTION

A NUMBER of recent papers[1] have dealt with the suggestion put forward by Sugawara[2] that a dynamical theory could be formulated entirely in terms of currents. In this approach the currents are regarded as the fundamental dynamical variables, and the theory is defined by stipulating the equal-time commutation algebra together with the explicit expression of the stress-energy tensor in terms of the currents. It was hoped that this theory would constitute an alternative to old-fashioned canonical Lagrangian field theories. In a previous paper,[3] however, Lurié and this author have shown that a Sugawara current theory for a single neutral current is completely equivalent to a canonical theory of a free neutral massless scalar field, the current being associated with the transformation

$$\phi\to\phi+\alpha\,. \tag{1.1}$$

In seeking to extend this result to Sugawara current theories based on non-Abelian groups, a hitherto unsuspected connection was found between the Sugawara theory on the one hand and spontaneously broken symmetries and massless particles on the other. That a connection does exist seemed to be implied by the association of the current with the transformation (1.1) It was precisely this transformation that was encountered by Umezawa,[4,5] Sen,[5] and Leplae[5] in their analysis of spontaneous symmetry breakdown.

They have shown how a symmetry operation applied to the basic fields in terms of which the theory is formulated can be dynamically rearranged into an entirely different symmetry operation on the asymptotic or "physical" fields which describe the quasiparticle and collective excitations of the system. As an example they consider the model of Nambu and Jona-Lasinio[6] characterized by the Lagrangian density

$$\mathcal{L}=-\bar\psi\gamma_\mu\partial_\mu\psi-g(\bar\psi\psi)^2+g(\bar\psi\gamma_5\psi)^2\,, \tag{1.2}$$

which exhibits invariance under the simple γ_5 gauge transformation

$$\psi\to e^{i\alpha\gamma_5}\psi \tag{1.3}$$

owing to the vanishing of a fermion bare mass and the γ_5-invariant form of the interaction term. When truncated in the chain or "random-phase" approximation, the above model exhibits a cutoff-dependent self-consistent solution characterized by a finite fermion mass m and a massless pseudoscalar bound state. (This is, in

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[1] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968); D. H. Sharp *ibid.* **165**, 1867 (1968); C. G. Callan, R. F. Dashen, and D. H. Sharp, *ibid.* **165**, 1883 (1968); C. M. Sommerfield, *ibid.* **176**, 2019 (1968).

[2] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[3] Y. Freundlich and D. Lurié, Phys. Rev. C **1**, 1660 (1970).

[4] H. Umezawa, Nuovo Cimento **38**, 1415 (1965); H. Umezawa, *ibid.* **40**, 450 (1965).

[5] R. N. Sen and H. Umezawa, Nuovo Cimento **50**, 53 (1967); L. Leplae, R. N. Sen, and H. Umezawa, *Problems of Fundamental Physics* (Kyoto, 1965), p. 637.

[6] Y. Nambu and C. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

fact, one of the first relativistic examples of Goldstone's theorem[7] linking spontaneous symmetry breakdown with the appearance of massless bosons.) It was then shown by Umezawa[4] that the original symmetry (1.3) for the basic fields is dynamically rearranged into the symmetry (1.1) for the Goldstone boson.

This mechanism seems to provide a very satisfying picture of what happens when a symmetry is spontaneously broken, for it provides a raison d'être for the Goldstone boson.[8] The latter is identified as that particle which carries away the original symmetry lost by the fields present in the Lagrangian. We believe that the correlation of Sugawara current theories and broken-symmetry models stems from this phenomenon.

An analogous result is obtained in Lagrangian models in which the symmetry-breaking scalar fields are elementary dynamic variables. If a symmetry inherent in the Lagrangian is spontaneously broken by imposing nonsymmetric vacuum conditions, massless bosons will appear in the theory. These massless bosons will form a nonlinear realization of the group to which the original symmetry is reduced. All reference to the field operators of the massless particles may then be eliminated in favor of the current operators associated with this nonlinear transformation. In particular, that part of the stress-energy tensor containing the massless fields goes over into the Sugawara form when reexpressed in terms of the currents. This model is, of course, not a "pure" Sugawara model. Reference must still be made to the field operator corresponding to massive particles. By suitable constraints, however, they may be eliminated from the theory, in which case a canonical Lagrangian representation of Sugawara's theory of currents will have been achieved. The crucial point is that the reverse is also ture. Any Sugawara theory is equivalent to the model thus obtained. Finally, it is of interest to note that if the currents are coupled to massless gauge fields, a simple massive Yang-Mills[9] field theory is obtained.

In Sec. II we will follow this prescription to obtain a $SU(2)$ current theory. In Sec. III we consider the equivalence of any $SU(2)$ theory to our model. We then examine an explicit realization of the Sugawara model which has appeared in the literature and exhibit the basic mechanism which is responsible for the equivalence. In Sec. IV vector gauge fields are added, while Sec. V contains a summary and conclusions.

II. FORMULATION OF MODEL

The $SU(2)$ Sugawara theory is defined by the equations

$$\theta_{\mu\nu}=(1/2C)[j_\mu{}^i j_\nu{}^i+j_\nu{}^i j_\mu{}^i-\delta_{\mu\nu}j_\nu{}^i j_\sigma{}^i], \tag{2.1}$$

$$[j_0{}^i(\mathbf{x},t),j_0{}^j(\mathbf{y},t)]=i\epsilon_{ijk}j_0{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{2.2a}$$

$$[j_0{}^i(\mathbf{x},t),j_a{}^j(\mathbf{y},t)]=i\epsilon_{ijk}j_a{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})-iC\delta_{ij}\partial_{a_x}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{2.2b}$$

$$[j_a{}^i(\mathbf{x},t),j_b{}^j(\mathbf{y},t)]=0, \tag{2.2c}$$

where C is a c-number constant.[10]

To obtain a canonical representation of this theory, we follow the procedure outlined in the Introduction. We consider a $U(2)=U(1)\times SU(2)$-symmetric theory and break the symmetry in the $U(1)$ direction. There then appears, in the theory, a triplet of massless bosons which form a nonlinear representation of the $SU(2)$ group. We therefore start with a complex Lorentz scalar and isospinor theory described by the Lagrangian

$$\mathfrak{L}=-\partial_\mu u^\dagger\partial_\mu u+V(u^\dagger u) \tag{2.3}$$

and the equal-time commutation relations (ETCR)

$$[u_\alpha(\mathbf{x},t),\dot{u}_\beta{}^\dagger(\mathbf{y},t)]=i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{2.4a}$$

$$[u_\alpha{}^\dagger(\mathbf{x},t),\dot{u}_\beta(\mathbf{y},t)]=i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{2.4b}$$

the other commutators being zero. This theory is clearly invariant under the constant $U(2)$ gauge transformation

$$u\rightarrow e^{i\phi^\alpha\tau^\alpha/2}u, \tag{2.5}$$

where α runs from 0 to 3, τ_0 is the unit 2×2 matrix, and τ_i are the usual isotopic spin matrices. When spontaneous symmetry breakdown is discussed, it will be convenient to make the polar decomposition[11]

$$u=e^{i\theta^\alpha\tau^\alpha}\chi, \tag{2.6}$$

where θ^α are real, and χ is a complex isospinor.[12] We also note, for further reference, the useful formula

$$\partial_\mu u=e^{i\theta^\alpha\tau^\alpha}(\partial_\mu\chi+i\theta_\mu{}^\alpha\tau^\alpha\chi), \tag{2.7}$$

where

$$\theta_\mu{}^\alpha\tau^\alpha=-ie^{-i\theta^\alpha\tau^\alpha}\partial_\mu(e^{i\theta^\alpha\tau^\alpha})=\theta_\mu{}^0\tau^0+\theta_\mu{}^i\tau^i. \tag{2.8}$$

The Lagrangian (2.3) goes over into

$$\mathfrak{L}=-(\partial_\mu\chi^\dagger\partial_\mu\chi+i\partial_\mu\chi^\dagger\theta_\mu{}^\alpha\tau^\alpha\chi-i\chi^\dagger\theta_\mu{}^\alpha\tau^\alpha\partial_\mu\chi+\chi^\dagger\theta_\mu{}^\alpha\tau^\alpha\theta_\mu{}^\beta\tau^\beta\chi)+V(\chi^\dagger\chi). \tag{2.9}$$

The middle terms correspond to undesirable mixed

[7] J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[8] This picture is marred, however, by the difficulty of identifying the charge as a generator of the symmetry [see Y. Freundlich and D. Lurié, Technion, Israel Institute of Technology report (unpublished)].

[9] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[10] We use $\delta_{\mu\nu}$ as a space-time metric. Fourth components of four vectors are imaginary. Four-dimensional space-time indices are indicated by μ, ν; spatial indices by a, b, $c=1, 2, 3$; and $SU(2)$ indices by i, j, $k=1, 2, 3$.

[11] P. W. Higgs, Phys. Rev. Letters 13, 508 (1964); Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967); Y. S. Kim and F. L. Markley, University of Maryland Report (unpublished). Of course, expressions such as these, to make sense, must be understood as expansions in terms of normal ordered products of the θ^i fields.

[12] Of course, only four of the eight variables are independent. The choice of the independent fields is dictated by the direction in which the symmetry group is broken. See Ref. 11 in connection with this point.

kinetic energy, and we eliminate them by imposing the conditions

$$\chi=(1/\sqrt{2})\rho\hat{\chi}\,, \tag{2.10}$$

where ρ is real and $\hat{\chi}$ is a constant unit isospinor. This together with the further constraint $\theta_0=0$ reduces the number of independent variables to four. In terms of these variables the Lagrangian may be cast into the form

$$\mathfrak{L}=-\tfrac{1}{2}(\partial_\mu\rho\partial_\mu\rho+\rho^2\theta_\mu{}^i\theta_\mu{}^i)+V(\rho^2)\,. \tag{2.11}$$

The broken symmetry condition is expressed by setting

$$\langle\rho\rangle_0=\eta\neq 0\,. \tag{2.12}$$

We finally write the Lagrangian as

$$\mathfrak{L}=-\tfrac{1}{2}[\partial_\mu\rho'\partial_\mu\rho'+(\eta+\rho')^2\theta_\mu{}^i\theta_\mu{}^i]+V(\rho^2)\,,\quad \rho'=\rho-\eta\,, \tag{2.13}$$

from which it is clear that $\eta^2\theta_\mu{}^i\theta_\mu{}^i$ is the kinetic-energy term for the θ^i fields. These fields create massless one-particle states from the vacuum and thus correspond to the Goldstone boson fields. To see this[13] we note that they form a nonlinear realization of the $SU(2)$ group, transforming as

$$e^{i\theta^i\tau^i}\longrightarrow e^{i\phi^i\tau^i/2}e^{i\theta^i\tau^i}; \tag{2.14a}$$

in terms of the infinitesimal transformation this becomes

$$\delta\theta^i=[\delta^{ij}\theta\cot\theta+(1-\theta\cot\theta)\theta^i\theta^j/\theta^2+\epsilon^{ijk}\theta^k]\tfrac{1}{2}\theta^j\,,\quad \theta^2=\theta^i\theta^i\,. \tag{2.14b}$$

The existence of massless particles in the physical spectrum is now proven in a manner completely analogous to the usual proof of the Goldstone theorem.[7]

From Lorentz invariance, we have

$$i\int d^4x\, e^{iqx}\langle 0|[j_\mu{}^i(x),\theta^j(0)]|0\rangle =q_\mu(\rho_{(1)}{}^{ij}(q^2)+\rho_{(2)}{}^{ij}(q^2)\epsilon(q_0))\,, \tag{2.15a}$$

where $j_\mu{}^i$ are the currents associated with the transformation (2.14) and whose forms will be exhibited shortly. From current conservation we conclude

$$\rho_{(1)}{}^{ij}(q^2)=a^{ij}\delta(q^2)\,,\quad \rho_2{}^{ij}(q^2)=b^{ij}(q^2)\delta(q^2)\,, \tag{2.15b}$$

where a^{ij} and b^{ij} are constants.

[13] Often [e.g., the last two authors in Ref. 11; also J. Honerkamp Nucl. Phys. **B12**, 227 (1969)] the existence of massless particles has been deduced from the fact that terms involving θ^i but not $\partial_\mu\theta^i$ do not appear in the Lagrangian. However, a counter example (though one which is not manifestly Lorentz-covariant) is supplied in Sec. IV as is obvious from Eq. (4.8). The proof supplied in the text depends upon the particular form of the symmetry transformation when reexpressed in terms of the θ^i fields. Thus not only does the original symmetry rearrange itself into a symmetry for the Goldstone bosons, but the very form of this symmetry operation reflects the massless character of these particles. We leave open the question of whether the theory allows for massive bound states.

On the other hand, from (2.14b) we have

$$i\phi^j\langle 0|[Q^j,\theta^i]|0\rangle=\langle 0|\delta\theta^i|0\rangle=\delta^{ij}\phi^j\,, \tag{2.16}$$

which combined with (2.15a) and (2.15b) yields

$$a^{ij}=2\pi\delta^{ij}\neq 0\,,$$

and hence the existence of massless one-particle states with the quantum numbers of $j_\mu{}^i$, θ^i.

We now turn our attention to the currents. The conserved $SU(2)$ currents associated with the transformations (2.5) are

$$j_\mu{}^i=\tfrac{1}{2}i(\partial_\mu u^\dagger\tau^i u-u^\dagger\tau^i\partial_\mu u)\,, \tag{2.17a}$$

which in terms of the polar variables are[14]

$$j_\mu{}^i=\tfrac{1}{2}\rho^2\theta'_\mu{}^i\,, \tag{2.17b}$$

$$\theta_\mu'=e^{i\theta}\theta_\mu e^{-i\theta}\,. \tag{2.18}$$

These currents may be used to eliminate all reference to the massless θ^i fields. The stress-energy tensor is

$$\theta_{\mu\nu}=\partial_\mu u^\dagger\partial_\nu u+\partial_\nu u^\dagger\partial_\mu u+\delta_{\mu\nu}\mathfrak{L} \tag{2.19a}$$

$$=\tfrac{1}{2}\{\partial_\mu\rho\partial_\nu\rho+\partial_\nu\rho\partial_\mu\rho-\delta_{\mu\nu}[\partial_\sigma\rho\partial_\sigma\rho-2V(\rho^2)]\} +\tfrac{1}{2}\rho^2(\theta_\mu{}^i\theta_\nu{}^i+\theta_\nu{}^i\theta_\mu{}^i-\delta_{\mu\nu}\theta_\sigma{}^i\theta_\sigma{}^i) \tag{2.19b}$$

$$=\tfrac{1}{2}\{\partial_\mu\rho\partial_\nu\rho+\partial_\nu\rho\partial_\mu\rho-\delta_{\mu\nu}[\partial_\sigma\rho\partial_\sigma\rho-2V(\rho^2)]\} +(2/\rho^2)(j_\mu{}^ij_\nu{}^i+j_\nu{}^ij_\mu{}^i-\delta_{\mu\nu}j_\sigma{}^ij_\sigma{}^i)\,, \tag{2.19c}$$

where we have used

$$\theta_\mu{}^i\theta_\nu{}^i=\theta'_\mu{}^i\theta'_\nu{}^i\,. \tag{2.20}$$

From the ETCR (2.4) and the explicit expression for the current (2.17), we derive the current-current commutation algebra

$$[j_0{}^i(\mathbf{x},t),j_0{}^j(\mathbf{y},t)]=i\epsilon_{ijk}j_0{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{2.21a}$$

$$[j_0{}^i(\mathbf{x},t),j_a{}^j(\mathbf{y},t)]=i\epsilon_{ijk}j_a{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y}) -i\delta_{ij}\partial_{a_x}\tfrac{1}{4}\rho^2\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{2.21b}$$

$$[j_a{}^i(\mathbf{x},t),j_b{}^j(\mathbf{y},t)]=0\,. \tag{2.21c}$$

These must be supplemented with the ETCR

$$[j_0{}^i(\mathbf{x},t),\rho(\mathbf{y},t)]=0=[j_a{}^i(\mathbf{x},t),\rho(\mathbf{y},t)]\,, \tag{2.22a}$$

$$[j_0{}^i(\mathbf{x},t),\rho(\mathbf{y},t)\dot\rho(\mathbf{y},t)]=0\,, \tag{2.22b}$$

$$[j_a{}^i(\mathbf{x},t),\rho(\mathbf{y},t)\dot\rho(\mathbf{y},t)]=2ij_a{}^i(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{2.22c}$$

$$[\rho(\mathbf{x},t),\dot\rho(\mathbf{y},t)]=i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{2.22d}$$

The similarity and contrast of this model with the Sugawara one is obvious. The form of the current-current commutation relations (2.2) and (2.21) in both theories is identical. In our model, however, the Schwinger term contains a q-number field operator. This difference is further expressed by additional terms in the stress-energy tensor (2.19) and in the need for the

[14] Where convenient, we will use a matrix notation. Hereafter θ is to be understood as the 2×2 matrix $\theta^i\tau^i$.

supplementary ETCR (2.22). It is now clear what must be done to formulate a pure Sugawara theory. We merely limit ρ to a constant c number and make the identification

$$\tfrac{1}{4}\rho^2=\tfrac{1}{4}\eta^2=C\,, \tag{2.23a}$$

which is equivalent to the constraint

$$u^\dagger u=2C\,. \tag{2.23b}$$

To sum up, our canonical representation of the $SU(2)$ Sugawara theory consists of interacting massless bosons and is defined by the Lagrangian[15]

$$\mathfrak{L}=-2C\theta_\mu{}^i\theta_\mu{}^i\,, \tag{2.24}$$

where

$$\theta_\mu=-ie^{-i\theta}\partial_\mu e^{+i\theta}\,, \tag{2.8$'$}$$

and by the canonical commutation relations

$$\left[4C\theta_0{}^k(\mathbf{x},t)\frac{\partial\theta_0{}^k(\mathbf{x},t)}{\partial(\partial_0\theta_i)},\,\theta^j(\mathbf{y},t)\right]=-i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{2.25}$$

The Sugawara currents are associated with the transformation

$$e^{i\theta}\longrightarrow e^{i\phi/2}e^{i\theta} \tag{2.14$'$}$$

and are given by

$$j_\mu{}^i=2C\theta'_\mu{}^i\,, \tag{2.17$'$}$$

where

$$\theta'_\mu=e^{i\theta}\theta_\mu e^{-i\theta}\,. \tag{2.18}$$

The equations of motion are

$$\partial_\mu\theta_\mu{}^i=\partial_\mu\theta'_\mu{}^i=0\,. \tag{2.26}$$

III. EQUIVALENCE THEOREM

We assert that any $SU(2)$ Sugawara theory must necessarily be equivalent to this model. To see this we apply the Heisenberg equations of motion

$$\partial_\mu j_\nu{}^i=i[j_\nu{}^i,P_\mu]\,, \tag{3.1}$$

with

$$P_\mu=-i\int\theta_{4\mu}(x)d^3x \tag{3.2}$$

to derive the equations of motion

$$\partial_\mu j_\nu-\partial_\nu j_\mu=(i/2C)[j_\mu,j_\nu]\,, \tag{3.3a}$$

$$\partial_\mu j_\mu=0\,. \tag{3.3b}$$

Equation (3.3a) is reduced to an identity by setting

$$j_\mu=2C\theta_\mu'=2Cie^{i\theta}\partial_\mu e^{-i\theta}\,. \tag{3.4}$$

Moreover, Bardakci and Halpern[15] have shown that this is the most general solution to Eq. (3.3a). Equation (3.3b) is the Lagrangian equation of motion (2.26) for the massless scalar fields. To establish complete equivalence we need merely show that the current algebra (2.2) implies the canonical commutation relations for the θ^i fields when the $j_\mu{}^i$ are given by Eq. (3.4).

To this end, we rewrite Eqs. (2.2) as

$$[j_0{}^i(\mathbf{x},t),j_0{}^j(\mathbf{y},t)\tau^j]=-\tfrac{1}{2}[\tau^i,j_0{}^j\tau^j]\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.5a}$$

$$[j_0{}^i(\mathbf{x},t),j_a{}^j(\mathbf{y},t)\tau^j]=-\tfrac{1}{2}[\tau^i,j_a{}^j\tau^j]\delta^{(3)}(\mathbf{x}-\mathbf{y})-iC\tau^i\partial_{a_x}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.5b}$$

and using Eq. (3.4) for the current we may derive[16]

$$[j_0{}^i(\mathbf{x},t),e^{i\theta(\mathbf{y},t)}]=-\tfrac{1}{2}\tau^i e^{i\theta}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.6a}$$

$$[j_0{}^i(\mathbf{x},t),\partial_0 e^{i\theta(\mathbf{y},t)}]=-\tfrac{1}{2}\tau^i\partial_0 e^{i\theta}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{3.6b}$$

To derive the canonical commutation relation, we must write an explicit expression for the canonical momenta. Accordingly, we adopt matrix notation and use the following definitions:

$$e^{-i\theta}\tau^i e^{i\theta}=D_{ij}(\theta)\tau^j\,, \tag{3.7a}$$

$$-ie^{-i\theta}\frac{\partial}{\partial\theta^i}e^{i\theta}=\Theta_{ij}\tau^j\,. \tag{3.7b}$$

From these we derive

$$\theta_\mu{}^i=\Theta_{ji}\partial_\mu\theta^j\,, \tag{3.8a}$$

$$\theta'_\mu{}^i=D_{ij}\Theta_{kj}\partial_\mu\theta^k\,, \tag{3.8b}$$

and from the Lagrangian (2.24) we have

$$\pi^i=\frac{\partial\mathfrak{L}}{\partial(\partial_0\theta^i)}=4C\Theta_{ij}\theta_0{}^j\,, \tag{3.9}$$

so that

$$j_0{}^i=-2CD_{ij}\theta_0{}^j=-\tfrac{1}{2}D_{ij}\Theta^{-1}{}_{jk}\pi^k\,. \tag{3.10}$$

Using these equations and Eq. (3.6), we derive

$$\begin{aligned}-\tfrac{1}{2}\tau^i e^{i\theta}\delta^{(3)}(\mathbf{x}-\mathbf{y})&=-\tfrac{1}{2}e^{i\theta}D_{il}\tau^l\delta^{(3)}(\mathbf{x}-\mathbf{y})\\&=-\tfrac{1}{2}D_{ij}\Theta^{-1}{}_{jk}\Theta_{kl}e^{i\theta}\tau^l\delta^{(3)}(\mathbf{x}-\mathbf{y})\\&=\tfrac{1}{2}iD_{ij}\Theta^{-1}{}_{jk}\frac{\partial e^{i\theta}}{\partial\theta^k}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\end{aligned} \tag{3.11a}$$

while

$$[j_0{}^i(\mathbf{x},t),e^{i\theta(\mathbf{y},t)}]=-\tfrac{1}{2}D_{ij}\Theta^{-1}{}_{jk}[\pi^k(\mathbf{x},t),e^{i\theta(\mathbf{y},t)}] \tag{3.11b}$$

When these two equations are compared, it is clearly seen that

$$[\pi^i(\mathbf{x},t),\theta^j(\mathbf{y},t)]=-i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{3.12}$$

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15 A canonical representation of the Sugawara theory consisting of the Lagrangian (2.24) with the constraint (2.23) was also found by Bardakci and Halpern [Phys. Rev. **172**, 1542 (1968)] using completely different methods.

16 From (3.5b) we derive (3.6a) which in turn allows us to derive (3.6b) from (3.5a).

We conclude that the Sugawara theory is necessarily equivalent to the interacting massless scalar field theory defined by Eqs. (2.24), (2.8), and (2.25).

In view of this theorem, it is of interest to examine an explicit realization of the Sugawara model and to exhibit the basic mechanism which is responsible for the equivalence. We have in mind the formal zero-mass limit of a massive Yang-Mills vector theory considered by Bardakci, Frishman, and Halpern.[17]

The massive $SU(2)$ Yang-Mills theory is defined by the equations of motion

$$F_{\mu\nu}{}^i=\partial_\mu B_\nu{}^i-\partial_\nu B_\mu{}^i+g\epsilon_{ijk}B_\mu{}^jB_\nu{}^k\,, \tag{3.13a}$$

$$\partial_\mu F_{\mu\nu}{}^i-m^2B_\nu{}^i=g\epsilon_{ijk}F_{\mu\nu}{}^jB_\mu{}^k\,, \tag{3.13b}$$

with the commutation relations

$$[B_a{}^i(\mathbf{x},t),B_b{}^j(\mathbf{y},t)]=0\,, \tag{3.14a}$$

$$[B_a{}^i(\mathbf{x},t),iF_{b4}{}^j(\mathbf{y},t)]=-i\delta_{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.14b}$$

$$[F_{a4}{}^i(\mathbf{x},t),F_{b4}{}^j(\mathbf{y},t)]=0\,, \tag{3.14c}$$

and the stress-energy tensor

$$\theta_{\mu\nu}=\tfrac{1}{2}[F_{\mu\rho}{}^iF_{\nu\rho}{}^i+F_{\nu\rho}{}^iF_{\mu\rho}{}^i+m^2B_\mu{}^iB_\nu{}^i+m^2B_\nu{}^iB_\mu{}^i]-\delta_{\mu\nu}[\tfrac{1}{4}F_{\rho\sigma}{}^iF_{\rho\sigma}{}^i+\tfrac{1}{2}m^2B_\sigma{}^iB_\sigma{}^i]. \tag{3.15}$$

The commutation relations (3.14) imply

$$[B_0{}^i(\mathbf{x},t),B_0{}^j(\mathbf{y},t)]=i(g/m^2)\epsilon_{ijk}B_0{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.16a}$$

$$[B_0{}^i(\mathbf{x},t),B_a{}^j(\mathbf{y},t)]=i(g/m^2)\epsilon_{ijk}B_a{}^k(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})-(i/m^2)\delta_{ij}\partial_{a_x}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.16b}$$

The limit to be taken is obtained by redefining the fields

$$\hat{B}_\mu{}^i=(m^2/g)B_\mu{}^i\,, \tag{3.17a}$$

$$\hat{F}_{\mu\nu}{}^i=(m^2/g)F_{\mu\nu}{}^i \tag{3.17b}$$

and demanding that $\hat{B}_\mu{}^i$ remain finite in the limit $m\rightarrow 0$ and $g\rightarrow 0$ with $m^2/g^2=C$, the constant in Sugawara's theory. Applying the transformation (3.17) to the commutation relations (3.14) and (3.16), we obtain

$$[\hat{B}_a{}^i(\mathbf{x},t),\hat{B}_b{}^j(\mathbf{y},t)]=0\,, \tag{3.18a}$$

$$[\hat{B}_a{}^i(\mathbf{x},t),i\hat{F}_{b4}{}^j(\mathbf{y},t)]=-i(m^4/g^2)\delta_{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}\cdot\mathbf{y})\,, \tag{3.18b}$$

$$[\hat{F}_{a4}{}^i(\mathbf{x},t),\hat{F}_{b4}{}^j(\mathbf{y},t)]=0\,, \tag{3.18c}$$

$$[\hat{B}_0{}^i(\mathbf{x},t),\hat{B}_0{}^j(\mathbf{y},t)]=i\epsilon_{ijk}\hat{B}_0{}^k\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.18d}$$

$$[\hat{B}_0{}^i(\mathbf{x},t),\hat{B}_a{}^j(\mathbf{y},t)]=i\epsilon_{ijk}\hat{B}_a{}^k\delta^{(3)}(\mathbf{x}-\mathbf{y})-i(m^2/g^2)\delta_{ij}\partial_{a_x}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.18e}$$

In the limit described above, $\hat{F}_{a4}$ commutes with all the canonically independent variables and is therefore a c number. Since $\langle F_{a4}\rangle_0=0$, it follows that $\hat{F}_{a4}\rightarrow 0$ and from (3.18b) vanishes like m^2. By Lorentz invariance all the components of $\hat{F}_{\mu\nu}$ vanish like m^2, and in this limit, the stress-energy tensor (3.15) reduces to

$$\theta_{\mu\nu}=(1/2C)[\hat{B}_\mu{}^i\hat{B}_\nu{}^i+\hat{B}_\nu{}^i\hat{B}_\mu{}^i-\delta_{\mu\nu}\hat{B}_\sigma{}^i\hat{B}_\sigma{}^i]. \tag{3.19}$$

The reason for this reduction of a massive Yang-Mills theory in this formal limit to a massless scalar field theory is that the massive Yang-Mills fields may be decomposed into a set of interacting massless transverse fields and massless scalar fields. These fields decouple in the limit described above. However, the stipulation that the transformed fields $\hat{B}_\mu$ remain finite in this limit is equivalent to discarding the vector components while retaining only the scalar ones. Rather than carry out this program directly by decomposing the massive Yang-Mills fields, we will approach the problem from the opposite point of view. It will be more instructive to show that the gauge-invariant interaction of massless vector fields with our massless scalar fields leads to the Yang-Mills fields. We will then reexamine the limit of Bardakci *et al.*

IV. ADDITION OF GAUGE FIELDS

We wish to couple the Sugawara currents to gauge fields and generalize the nonlinear transformations involved to include space-time dependence. We do this first for a single neutral current theory where the ideas are simpler. The Sugawara theory for a single neutral current is equivalent[3] to a free massless scalar field theory consisting of the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}\partial_\mu\theta\partial_\mu\theta \tag{4.1}$$

and the current

$$j_\mu=\partial_\mu\theta \tag{4.2}$$

associated with the symmetry transformation

$$\theta\rightarrow\theta+\alpha\,. \tag{4.3}$$

This transformation may be considered a nonlinear realization of the gauge transformation

$$S\rightarrow e^{i\alpha}S \tag{4.4}$$

obtained by applying the constraint $S^*S=1$ and defining

$$S=e^{i\theta}. \tag{4.5}$$

The extension of the transformation (4.4) to a space-time–dependent transformation is obtained by replacing the ordinary derivative in the Lagrangian

$$\mathcal{L}_0=-\tfrac{1}{2}\partial_\mu S^*\partial_\mu S \tag{4.6}$$

with the covariant derivative

$$D_\mu S=(\partial_\mu-igA_\mu)S\,. \tag{4.7}$$

The full Lagrangian, in terms of the polar variable θ, is then

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu\theta-gA_\mu)(\partial_\mu\theta-gA_\mu)-\tfrac{1}{4}f_{\mu\nu}f_{\mu\nu}\,, \tag{4.8a}$$

where

$$f_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu; \tag{4.8b}$$

[17] K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).

it is invariant under the combined transformation

$$\theta(x) \to \theta(x) + g\alpha(x)\,, \tag{4.9a}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)\,. \tag{4.9b}$$

By means of the simple substitution

$$B_\mu = A_\mu - (1/g)\partial_\mu \theta\,, \tag{4.10}$$

the Lagrangian (4.8) goes over into

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial_\mu B_\nu - \partial_\nu B_\mu) - \tfrac{1}{2}g^2 B_\mu B_\mu\,, \tag{4.11}$$

which is the Lagrangian for a free massive vector field of mass g. Upon verifying the commutation relations, we must remember that our choice for the kinematic part of the massless vector field in (4.8) implies that we are working in the radiation gauge and the commutators must be chosen accordingly. Using the commutators

$$[A_a(\mathbf{x},t), if_{b4}(\mathbf{y},t)] = -i(\delta_{ab} - \partial_a\partial_b/\nabla^2)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{4.12a}$$

$$[\theta(\mathbf{x},t), (\delta_{ab} - \partial_a\partial_b/\nabla^2) f_{b4}(\mathbf{y},t)] = 0\,, \tag{4.12b}$$

$$[\theta(x,t), \dot\theta(\mathbf{y},t) + gA_0(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{4.12c}$$

and the equation of motion

$$\partial_a f_{a4} = ig(\dot\theta + gA_0)\,, \tag{4.13}$$

we may derive

$$[B_a(\mathbf{x},t), if_{b4}(\mathbf{y},t)] = -i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,. \tag{4.14}$$

Coupling the $SU(2)$ currents to gauge fields is a bit more complicated, but the procedure is analogous to the one outlined above. The model defined by Eqs. (2.24), (2.8), and (2.25) is invariant under the transformation (2.14). As previously shown, this transformation may be considered a nonlinear realization of the gauge transformation

$$u \to e^{ig\phi^i\tau^i/2} u \tag{4.15}$$

obtainied by applying the constraint $u^\dagger u = 2C$ and defining

$$u = (2C)^{1/2} e^{i\theta^i \tau^i} \chi\,. \tag{4.16}$$

The transformation (4.15) is extended to include space-time transformation by substituting the covariant derivative

$$D_\mu u = (\partial_\mu - \tfrac{1}{2} ig A_\mu{}^i \tau^i) u \tag{4.17}$$

for the ordinary derivative in the Lagrangian

$$\mathcal{L}_0 = -2C\partial_\mu u^\dagger \partial_\mu u\,. \tag{4.18}$$

The total Lagrangian in terms of the polar variables is therefore

$$\mathcal{L} = -2C(\theta_\mu{}^i - \tfrac{1}{2}gA'_\mu{}^i)(\theta_\mu{}^i - \tfrac{1}{2}gA'_\mu{}^i) - \tfrac{1}{4}F_{\mu\nu}{}^i F_{\mu\nu}{}^i\,, \tag{4.19a}$$

where[13]

$$A'_\mu = e^{-i\theta} A_\mu e^{+i\theta}\,, \tag{4.19b}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \tfrac{1}{2} ig[A_\mu, A_\nu]\,. \tag{4.19c}$$

This Lagrangian is invariant under the space-time-dependent transformations

$$e^{i\theta} \to e^{ig\phi/2} e^{i\theta}\,, \tag{4.20a}$$

$$A_\mu \to e^{ig\phi/2}(A_\mu + (2/g)\phi_\mu(g)) e^{-ig\phi/2}\,, \tag{4.20b}$$

with

$$\phi_\mu(g) = -ie^{-ig\phi/2}\partial_\mu e^{ig\phi/2}\,. \tag{4.20c}$$

Introducing the new variables

$$B_\mu = e^{-i\theta} A_\mu e^{i\theta} - (2/g)\theta_\mu \tag{4.21}$$

transforms the theory into a massive Yang-Mills field theory.[18] To see this, we first note that

$$F_{\mu\nu} = e^{i\theta} F_{\mu\nu}{}^B e^{-i\theta}\,, \tag{4.22a}$$

where

$$F_{\mu\nu}{}^B = \partial_\mu B_\nu - \partial_\nu B_\mu - \tfrac{1}{2} ig[B_\mu, B_\nu]\,, \tag{4.22b}$$

and we have used the identities

$$\partial_\mu \theta_\nu - \partial_\nu \theta_\mu + i[\theta_\mu, \theta_\nu] = 0 \tag{4.23a}$$

and

$$\partial_\mu(e^{i\theta}\rho e^{-i\theta}) = e^{i\theta}(\partial_\mu \rho + i[\theta_\mu, \rho])e^{-i\theta}\,, \tag{4.23b}$$

which is true for arbitrary ρ^i,

The Lagrangian (4.19) can thus be written as

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}{}^{Bi} F_{\mu\nu}{}^{Bi} - \tfrac{1}{2}g^2 C B_\mu{}^i B_\mu{}^i\,, \tag{4.24}$$

since

$$F_{\mu\nu}{}^i F_{\mu\nu}{}^i = F_{\mu\nu}{}^{Bi} F_{\mu\nu}{}^{Bi}\,.$$

This is, of course, the Lagrangian for a massive Yang-Mills field of mass $m^2 = g^2 C$. We further note that there are no longer any massless particles in the theory. This is in line with the work of Higgs and Kibble,[11] who have formulated the idea that massless vector gauge particles may absorb the Goldstone bosons to become massive vector particles, with the Goldstone bosons providing the longitudinal mode.

We return to the considerations of Sec. III. We can now understand why the limit of Bardakci *et al.* leads to our massless scalar field theory. Their stipulation that the transformed fields

$$\hat{B}_\mu = (m^2/g) e^{-i\theta} A_\mu e^{i\theta} - (2m^2/g^2)\theta_\mu \tag{4.25}$$

remain finite in the m, $g \to 0$ limit with $m^2/g^2 = C$

[18] D. Boulware and W. Gilbert [Phys. Rev. **126**, 1863 (1962)] have shown that in the zero-mass limit the transverse part goes over to a massless vector field in the radiation gauge, while the longitudinal part goes over to a massless scalar field. Similarly, R. Finkelstein and L. Staunton [Ann. Phys. (N. Y.) **54**, 97 (1969)] have pointed out that the massive Yang-Mills field may be considered the Lagrangian equivalent of a coupled massless pseudoscalar and vector field. Here we wish to stress the relation of this phenomenon to the Sugawara model on the one hand and the Goldstone bosons on the other. In particular, we show why the Yang-Mills field, in the formal limit of Bardakci *et al.* yields a realization of the Sugawara model, implying, according to our theorem, its reduction to a massless scalar field theory.

effectively discards the transverse modes of the vector field and retains only the longitudinal ones.

V. SUMMARY AND CONCLUSIONS

We have shown that dynamical theory of currents of the kind suggested by Sugawara is equivalent to a canonical theory of massless scalar particles, provided that the currents are associated with nonlinear transformations of the field. These nonlinear transformations arise naturally in the context of spontaneous breakdown of symmetry, and it is interesting to note that in Lagrangian models of broken symmetry the Goldstone bosons fulfill the Sugawara criteria. This allows for a general procedure for obtaining canonical representations of Sugawara models for any group, and we have exhibited this mechanism specifically for a $SU(2)$ model. Finally, we have shown that if the Sugawara currents are coupled to gauge fields, the resulting theory is a massive Yang-Mills one.

We conclude that Sugawara models as formulated contain massless scalar particles and as such are unrealistic. We also conclude that any attempt to resolve the contradiction involved in associating a massive scalar particle with a Goldstone boson must fail, since this procedure removes the scalar particle altogether.[19]

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[19] This phenomenon appears also in theories in which the symmetry-breaking scalar fields are not elementary dynamic variables but bilinear combinations of Fermi fields. Y. Freundlich and D. Lurié, Nucl. Phys. (to be published).

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Momentum-Space Behavior of Integrals in Nonpolynomial Lagrangian Theories

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Methods are developed for constructing momentum-space amplitudes corresponding to nonpolynomial nonderivative interactions of a real scalar field. The methods give rise to a supergraph technique and rules for writing down matrix elements very similar to Feynman techniques. The methods are not established rigorously; at several points the argument requires certain analytic properties of Feynman integrands which, though plausible, can only be demonstrated rigorously for the zero-mass case. Asymptotic behavior, both in spacelike and timelike directions, is discussed. Rough arguments are given that indicate that the singularity structure of the amplitudes is likely to be consistent with unitarity.

I. INTRODUCTION

IF it is to have any future, Lagrangian field theory must learn to cope with nonrenormalizable interactions. This becomes apparent when one examines what we currently believe are Lagrangians of physical interest.

1. These Lagrangians include the following.

(a) *Chiral $SU(2)\times SU(2)$ Lagrangians for strong interactions.* A typical example is Weinberg's Lagrangian for π mesons:

$$\mathcal{L}=(\partial_\mu\phi)^2/(1+f\phi^2)^2.$$

(b) *Intermediate-boson-mediated weak Lagrangian.* An example is an intermediate neutral vector meson U_μ interacting with quarks Q. As is well known in Stückelberg's representation $(U_\mu=A_\mu+\kappa^{-1}\partial_\mu B)$, $\mathcal{L}_{\text{int}}$ can be written in the typical form

$$\mathcal{L}_{\text{int}}=f\bar{Q}\gamma_\mu(1+\gamma_5)QA_\mu+m\bar{Q}(e^{i\gamma_5(f/\kappa)B}-1)Q.$$

(c) *The gravitational Lagrangian of Einstein* expressed in terms of the contravariant tensor $g^{\mu\nu}$

$$L=\kappa^{-2}(\sqrt{-g})g^{\mu\nu}(\Gamma_{\mu\rho}{}^\lambda\Gamma_{\nu\lambda}{}^\rho-\Gamma_{\mu\nu}{}^\lambda\Gamma_{\lambda\rho}{}^\rho),$$

where

$$\Gamma_{\mu\nu}{}^\lambda=\tfrac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}).$$

The covariant components $g_{\mu\nu}$ which enter the expression for $g=\det g_{\alpha\beta}$ are expressed as a ratio of two polynomials in $g^{\mu\nu}$.

The interaction Lagrangians in all these theories are typically of a nonpolynomial form in field variables. These Lagrangians can be expanded in power series of the type

$$\mathcal{L}_{\text{int}}(\phi)=G\sum_n\frac{v(n)}{n!}(-\phi)^n. \tag{1.1}$$

(Here ϕ is a scalar field, and for simplicity we are ignoring derivatives.) The coefficients $v(n)$ are proportional to f^n, where f is a coupling constant.[1] All terms in such

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[1] In this paper we distinguish between the coupling constants G and f: G will be called the *major* coupling constant and f the *minor*. We shall be considering fixed order in G and all orders in f.