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Superselection Rule for Charge*

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The customary notion of superselection rule stipulates the vanishing of those matrix elements of the density matrix that connect states separated by the superselection rule, i.e., the relative phases of which are, according to the superselection rule, meaningless. It is shown that, if any two states for which an additive conserved quantity has different values are separated for all physical systems by a superselection rule at one time, the separation will persist for all times. It is concluded that the states with different electric (or baryonic) charges are so separated. The reason for the difference between electric charge and other additive conserved quantities, such as momentum, is brought out.

INTRODUCTION

IT has been recently claimed that coherent superpositions of states with different charges can be produced,[1] i.e., that there is no charge superselection rule. The same question was discussed before[2] and the opposite conclusion arrived at. Since, however, the earlier discussion's conclusion was part of a much more general argument, it may be worthwhile to repeat it in the much simpler form in which it applies only to the possibility of the production of states which are superpositions of states with different charges.

The argument of the authors[1] amounts to the following. They consider systems C, composed of two subsystems A and B, and the time evolution of a state of C under a charge-conserving interaction. A is supposed to be, initially, in a state consistent with the superselection rule. The authors then show that after some time it can be arranged that A is in a superposition of states with different charges, i.e., in a state violating the charge superselection rule, *provided that B is initially in a state that is a coherent superposition of states with different charges.* It was one of the main points of the previous paper[2] that nature does not seem to admit the italicized assumption. The principal point of the present paper is to show that the italicized assumption is *necessary* in order that the state of A should evolve from a state satisfying, into a state violating, the charge superselection rule. Hence, we conclude that all present evidence points to the universal validity of the charge superselection rule.

In addition to the assumption concerning the initial state of all objects, mentioned before, we shall use two other assumptions which may as well be stated here. The first of these is the conservation law for the electric charge. The second one is less commonly stated explicitly: It refers to the charge operators of the two subsystems into which a composite system can be separated. If we denote the Hilbert spaces of the two subsystems by H_1 and H_2, the Hilbert space of the composite system will be the direct product $H_1 \times H_2$ of these. We assume that the charge operator Q in this composite system is the sum $Q = Q^1 + Q^2$ of the charge operators which give the charges of the two subsystems and, further, that the expectation value of the charge in the first subsystem is, for a pure state thereof, independent of the state of the second subsystem, and conversely.

PROPERTIES OF CHARGE OPERATOR

It may be appropriate to start the discussion by deriving the form of the charge operator of a composite system and to relate it to the charge operators Q^1 and Q^2 of the constituent systems. As one expects, the form of the latter will be[3]

$$Q^1 = Q_1 \times 1, \quad Q^2 = 1 \times Q_2. \tag{1}$$

The first factor of both expressions on the right side operate in the Hilbert space H_1 of the first, the second factor in the Hilbert space H_2 of the second subsystem. In order to prove (1), one has only to consider states $\Phi \times \Psi$ of the composite system for which both subsystems individually are in pure states. The expectation value of the charge operator Q^1 of the first subsystem must be, then, according to our assumption, independent of the state Ψ of the second subsystem. Introducing a discrete coordinate system in both Hilbert spaces H_1 and H_2, and denoting the coordinate axes in the first one by $\alpha, \beta, \ldots$, in the second one by $a, b, \ldots$, our postulate for Q^1 demands that the left side of

$$\frac{\sum \Phi_\alpha^* \Psi_a^* (Q^1)_{\alpha a;\beta b} \Phi_\beta \Psi_b}{\sum \Psi_a^* \Psi_a} = \sum (Q_1)_{\alpha\beta} \Phi_\alpha^* \Phi_\beta \tag{2}$$

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[1] R. Mirman, Phys. Rev. **186**, 1380 (1969); Y. Aharonov and L. Susskind, *ibid.* **155**, 1428 (1967).

[2] E. P. Wigner, in *Physikertag. Hauptvortraege Jahrestag. Verbandes Deut. Physik. Ges.* (Physik Verlag, Mosbach/Baden, 1962). The superselection rule for electric charge was first proposed by G.-C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

[3] The reader who is familiar with (1), or is willing to take it for granted, should skip the present section.

be independent of Ψ. All summations in (2) are over all indices, α, a, β, and b, which occur in the expression following the summation sign. In order to obtain (2), we note that since the left side of (2) is independent of Ψ, we can equate it with its value for $\Psi_a=\delta_{a1}$. Denoting then $(Q^1)_{\alpha 1\,\beta 1}$ by $(Q_1)_{\alpha\beta}$, one obtains (2), valid for all Φ and Ψ. Multiplication of (2) with the denominator of the left side gives

$$\sum \Phi_\alpha{}^*\Psi_a{}^*(Q^1)_{\alpha a;\beta b}\Phi_\beta\Psi_b=\sum (Q_1)_{\alpha\beta}\delta_{ab}\Phi_\alpha{}^*\Phi_\beta\Psi_a{}^*\Psi_b. \quad (2')$$

Both sides of (2′) are Hermitian quadratic forms of Ψ. They can be equal only if the matrices of the two forms are equal. Hence, one can drop the summation over a and b if one also drops the Ψ factors on both sides. One then obtains Hermitian forms of Φ on both sides; these can be equal again only if the matrices are equal. Hence, (2′) implies

$$(Q^1)_{\alpha a;\beta b}=(Q_1)_{\alpha\beta}\delta_{ab}, \quad (3)$$

which is the explicit form of the first part of (1). The second part thereof,

$$Q^2=1\times Q_2, \quad (3')$$

is obtained in the same way.

DEMONSTRATION OF SUPERSELECTION RULE

Before proceeding with the demonstration of the superselection rule for charge, let us ascertain what its explicit form is for both a pure case and a mixture. In order to do this, we introduce particular coordinates in our Hilbert spaces. These will have double indices q and n, the first giving the value of the charge (i.e., the characteristic value of the charge operator Q, or Q^1 or Q^2, whichever Hilbert space we consider); the second, n, distinguishes between the various states with the same q. Evidently, the charge operators have highly degenerate characteristic values and n enumerates the degeneracy. The most general pure state compatible with the charge superselection rule is then (by definition)

$$\Psi_{qn}=\delta_{qq_0}\psi_n, \quad (4)$$

where q_0 is the charge of the state. Hence, the form of the most general density matrix becomes

$$R_{qn;q'n'}=\delta_{qq'}\rho_{nn'}{}^q, \quad (5)$$

where the $\rho_{nn'}{}^q$ are positive semidefinite Hermitian matrices; the sum of their traces is 1. Conversely, every density matrix of the form (5) is compatible with the charge superselection rule, since it can be considered as a mixture of pure states of the form (4). In order to decompose it into such states, one first decomposes a particular ρ^q, e.g., the ρ with $q=q_0$, into states (4); the ψ_n thereof is a characteristic vector of the ρ^{q_0}. The pure states contained in the mixture R are then all characteristic vectors of all ρ^q in (5), the characteristic values of which do not vanish. What we shall have to prove, therefore, is that the density matrices of all states which can result from states satisfying our postulates are of the form (5), i.e., do commute with the charge operator. We shall call a density matrix of the form (5), i.e., a density matrix consistent with the charge superselection rule, a permissible density matrix.

We now proceed with the demonstration. Three processes will be considered: the union of two systems into a composite system, the change of a system (composite or not) in time, and the separation of a composite system into its parts. Each set of events can be decomposed into a succession of these three processes. Since a permissible density matrix will be replaced by one or more permissible density matrices by each of the three processes, the same will hold for any succession of them and thus for any set of events. However, when proceeding with the decomposition it is important not to replace a composite system's density matrix with the density matrices of its constituents if these will interact again at a later time, i.e., if they are to be united again into a composite system. This means that the separation of composite systems into their constituents should always be carried out last. The reason for this is that, when considering the process of the union of two systems, it will be assumed that these were, before the union, statistically independent, i.e., that the density matrix of the composite system is the direct product of the density matrices of the constituents. As is well known, this is not true, in general, if they had interacted before either directly or indirectly. This point has been greatly emphasized by Aharonov and Susskind.[4] The point is that the density matrix of the composite system (or its state vector if it is in a pure state) contains more information than the density matrices of the constituents. One replaces the density matrix of the composite system by those of the constituents if one wishes to restrict one's attention, henceforth, to one of the constituents and ascertain, for instance, that it is in a permissible state. Hence, the postponement of the separation process, so as not to be followed by any process of uniting two subsystems, does not affect the validity of the demonstration which follows.

Let us now consider each of the three processes separately. That the first two of them do not change the permissible nature of density matrices is pretty obvious. In particular, if R^1 and R^2 are permissible, their direct product is also. This can be seen either by writing out the matrix for the direct product, with four symbols labeling both rows and columns, or by observing that if R^1 commutes with Q^1, and R^2 commutes with Q^2, their direct product $R^1\times R^2$ will commute with the Q of (1), i.e., with $Q^1\times 1+1\times Q^2$. As to the second point, the conservation law for R means that R commutes with the Hamiltonian. It then also commutes with the

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[4] Y. Aharonov and L. Susskind, Ref. 1. This article already contains the form (5) for the "permissible" density matrix.

time displacement operator $U=e^{-iHt/\hbar}$. Hence, if Q commutes with R, it also will commute with URU^{-1} since it commutes with each of the three factors.

Only the verification of the third statement is not present in commonly used form and we presume that this is the statement which is being questioned. It claims that the separation of a composite system with a permissible state vector or density matrix into its two components results in permissible density matrices for each. The point is that the density matrix for the composite system may not have the form of a direct product of density matrices referring to the constituent systems. It can have arisen, for instance, from such a direct product by the time development operator of the composite system. However, the verification is straightforward also in this case.

We again introduce a coordinate system in the Hilbert space $H_1\times H_2$ of the composite system the axes of which are direct products of orthonormal vectors in the component spaces. The labeling αa, etc., used in (2) is then applicable for the axes chosen. We further choose the axes a, b, ..., in H_2 to be characteristic vectors of the charge operator Q_2 in H_2 so that the labels a, b, ..., are appropriately replaced by double labels qn, where q is the charge of the state. We similarly replace the labels α, β, ..., by double labels $\kappa\nu$, the first of which, κ, is again the charge of the corresponding state, i.e., the characteristic value of Q_1 of the unit vector labeled by $\kappa\nu$ in the Hilbert space H_1. The elements of the density matrix in the composite Hilbert space will then have eight labels, such as $R_{\kappa\nu qn;\kappa'\nu'q'n'}$. The most general permissible density matrix in $H_1\times H_2$ will have the form

$$R_{\kappa\nu qn;\kappa'\nu'q'n'}=\delta_{\kappa+q;\kappa'+q'}\rho_{\nu n;\nu'n'}{}^{\kappa q\kappa'q'}. \tag{6}$$

The density matrix R^2 of the second component will be

$$\begin{aligned} R^2{}_{qn;q'n'} &= \sum_{\kappa\nu} R_{\kappa\nu qn;\kappa\nu q'n'} \\ &= \sum_{\kappa\nu} \delta_{\kappa+q;\kappa+q'}\rho_{\nu n;\nu n'}{}^{\kappa q\kappa q'}. \end{aligned} \tag{7}$$

Since $\delta_{\kappa+q;\kappa+q'}=\delta_{qq'}$, the R^2 of (7) indeed has the permissible form (5) with the second factor on the right side equal to

$$\rho_{nn'}{}^{q}=\sum_{\kappa\nu}\rho_{\nu n;\nu n'}{}^{\kappa q\kappa q}. \tag{7'}$$

A similar calculation of the density matrix of the first constituent of the composite system leads, naturally, to the same conclusion.

CONCLUSION

The above concludes the demonstration. It may be well, however, to reemphasize the critical assumption on which our analysis is based: that we have, to begin with, only "permissible" states, i.e., that we have no states naturally given which are superpositions (rather than mixtures) of states with different charges. There is, in this regard, a fundamental difference between conserved quantities, such as linear and angular momentum on the one hand,[5] and electric (and baryonic) charge on the other. We have naturally been given states which are superpositions of states with different momenta; all more or less localized states are of this nature. In particular, all macroscopic systems are. We also have an interpretation of the relative phases of these states: They give the position in space. We were not naturally given states which are superpositions of states with different charges (nor do we know how the relative phases could be interpreted). What was called skew information[6] is well available with respect to momentum and it has a classical interpretation. This is not true with respect to charge, electric or baryonic. We do not know the deeper reason for this difference, but it seems to be present.

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[5] Cf. Ref. 2.

[6] E. P. Wigner and M. Yanase, Proc. Natl. Acad. Sci. U. S. **49**, 910 (1963). The more general statement referred to in the Introduction (and Ref. 2) is that if, with respect to an additive conserved quantity, the skew information is zero to begin with, it will remain zero throughout.