Relativistic Quantum Mechanics of Two Interacting Particles*

C. FRONSDAL† AND L. E. LUNDBERG‡
International Centre for Theoretical Physics, Trieste, Italy

(Received 5 February 1970)

A variation of the quasipotential approach of Logunov and Tavkhelidze is investigated. Two relativistic particles with or without spin are subjected to a mutual interaction that can be described (i) as generated by the exchange of field quanta or (ii) as a two-particle potential $V(\mathbf{r})$ that depends on the relativistic three-dimensional coordinates introduced by Kadyshevsky. The framework includes the complete axiomatic structure of nonrelativistic quantum mechanics (except that the metric in Hilbert space is indefinite in the same way as in the static Klein-Gordon or Dirac theories) and is at the same time fully and explicitly covariant. Both the nonrelativistic and the classical limits exist and are susceptible to detailed interpretation. The case of two spinless particles with a "relativistic Coulomb" interaction is examined in detail. The wave equation, as well as the equation for the T matrix, is in this case exactly soluble. Closed analytic expressions are given for transition form factors, including "photoproduction," the elastic scattering amplitude, and a production amplitude (bremsstrahlung).

I. INTRODUCTION

CONSIDER two noninteracting relativistic particles with or without spin, described by wave functions $\psi_1(x_1)$ and $\psi_2(x_2)$ that satisfy the free wave equations

$$K_1\psi_1=0, \qquad K_2\psi_2=0, \tag{1.1}$$

where K_1 and K_2 are either Klein-Gordon operators

$$K_1=-(\partial/\partial x_{1\mu})^2-m_1^2, \qquad K_2=-(\partial/\partial x_{2\mu})^2-m_2^2 \tag{1.2}$$

or Dirac operators

$$K_1=i\gamma_\mu\partial/\partial x_{1\mu}-m_1, \qquad K_2=i\gamma_\mu\partial/\partial x_{2\mu}-m_2. \tag{1.3}$$

Following the example of nonrelativistic quantum mechanics, one wishes to replace ψ_1 and ψ_2 by a single two-particle wave function $\psi(x_1,x_2)$ satisfying the two equations

$$K_1\psi=0, \qquad K_2\psi=0. \tag{1.4}$$

How can mutual interactions between the two particles be introduced into this framework? The obvious procedure, to add interaction terms to both equations, leads to two serious difficulties: (i) The two equations

$$(K_1-V_1)\psi=0, \qquad (K_2-V_2)\psi=0 \tag{1.5}$$

are mutually inconsistent unless V_1 and V_2 are chosen with great care, and (ii) no Lagrangian variational principle can be found from which the two equations can be derived.

Attempts to resolve these difficulties are of two kinds. It is possible, though difficult, to choose V_1 and V_2 such as to obtain two mutually consistent equations. This has been shown by Wigner and Van Dam[1] and others[2] on the classical level, and by Logunov and Tavkhelidze,[3] Matveev, Muradyan, and Tavkhelidze,[4] and by Bogoliubov[5] in quantum theory. The difficulties of developing a complete physical theory are considerable in either case. The simplest example of this approach is the case of two spinless particles of equal mass. Starting with the free equations, one forms the sum and the difference; in momentum space,

$$(K_1+K_2)\psi=(\tfrac{1}{2}p^2+\tfrac{1}{2}q^2-2m^2)\psi=0, \tag{1.6}$$

$$(K_1-K_2)\psi=p_\mu q^\mu\psi=0. \tag{1.7}$$

Here $p=p_1+p_2$ and $q=p_1-p_2$ are the total and relative momenta. The second equation has the simple intuitive content of setting the relative energy equal to zero in the center-of-mass system and has the effect of reducing the first equation to an ordinary nonrelativistic Schrödinger equation in that frame. In the quasipotential approach of Tavkhelidze *et al.*,[3,4] one retains Eq. (1.7) in the presence of interactions as well, and introduces interactions in Eq. (1.6) only; this can be done without violating the "subsidiary condition" (1.7). The difficulty with this approach is that Eq. (1.7) contains the total momentum p_μ; as a consequence, one has no relativistic local Lagrangian, no conserved current, and trouble with gauge invariance. The Bethe-Salpeter equation[6] represents a different kind of approach to relativistic two-particle dynamics. This equation is of the form

$$(K_1K_2-V)\psi=0, \tag{1.8}$$

where V is an operator that represents the interaction. It has often been said that the Bethe-Salpeter equation contains objectionable features associated with the

* Supported in part by the National Science Foundation.

† On leave of absence from University of California, Los Angeles, Calif.
‡ On leave of absence from Umeå University, Sweden.

[1] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965). See also E. P. Wigner, in *Proceedings of the First Coral Gables Conference on Fundamental Interactions at High Energy* (Freeman, San Francisco, 1969).

[2] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963); D. G. Currie, J. Math. Phys. **4**, 1470 (1963); J. T. Cannon and T. F. Jordan, *ibid.* **5**, 299 (1964); L. L. Foldy, Phys. Rev. **122**, 175 (1961).

[3] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 230 (1963). See also A. N. Tavkhelidze, Lectures on the Quasi-Potential Approach, Tata Institute, Bombay, 1963 (unpublished).

[4] V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, JINR, Dubna Report No. E2-3498, 1967 (unpublished).

[5] P. Bogoliubov, Trieste Report No. IC/69/76 (unpublished).

[6] H. A. Bethe and E. E. Salpeter, Phys. Rev. **84**, 1232 (1951). A comprehensive review article has appeared recently: N. Nakanishi, Progr. Theoret. Phys. (Kyoto) Suppl. (to be published).

interpretation of the relative time and with the normalization of the bound-state wave functions. In addition—and this is probably more serious—it may have too many solutions. This can be seen in the simplest case of scalar massless exchange in the ladder approximation, or more easily by examining the limit $V\to 0$. To its credit, Eq. (1.8) is derivable from a relativistic local Lagrangian and thus it fits into a relativistic c-number field theory. We now present a variation on the quasipotential approach of Tavkhelidze *et al.*[3,4] that combines the most satisfactory features of both theories.

Note: The most obvious objection to our procedure is the apparent asymmetry between the roles played by the two particles. It will be shown that this asymmetry is merely a feature of the formulation, and that the two-particle on-shell scattering amplitude is symmetrical. (See the end of Sec. VII and the last part of the Appendix.)

II. SUBSIDIARY CONDITION AND WAVE EQUATION

Variation of a Lagrangian $L[\psi]$ with respect to $\psi(x_1, x_2)$ can give only one field equation, and yet it is necessary to impose the pair of equations

$$K_1\psi=0, \qquad K_2\psi=0,$$

in the limit of no interaction between the particles. It is therefore necessary to retain one of the two equations (or some combination of them) in the presence of the interaction; that is, the two equations must be rearranged as one field equation, to be modified by the interaction, and one subsidiary condition. In order that the result be a local Lagrangian field theory it is necessary that the subsidiary condition contain only "internal" variables and not the total momentum p_μ.[7] However, this criterion has meaning only if the internal momentum q_μ has been defined. Consider the following choice of canonical position and momentum variables:

$$p=p_1+p_2, \qquad x=cx_1+(1-c)x_2,$$
$$r=x_1-x_2, \qquad q=(1-c)p_1-cp_2.$$

One conventional choice is to take $c=m_1/(m_1+m_2)$, in which case x is the position of the center of mass; this is convenient in case external gravitational fields are of importance. Having different priorities, we take

$$c=0, \qquad x=x_2, \qquad q=p_1,$$

the advantage of which is that K_1 depends on q_μ only, and not on p_μ, so that the equation

$$K_1\psi=0,$$

i.e.,

$$(q^2-m_1^2)\psi=0 \quad \text{or} \quad (\gamma q-m_1)\psi=0,$$

can be treated as a subsidiary condition.[8]

In order to simplify this discussion we suppose from now on that particle 1 is spinless, although the complications that arise in case of spin $\frac{1}{2}$ are purely technical.

Notation: From now on $\psi(p)$ will indicate the wave function for a state with total momentum p_μ, defined on the two-sheeted hyperboloid $q^2=m_1^2$. The q dependence will be suppressed whenever possible, in favor of an operator or matrix notation. The q dependence is analogous to the dependence of the Dirac wave function on the four-spinor index, and the restriction of q_μ to the mass hyperboloid corresponds to the restriction of the spinor index to the values 1, 2, 3, and 4.

The subsidiary condition $K_1\psi=0$ has now been expressed by the stated range of variation of the internal variable q_μ, and no further reference to it is necessary. All dynamics is contained in the equation $K_2\psi=0$, which in the presence of interaction will be modified to read

$$(K_2-V)\psi\equiv L\psi=0. \tag{2.1}$$

The potential V is an operator in the space of wave functions. If K_2 is expressed in terms of p and q, then

$$L=p^2-2p_\mu q^\mu+m_1^2-m_2^2-V \tag{2.2}$$

if particle 2 has spin zero, and

$$L=p\cdot\gamma-q\cdot\gamma-m_2-V \tag{2.3}$$

if particle 2 has spin $\frac{1}{2}$. Generalizations to higher spins are of course possible. Equation (2.1) may be derived from the Lagrangian

$$\mathfrak{L}=\int(dq)d^4p\,\psi^*(p)L(p)\psi(p), \tag{2.4}$$

where the complex conjugate ψ^* should be replaced by $\psi^*\gamma_0$ in the case of spin $\frac{1}{2}$. The measure (dq) is determined by the following considerations. We first require that the contribution of K_2 to $\mathfrak{L}$ be real; this means that K_2 must be Hermitian with respect to the measure and implies that the latter is local, or one-point. Lorentz invariance leads to the form

$$(dq)=\delta(q^2-m_1^2)\epsilon(q_0)d^4q \tag{2.5}$$

on each of the two sheets of the mass hyperboloid, with the possibility of different weights for the two sheets. Positive weight factors may be absorbed in the wave function; the sign factor $\epsilon(q_0)$ is included in (2.5) for future convenience.

Some important operators, including q_μ, K_2, and the generators of Lorentz transformations, are Hermitian operators in a Hilbert space whose elements are the

[7] Compare higher-spin theories with momentum-dependent transversality conditions. Inconsistencies in the earliest attempts were removed by arranging for these "subsidiary conditions" to be included in the Lagrangian variational equations. See M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[8] Another important point is the physical interpretability of the time coordinate x_0, which is the time measured by an observer following particle 2. What, in fact, is the meaning of the "center-of-mass time"?

wave functions $\psi(p)$, and in which the inner product is defined by[9]

$$(\psi(p), \psi'(p')) = \int (dq)\psi^*(p)\psi'(p'). \qquad (2.6)$$

The Hermitian conjugate of an operator A in $\mathcal{H}_L$ is denoted A^*. Now that $\mathcal{H}_L$ has been determined by the requirement that the contribution of K_2 to $\mathcal{L}$ be real, we must require that $V^*=V$ in order that the contribution of V to $\mathcal{L}$ be real as well. (See Table I.)

In addition to the "Lagrangian Hilbert space," we must define a physical Hilbert space $\mathcal{H}$, whose elements are solutions of the wave equation, and whose inner product can be interpreted as a probability. This can be done in the standard way, provided the potential V is a polynomial in the total momentum p_μ. In this case it is possible to construct a conserved canonical current whose time component provides the metric in $\mathcal{H}$.

For simplicity, suppose that V is a polynomial $V(p^2)$ in p^2, and define

$$(\psi(p), J_\mu\psi'(p')) = \int (dq)\psi^*(p)I_\mu(p,p')\psi'(p'), \qquad (2.7)$$

where

$$I_\mu(p,p') = (p+p')_\mu - 2q_\mu - \frac{V(p^2)-V(p'^2)}{p^2-p'^2}(p+p')_\mu \qquad (2.8)$$

in the spin-zero case and

$$I_\mu(p,p') = \gamma_\mu - \frac{V(p^2)-V(p'^2)}{p^2-p'^2}(p+p')_\mu \qquad (2.9)$$

in the case of spin $\frac{1}{2}$. In either case

$$(p-p')^\mu I_\mu(p,p') = L(p) - L(p'). \qquad (2.10)$$

The physical inner product is defined as[10]

$$\langle\psi(p)\,|\,\psi'(p')\rangle = (\psi(p), J_0\psi'(p')). \qquad (2.11)$$

If ψ and ψ' are any two solutions of the wave equation, then it follows from (2.10) that the current is conserved:

$$(p-p')^\mu(\psi(p), J_\mu\psi'(p')) = (\psi(p), [L(p)-L(p')]\psi'(p')) = 0. \qquad (2.12)$$

In particular, if $\mathbf{p}=\mathbf{p}'$, then (2.12) reads

$$(p_0-p_0')\langle\psi(p_0,\mathbf{p})\,|\,\psi'(p_0',\mathbf{p})\rangle = 0, \qquad (2.13)$$

so that two wave functions with the same three-momentum but different energies are orthogonal in the metric (2.11). (See Table I.)

The structure encountered here is very similar to that of ordinary nonrelativistic quantum mechanics with an energy-dependent potential; in both cases it is necessary to restrict the potential in order to ensure a complete physical interpretation. We have already mentioned that $V^*=V$; this was used in (2.12) to prove current conservation, without which the probability interpretation breaks down. Note that it is Hermiticity in $\mathcal{H}_L$ which is important here, not Hermiticity in $\mathcal{H}$. Observables, on the other hand, are Hermitian in $\mathcal{H}$, and (2.13) is a necessary condition for the energy to be a physical observable. In nonrelativistic quantum mechanics with energy-independent potential, $I_0(p,p')$ reduces to unity, and the Hermitian conjugate A^* of an operator A in $\mathcal{H}_L$ is also the Hermitian conjugate in $\mathcal{H}$ (provided the solutions of the wave equation are complete). Another restriction on V is that the metric in $\mathcal{H}$, which depends on V according to (2.8), must be positive definite. Actually, it is possible to make $I_0>0$ in the case of spin $\frac{1}{2}$ and $p_0I_0>0$ in the case of spin zero, as in the theory of a Dirac or Klein-Gordon electron in a static field. Finally, it is desirable that the solutions of the wave equation should form a complete set.

Perhaps the most important property of nonrelativistic quantum mechanics is the existence of soluble models, by means of which a really detailed study is possible. It turns out that there exists a realistic choice of the relativistic potential operator V in Eq. (2.1) for which that theory is exactly soluble. The remainder of this paper is concerned with that particular case.

III. SOLUBLE EXAMPLE

Consider the case of two spinless particles, and let the potential V in (2.2) have the form

$$V = \gamma(p^2)\Gamma_4^{-1}, \qquad (3.1)$$

where $\gamma(p^2)$ is a polynomial in p^2 with numerical coefficients and

$$\Gamma_4^{-1}\psi_q(p) = -\pi^{-2}\int[(dq')/(q-q')^2]\psi_{q'}(p). \qquad (3.2)$$

Here we have indicated the "index" q in order to show explicitly how the integral operator Γ_4^{-1} acts on the wave function. Note the strong suggestion that this potential represents the exchange of a massless field quantum between the two particles. That the similarity is not illusory will be seen below by the property of the solutions. A full discussion of the relation with quantum field theory will be given in another paper.

The name Γ_4^{-1} given to the operator (3.2) derives from the existence of an irreducible representation of $SO(4,1)$ in the Hilbert space $\mathcal{H}_L$, and a set Γ_A ($A=0,1,2,3,4$) of operators that transform among themselves like the components of a five-vector. It is useful to introduce a notation that exploits this situation. The generators of $SO(4,1)$ will be denoted by s_{AB} ($A, B=0,1,2,3,4$). The subset $s_{\mu\nu}$ ($\mu,\nu=0,1,2,3$) are the generators of infinitesimal Lorentz rotations of q_μ, and $s_{\mu4}$ are "Runge-Lenz operators":

$$s_{\mu\nu} = i[q_\mu(\partial/\partial q^\nu) - q_\nu(\partial/\partial q^\mu)], \qquad (3.3)$$

$$s_{\mu4} = m_1^{-1}(q^\nu s_{\mu\nu} + 2iq_\mu) = -s_{4\mu}. \qquad (3.4)$$

[9] The inner product (2.6) is not positive definite. Whenever necessary for the purpose of defining limits and other topological properties of the space, we use the modified norm obtained by dropping the factor $\epsilon(q_0)$. This procedure is familiar in the case of the Klein-Gordon or Dirac theories with a static potential.

[10] Like (2.6), this inner product is not positive definite in the spin-zero case. The same comments apply—see Ref. 9.

TABLE I. Spaces and notation.

Context	Real Lagrangian	Physical properties	Group representation
Designation	$\mathcal{H}_L$	$\mathcal{H}$	$\mathcal{H}_R$
Metric (bosons)	$(\psi,\phi)\equiv\int\psi_q{}^*\phi_q(dq)$	$\langle\psi\mid\phi\rangle\equiv(\psi,I_0\phi)$	$\psi^\dagger\phi\equiv(\psi,\Gamma_4{}^{-1}\phi)$
Properties of metric	$(\psi,q_0\psi)>0$	$\langle\psi\mid p_0\mid\psi\rangle>0$	$\psi^\dagger\psi>0$
Hermitian conjugate	ψ^*, A^*	No symbol	$\psi^\dagger$, $A^\dagger$
Hermitian operators	p_μ, q_μ, V	Physical observables	Group generators

$$(dq)=\delta(q^2-m_1{}^2)\epsilon(q_0)d^4q$$

Five-vectors: $P_A,\lambda_A,\ldots$ $(A=0,1,2,3,4)$
Minkowsky vectors: $p_\mu,q_\mu,\ldots$ $(\mu=0,1,2,3)$
Fock vector: u_a $(a=1,2,3,4)$
Space vectors: **p**, **q**, boldface
$\psi(p,nlm)$ = off-shell eigenvectors
$\psi(\mathbf{p},nlm)$ = on-shell eigenvectors
$\psi_q(p)$ = realization of vector $\psi(p)$ by function of q_μ
$\psi(\mathbf{p},\mathbf{q})$ = asymptotic distorted wave solution
m_1, m_2 = constituent masses, $m_\pm=m_1\pm m_2$
$\mu=m_1m_2/m_+$, M_n = bound-state masses
$\tilde{m}$ = arbitrary normalizer with dimension of mass

The operators Γ_A, which together with the s_{AB} are the generators of an irreducible representation of $SO(4,2)$, are defined by (3.2) and by

$$m_1\Gamma_\mu=\Gamma_4q_\mu. \tag{3.5}$$

Some of these operators, namely, Γ_4 and $s_{\mu\nu}$, are Hermitian operators in $\mathcal{H}_L$, but all of them are Hermitian in the metric (see the Appendix):

$$\psi^\dagger\psi'\equiv-\pi^{-2}\int[(dq)(dq')/(q-q')^2]\psi_q{}^*\psi'_{q'}. \tag{3.6}$$

This metric is positive definite and defines a Hilbert space $\mathcal{H}_R$ in which the operators generate a unitary irreducible representation of $SO(4,2)$. The proofs of these statements are given in the Appendix.

The metric (3.6) differs from the metric in $\mathcal{H}_L$, given by (2.6), by a factor $\Gamma_4{}^{-1}$:

$$\psi^\dagger(p)\psi'(p')=(\psi(p),\Gamma_4{}^{-1}\psi'(p')). \tag{3.7}$$

The introduction of this new metric, the third one, is not as fundamental as the first two; rather it is part of the special technique used to solve the wave equation in the case of the potential (3.1). Table I is a summary of the notation related to $\mathcal{H}_L$, $\mathcal{H}$, and $\mathcal{H}_R$. It is convenient to adapt the notation to the space $\mathcal{H}_R$. To every operator A in $\mathcal{H}_L$ there is an operator $\Gamma_4A=\hat{A}$ in $\mathcal{H}_R$ such that the matrix elements of A in $\mathcal{H}_L$ are the same as the matrix element of $\hat{A}$ in $\mathcal{H}_R$:

$$(\psi,A\psi')=\psi^\dagger\hat{A}\psi'. \tag{3.8}$$

In particular, the Lagrangian (2.4) is

$$\mathcal{L}=\int d^4p\,\psi^\dagger(p)\hat{L}(p)\psi(p) \tag{3.9}$$

and the current (2.7) is

$$(\psi(p),J_\mu\psi'(p'))=\psi^\dagger(p)\hat{I}_\mu(p,p')\psi'(p'), \tag{3.10}$$

with

$$\hat{L}(p)=\Gamma_4L(p)=(p^2+m_1{}^2-m_2{}^2)\Gamma_4-2m_1p_\mu\Gamma_\mu-\gamma(p^2), \tag{3.11}$$

$$\begin{aligned}\hat{I}_\mu(p,p')&=\Gamma_4I_\mu(p,p')\\&=\{\Gamma_4+[\gamma(p^2)-\gamma(p')^2]/(p^2-p'^2)\}\\&\qquad\times(p+p')_\mu-2m_1\Gamma_\mu.\end{aligned} \tag{3.12}$$

The physical inner product is

$$\langle\psi(p)\mid\psi'(p')\rangle=\psi^\dagger(p)\hat{I}_0(p,p')\psi'(p'). \tag{3.13}$$

Finally, the wave equation

$$\begin{aligned}0&=\hat{L}(p)\psi(p)\\&=[(p^2+m_1{}^2-m_2{}^2)\Gamma_4-2m_1p_\mu\Gamma_\mu-\gamma(p^2)]\psi(p)=0\end{aligned} \tag{3.14}$$

is seen to be of the type already studied by Majorana and others.[11,12]

This model has a close relationship with the work of Kadyshevsky *et al.*[13] In particular, the operator Γ_4 is essentially the variable that Kadyshevsky calls r and which he has proposed as a covariant definition of the internal "distance" of the two-particle system. The definition is indeed attractive; in the classical limit it is just the inverse of the (electrostatic or gravitational) potential due to particle 1 and measured at particle 2.

---

[11] E. Majorana, Nuovo Cimento **9**, 335 (1932); I. M. Gel'fand, A. M. Yaglom, and M. A. Naimark, summarized in A. M. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, London, 1964); Y. Nambu, Phys. Rev. **160**, 1171 (1967); and others mentioned below.

[12] C. Fronsdal, Phys. Rev. **171**, 1811 (1968).

[13] V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Nuovo Cimento **55A**, 223 (1968); C. Itzykson, V. G. Kadyshevsky, and I. T. Todorov, Phys. Rev. D **1**, 2823 (1970).

IV. PROPERTIES OF DISCRETE SPECTRUM

Let P_A and λ_A be the five-vectors

$$P_A=\{2m_1p_\mu,\, p^2+m_1^2-m_2^2\}, \tag{4.1}$$

$$\lambda_A=P^{-1}P_A, \tag{4.2}$$

where $P=(P^2)^{1/2}$ and

$$P^2=P_\mu^2-P_4^2=(p^2-m_-^2)(m_+^2-p^2), \tag{4.3}$$

$$m_\pm=m_1\pm m_2. \tag{4.4}$$

Let n be the operator

$$n=\lambda^A\Gamma_A. \tag{4.5}$$

The spectrum of n in $\mathcal{H}_R$ is unaffected by $SO(4,1)$ rotations of λ_A and depends only on whether P_A is timelike, spacelike, or lightlike:

$$n=1,2,\ldots, \qquad \text{if } P^2>0$$
$$=\text{pure imaginary} \qquad \text{if } P^2<0. \tag{4.6}$$

For any choice of p_μ we can diagonalize $\hat{L}$ by diagonalizing n. From (3.11) and (4.5),

$$\hat{L}(p)=-P^A\Gamma_A-\gamma(p^2),$$
$$\Rightarrow \hat{L}_n(p^2)=-Pn-\gamma(p^2). \tag{4.7}$$

We now calculate the eigenvectors corresponding to the discrete eigenvalues of n.

Let $m_-^2<p^2<m_+^2$, so that $P^2>0$; and $p_0>0$, $\mathbf{p}=0$, so that

$$n=\lambda_0\Gamma_0-\lambda_4\Gamma_4, \tag{4.8}$$

$$\lambda_0=2m_1p_0P^{-1}, \qquad \lambda_4=(p_0^2+m_1^2-m_2^2)P^{-1}. \tag{4.9}$$

Define

$$\Gamma_0'=\lambda_0\Gamma_0-\lambda_4\Gamma_4, \tag{4.10}$$

$$\Gamma_4'=-\lambda_4\Gamma_0+\lambda_0\Gamma_4, \qquad \mathbf{\Gamma}'=\mathbf{\Gamma}, \tag{4.11}$$

and

$$u_a=\Gamma_0'^{-1}\Gamma_a', \qquad a=1,2,3,4. \tag{4.12}$$

From (4.12) and (3.5) it follows that

$$\mathbf{u}=-\mathbf{q}Pm_1^{-1}(p_2^2-m_2^2)^{-1}, \tag{4.13}$$

$$u_4=-[2m_1^2p_0-q_0(p_0^2+m_1^2-m_2^2)]m_1^{-1}(p_2^2-m_2^2)^{-1}. \tag{4.14}$$

The complete set of simultaneous eigenstates of n and angular momentum is given by (see the Appendix)

$$\psi_q(p,nlm)=d_n(p_2^2-m_2^2)^{-2}Y_{nlm}(u), \tag{4.15}$$

where $Y_{nlm}(u)$ are four-dimensional spherical harmonics and $l=0,1,\ldots,n-1$. The normalization constant d_n will be determined next.

The expectation values of Γ_0' and Γ_4' in the $\mathcal{H}_R$ metric are n and 0, respectively; hence

$$\psi(p,nlm)^\dagger\Gamma_A\psi(p,nlm)=n\lambda_A\psi(p,nlm)^\dagger\psi(p,nlm), \tag{4.16}$$

and in particular,

$$(\psi(p,nlm),\psi(p,nlm))=n\lambda_4\psi^\dagger(p,nlm)\psi(p,nlm). \tag{4.17}$$

We shall determine d_n by setting

$$1=\psi(p,nlm)^\dagger\psi(p,nlm)$$
$$=(n\lambda_4)^{-1}\int(dq)\,|\psi(p,nlm)|^2$$
$$=\tfrac{1}{2}m_1^2n^{-1}P^{-2}(p^2+m_1^2-m_2^2)^{-1}\times\int\{d\Omega/[(p-q)^2-m_2^2]\}\,|d_nY_{nlm}|^2$$
$$=\tfrac{1}{2}m_1^2n^{-1}P^{-4}\times\int d\Omega\{1+[2m_1p_0/(p^2+m_1^2-m_2^2)]u_4\}\,|d_nY_{nlm}|^2. \tag{4.18}$$

Here $d\Omega=d^3u/u_4$. The Y_{nlm} are normalized to unity and the expectation value of u_4 vanishes. Thus

$$|d_n|^2=2nP^4/m_1^2. \tag{4.19}$$

Finally, it is necessary to generalize (4.15) to an arbitrary Lorentz frame. The result is

$$\psi(p,nlm)=(2n)^{1/2}m_1(\lambda Q)^{-2}Y_{nlm}(\lambda,Q). \tag{4.20}$$

The meaning of the notation is as follows. First, λ and Q stand for the five-vectors (4.2) and

$$Q_A=\{q_\mu,m_1\}. \tag{4.21}$$

These determine a four-vector u_a which is defined as the projection of Q_A in the plane normal to λ_A, and $Y_{nlm}(\lambda,Q)$ is a spherical harmonic depending on the direction of u_a. The quantization axes are as follows: for n, λ_A; for l, in the $(4,p_\mu)$ plane normal to λ_A; for m, in the $(0,\mathbf{p})$ plane normal to p_μ. Finally, $(\lambda Q)\equiv\lambda^AQ_A$.

The vectors (4.20) are off-shell wave functions; they become physical on-shell wave functions when p_0 takes a value such that $m_-^2<p^2<m_+^2$ and such that the eigenvalue (4.7) of the wave operator vanishes. Consider first the case when $\gamma(p^2)$ is a negative constant.[14] Then for every positive integer n and for every three-vector $\mathbf{p}$ there are two values of p_0 such that $m_-^2<p^2<m_+^2$ and such that $\hat{L}_n(p^2)=0$, namely,

$$p_0=+(\mathbf{p}^2+M_{n+}^2)^{1/2} \quad\text{and}\quad p_0=+(\mathbf{p}^2+M_{n-}^2)^{1/2},$$
$$M_{n\pm}^2=m_1^2+m_2^2\pm(4m_1^2m_2^2-\gamma^2/n^2)^{1/2}.$$

The physical norm is given by (3.13); in the frame $\mathbf{p}=0$,

$$\langle\psi(p,nlm)\,|\,\psi(p,nlm)\rangle$$
$$=\psi(p,nlm)^\dagger(-2m_1\Gamma_0+2p_0\Gamma_4)\psi(p,nlm)$$
$$=2n(p_0\lambda_4-m_1\lambda_0)\psi(p,nlm)^\dagger\psi(p,nlm)$$
$$=2np_0P^{-1}(p^2-m_1^2-m_2^2).$$

[14] In this case our wave equation coincides with one that has been used by Barut and co-workers; see, e.g., A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. **167**, 1527 (1968).

We see that the states with mass M_{n+} have positive norm and the states with mass M_{n-} have negative norm. We therefore reject this theory[15] and consider the next simplest choice of $\gamma(p^2)$.

Let

$$\gamma(p^2)=-\tfrac{1}{2}e^2(p^2-m_-^2), \tag{4.22}$$

where e is a real constant.[16] The zeros of $\hat{L}_n(p^2)$ are now at

$$p_0=+(\mathbf{p}^2+M_n^2)^{1/2}, \tag{4.23}$$

$$M_n^2=m_+^2\frac{1+(m_-/m_+)^2(e^2/2n)^2}{1+(e^2/2n)^2}. \tag{4.24}$$

The physical on-shell bound state wave functions are sufficiently labeled by nlm and $\mathbf{p}$, since p_0 is determined by (4.23). We may therefore distinguish on-shell from off-shell wave functions by leaving out the argument p_0; from now on $\psi(\mathbf{p}, nlm)$ stands for a discrete solution of the wave equation. The physical norm is given by (3.13). In the frame $\mathbf{p}=0$,

$$\begin{aligned}\eta_n &\equiv \langle\psi(\mathbf{p}, nlm)\,|\,\psi(\mathbf{p}, nlm)\rangle \\ &=2n(p_0\lambda_4-m_1\lambda_0)+p_0e^2 \\ &=4m_1m_2np_0P^{-1},\end{aligned} \tag{4.25}$$

and this is positive definite.

The continuum solutions may be found in the same way, except that p^2 must now be outside the interval from m_-^2 to m_+^2, and the spectrum of n is the entire imaginary axis. However, this naive procedure cannot be justified in general—it is usually incorrect, since a normalizable wave packet in $\mathcal{H}_R$ may be quite different from a normalizable wave packet in $\mathcal{H}$. The proper tool for understanding the continuum is the resolvant operator.

V. GREEN'S FUNCTION, CONTINUUM

The Green's function $G_{qq'}(p)$ is a matrix element of the operator

$$G(p)=L(p)^{-1}=\hat{L}(p)^{-1}\Gamma_4. \tag{5.1}$$

To evaluate $G_{qq'}(p)$ we begin with p_μ in the range $m_-^2<p^2<m_+^2$; then the functions (4.20) satisfy the completeness relation

$$\sum_{nlm}\psi_q(p,nlm)\psi_{q'}(p,nlm)^*=\delta_{qq'}\Gamma_4. \tag{5.2}$$

Here $\delta_{qq'}$ is the Dirac δ function associated with the measure (dq) and $\delta_{qq'}\Gamma_4$ is the unit operator in $\mathcal{H}_R$. Inserting (5.2) into (5.1) we get

$$G_{qq'}(p)=\sum_{nlm}\hat{L}_n(p^2)^{-1}\psi_q(p,nlm)\psi_{q'}(p,nlm)^*. \tag{5.3}$$

[15] When our model is studied in the context of quantum field theory it becomes clear that the domain of greatest physical relevance is p^2 near or above m_+^2. Consequently, we are not really justified in rejecting the choice of a constant $\gamma(p^2)$ on the basis of anomalies near m_-^2.

[16] In this case the wave equation is the same as that studied in Ref. 12.

Inserting (4.20) and summing over l, m, one obtains

$$G_{qq'}(p)=[m_1^2/\pi^2(\lambda Q)^2(\lambda Q')^2]\sum_n[n^2/\hat{L}_n(p^2)]P_{n,4}(\phi). \tag{5.4}$$

Here, ϕ is the angle between the projections of Q and Q' into the four-space normal to λ, and $P_{n,4}$ is a four-dimensional Legendre function:

$$\cos\phi=1-(QQ')/(\lambda Q)(\lambda Q'), \tag{5.5}$$

$$P_{n,4}(\phi)=\sin(n\phi)/\sin\phi. \tag{5.6}$$

The series (5.4) is a difference between two hypergeometric series. It converges when $m_-^2<p^2<m_+^2$ but is easily continued analytically by a Watson-Sommerfeld transformation—or by using Barnes's integral representation for the hypergeometric functions. The result is

$$G_{qq'}(p)=(m_1/\pi)^2(\lambda Q)^{-2}(\lambda Q')^{-2}\times\tfrac{1}{2}i\int_{-i\infty}^{+i\infty}dn\,\csc\pi n\,n^2\hat{L}_n(p^2)^{-1}P_{n,4}'(\phi), \tag{5.7}$$

where

$$P_{n,4}'(\phi)=[(-e^{i\phi})^n-(-e^{-i\phi})^n]/[e^{i\phi}-e^{-i\phi}]. \tag{5.8}$$

The representation (5.7) is valid when $-e^{\pm i\phi}$ is in the complex plane cut along the negative real axis. The integration contour passes to the left of the pole of $\hat{L}_n(p^2)^{-1}$ and intersects the real axis between -1 and $+1$.

Equation (5.7) defines, for fixed q, q', and $\mathbf{p}$, an analytic function[17] of p_0, with poles at the positions of the discrete solutions (4.23) and cuts along the following portions of the real axis:

$$p_0^2>m_+^2+\mathbf{p}^2 \quad (\text{"normal" cuts}), \tag{5.9}$$

$$p_0^2<m_-^2+\mathbf{p}^2 \quad (\text{"left" cuts}). \tag{5.10}$$

Asymptotically, as $|p_0|\to\infty$, $p_0G_{qq'}(p)\to 0$; hence we have the Cauchy formula

$$G_{qq'}(p_0,\mathbf{p})=(2\pi i)^{-1}\int_C\frac{dz}{z-p_0}G_{qq'}(z,\mathbf{p}). \tag{5.11}$$

The contour C runs around the poles and the cuts in the clockwise sense. The residues of the poles are given by (5.4), and the discontinuities across the cuts are obtained from (5.7) by replacing the integration contour by a circle surrounding the pole of $L_n(p^2)^{-1}$. Thus[18]

$$\begin{aligned}G_{qq'}(p)=&(m_1/\pi)^2\sum_n(E_n-p_0)^{-1}\\ &\times[(\lambda Q)^{-2}(\lambda Q')^{-2}n^2P_{n,4}(\phi)\eta_n^{-1}]_{p_0=E_n}\\ &+(m_1/\pi)^2(\tfrac{1}{2}i)\int[dz/(z-p_0)]\\ &\times[(\lambda Q)^{-2}(\lambda Q')^{-2}(\nu^2/\sin\pi\nu)P_{\nu,4}'(\phi)P^{-1}]_{p_0=z+i\epsilon},\end{aligned} \tag{5.12}$$

[17] The analytic properties of $G_{qq'}(p)$ are the same as those of the Compton scattering amplitude already studied in connection with current algebra; C. Fronsdal, Phys. Rev. 182, 1564 (1969).

[18] For further details see Ref. 12.

where E_n are the energies (4.23) of the bound states and

$$\nu(p)=-\gamma(p^2)/P. \quad (5.13)$$

The integration runs along the cuts from left to right.

To write (5.12) in terms of discrete and continuum solutions of the wave equation we expand $P_{n,4}$ and $P_{\nu,4}'$ in four-dimensional spherical harmonics (see the Appendix) and obtain

$$G_{qq'}(p)=2m_1^2\sum_{nlm}(E_n-p_0)^{-1}$$
$$\times[(\lambda Q)^{-2}(\lambda Q')^{-2}nY_{nlm}(\lambda,Q)Y_{nlm}^*(\lambda,Q')\eta_n^{-1}]_{p_0=E_n}$$
$$+2m_1^2\sum_{lm}\tfrac{1}{2}i\int\frac{dz}{z-p_0}[(\lambda Q)^{-2}(\lambda Q')^{-2}(\nu/\sin\pi\nu)$$
$$\times Y_{nlm}(\lambda Q)Y_{nlm}'^*(\lambda,Q')P^{-1}]_{p_0=z+i\epsilon} \quad (5.14)$$
$$=\sum_{nlm}\frac{\psi_q(\mathbf{p},nlm)\psi_{q'}(\mathbf{p},nlm)^*}{\eta_n}(E_n-p_0)^{-1}$$
$$+\sum_{lm}\int\frac{\psi_q(p,\nu lm)\psi'_{q'}(p,\nu lm)^*}{2iP\sinh\pi|\nu|}\frac{dz}{z-p_0}. \quad (5.15)$$

This result shows that the bound states are completed by a continuum whose wave functions are obtained by analytic continuation in n to $n=-\gamma(p^2)P^{-1}$ and which cover the cuts (5.9) and (5.10). The right-hand portion of (5.9) is the conventional scattering continuum. The remainder of the continuum disappears in the nonrelativistic limit.

VI. STATIC AND NONRELATIVISTIC LIMITS

In the static limit $m_1\to\infty$ and particle 1 acts as a fixed potential. Our wave equation, in the form (2.1), reduces to either the Klein-Gordon equation or the Dirac equation in the external potential V. In particular, the potential (3.1) becomes

$$\gamma(p^2)\pi^{-2}\int[d^3q'/2m_1(\mathbf{q}-\mathbf{q}')^2]\psi_{q'}.$$

With the choice (4.22) for $\gamma(p^2)$, i.e.,

$$\gamma(p^2)\to-m_1e^2(p_{20}+m_2),$$

this is the same as

$$-(e^2/2\pi^2)(p_{20}+m_2)\int[d^3q'/(\mathbf{q}-\mathbf{q}')^2]\psi_{q'}$$

or

$$V=V(r)=-(p_{20}+m_2)(e^2/r).$$

In the nonrelativistic limit, both m_1 and m_2 are finite, but $\mathbf{p}_1$ and $\mathbf{p}_2$ (or $\mathbf{p}$ and $\mathbf{q}$) are small compared with either mass. The nonrelativistic representation of the Γ matrices is[19]

$$\Gamma_0-\Gamma_4=\mu r,\qquad \Gamma_0+\Gamma_4=\mu^{-1}r\mathbf{q}^2,$$
$$\mathbf{\Gamma}=r\mathbf{q},$$

[19] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

where $\mathbf{r}$ and $\mathbf{q}$ are the usual relative coordinates and μ is the reduced mass. The connection between the relativistic and the nonrelativistic realizations is given by a rotation in the (0, 4) plane (we are working in the center-of-mass frame):

$$\Gamma_0\to(\mu/4m_1)(\Gamma_0+\Gamma_4)+(m_1/\mu)(\Gamma_0-\Gamma_4),$$
$$\Gamma_4\to-(\mu/4m_1)(\Gamma_0+\Gamma_4)+(m_1/\mu)(\Gamma_0-\Gamma_4).$$

Thus

$$\Gamma_0\to m_1r(1+\mathbf{q}^2/4m_1^2),$$
$$\Gamma_4\to m_1r(1-\mathbf{q}^2/4m_1^2),$$
$$\mathbf{\Gamma}\to\mathbf{\Gamma},$$

and to lowest order in inverse masses,

$$q_0\to m_1+\mathbf{q}^2/2m_1,$$
$$\mathbf{q}\to\mathbf{q}.$$

We also define the energy by

$$p_0=p_{10}+p_{20}=m_++E,$$

so that, finally,

$$p_2^2-m_2^2\to2m_2(E-\mathbf{p}_1^2/2m_1-\mathbf{p}_2^2/2m_2)$$
$$=2m_2(E-\mathbf{q}^2/2\mu)$$

as expected. Our wave equation thus turns into the nonrelativistic Schrödinger equation for two particles with masses m_1 and m_2 and with the potential $-e^2/r$.

VII. SCATTERING MATRIX

The integral equation for the scattering matrix is[20]

$$T(p_1p_2;p_1'p_2')=V(p_1,p_1')$$
$$+\int(dq)V(p_1,q)[(p-q)^2-m_2^2]^{-1}T(q,p-q;p_1'p_2'), \quad (7.1)$$

and the solution is

$$T=(p_2^2-m_2^2)(L^{-1}-K_2^{-1})(p_2'^2-m_2^2). \quad (7.2)$$

Since we are mostly interested in the on-shell limit we may drop the second term. Using $L^{-1}=G$ and (5.4) we get

$$T=[P^2/\pi^2(\lambda Q)(\lambda'Q)]\sum_n[n^2/\hat{L}_n(p^2)]P_{n,4}(\cos\phi), \quad (7.3)$$

where ϕ is the angle between the projections of Q and Q' in the plane normal to λ:

$$\cos\phi=1-[QQ'/(\lambda Q)(\lambda'Q)]. \quad (7.4)$$

When $p_2^2\to m_2^2$ and $p_2'^2\to m_2^2$ this tends to infinity, since

$$\lambda Q=m_1P^{-1}(p_2^2-m_2^2). \quad (7.5)$$

[20] This may be shown within the quantum-mechanical framework. In another paper we shall show that Eq. (7.1) may be derived directly from quantum field theory.

Consequently, only the contribution of the Regge pole at $n=\nu\equiv-\gamma(p^2)/P$ survives the limit[21,22]

$$\sum_n \frac{n^2 P_{n,4}(\phi)}{-Pn-\gamma(p^2)}$$

$$=(2\pi i)^{-1}\int \frac{\Gamma(1-n)\Gamma(1+n)ndn}{Pn+\gamma(p^2)} P_{n,4}(-\cos\phi)$$

$$\to \nu P^{-1}\Gamma(1-\nu)\Gamma(1+\nu)\left[\frac{2QQ'}{(\lambda Q)(\lambda' Q)}\right]^{\nu-1}. \quad (7.6)$$

In the on-shell limit there appears an infinite angle-independent phase factor that is familiar in the case of the nonrelativistic Coulomb problem. Following Finkelstein and Levy,[21] we introduce incoming and outgoing wave packets by

$$\left(\frac{-i\tilde{m}^2}{p_2^2-m_2^2}\right)^\nu \to \lim_{\Delta\to 0}\int_0^\infty \left(\frac{-i\tilde{m}^2}{p_2^2-m_2^2}\right)^\nu \times \frac{\sin[(|p_2|-m_2)/\Delta]}{\pi(|p_2|-m_2)} d|p_2|, \quad (7.7)$$

where $\tilde{m}$ is an arbitrary unobservable quantity with the dimension of a mass that may depend on energy but not on angle. After a change of the variable of integration the integral turns into a well-known representation for $1/\Gamma(1+\nu)$, except for an infinite phase factor that we drop. The on-shell limit of the T matrix (7.3) is thus

$$T=\frac{\gamma(s)}{\pi^2}t^{-1}\frac{\Gamma(1-\nu(s))}{\Gamma(1+\nu(s))}\left(\frac{t}{\tilde{m}}\right)^{\nu(s)}, \quad (7.8)$$

where $s=(p_1+p_2)^2$, $t=(p_1-p_1')^2=(p_2-p_2')^2$, and

$$\nu(s)=+ie^2[(s-m_-^2)/(s-m_+^2)]^{1/2}. \quad (7.9)$$

The final expression for the scattering matrix is completely symmetric in the two particles, which is surprising in view of the different roles assigned to them by the dynamics. Such a situation is not unthinkable, however, since the dynamical equations are no more than a mathematical model for the interpolating states. A more general result is obtained in the last part of the Appendix.

VIII. EXTERNAL INTERACTIONS

Interactions with an external electromagnetic field may be introduced by making the usual gauge-invariant substitution in the Lagrangian (2.2), or in (3.11). To first order A_μ is coupled to the conserved canonical current, Eq. (3.10). This current is local and generates a local Gell-Mann current algebra.[17]

The electromagnetic form factor for a transition between two arbitrary states is

$$\psi^\dagger(p)\hat{I}_\mu(p,p')\psi'(p'). \quad (8.1)$$

If the two states are stationary bound states this is given by Eqs. (A.20)–(A.22) of the Appendix. In particular, the elastic form factor for the ground state is

$$\frac{(p+p')_\mu}{2(p_0p_0')^{1/2}}\frac{1-(m_1/2m_2)t/(p^2-m_-^2)}{(1-tm_1^2/\gamma^2)^2}. \quad (8.2)$$

The analytic structure of this and other form factors was investigated earlier,[12] before the detailed physical content of the wave equation was understood. It was pointed out that the singularity at $t=\gamma^2/m_1^2$ is the anomalous threshold singularity that is expected for a compound in which only particle 2 is charged. The reason for this is now completely clear. The gauge-invariant minimal coupling that is obtained by replacing p_μ by $p_\mu-eA_\mu$ in the wave equation (3.14) actually means that $p_{2\mu}\to p_{2\mu}-eA_\mu$ and $p_{1\mu}\to p_{1\mu}$. We have coupled locally at x_μ, which is the position coordinate of particle 2. If both particles are charged, then the coupling will not be minimal. The current will have an extra nonlocal contribution.[23] Hence we see that the electromagnetic current is not symmetric in the two particles. However, there exists a nonlocal unitary transformation that interchanges the roles of the particles, which shows that the asymmetry is not fundamental. (These remarks have some obvious relevance to the problem of constructing a nonfactored current algebra.)

The matrix element for Compton scattering is a sum of two terms illustrated by the Feynman diagrams of Fig. 1(a). We have

$$\mathfrak{M}_{\mu\nu}^{(1)}$$
$$=\psi(p'',\text{out})^\dagger\hat{I}_\mu(p'',p')[1/\hat{L}(p')]\hat{I}_\nu(p',p)\psi(p,\text{in})$$
$$=\hat{I}_\mu'(p'',p')\hat{I}_\nu'(p',p)\psi(p'',\text{out})^\dagger[1/\hat{L}(p')]\psi(p,\text{in}). \quad (8.3)$$

Here the $\hat{I}_\mu$ have been pulled out with the help of (A.22); hence $\hat{I}_\mu'(p'',p')$ is related to $\hat{I}_\mu(p'',p')$ by the substitution of $\partial/\partial\lambda''^\mu+\partial/\partial\lambda'^\mu$ for Γ_μ.[24] The scalar matrix element is evaluated in the same way as $G_{qq'}(p)$; in the case of elastic scattering from the ground state we easily find[12]

$$\psi(p'',1)^\dagger[1/\hat{L}(p')]\psi(p,1)$$
$$=2iP^{-1}D^{-1/2}[{}_2F_1(1,-b;1-b;u)-{}_2F_1(1,-b;1-b;v)], \quad (8.4)$$

where D is the Kibble determinant formed from λ'', λ', and λ, and

$$\begin{Bmatrix}u\\v\end{Bmatrix}=\left(\frac{\lambda''\lambda'-1}{\lambda''\lambda'+1}\frac{\lambda'\lambda-1}{\lambda'\lambda+1}\right)^{1/2}\begin{Bmatrix}e^{i\phi}\\e^{-i\phi}\end{Bmatrix},$$
$$b=-\gamma(p^2)/P.$$

[21] R. J. Finkelstein and D. Levy, J. Math. Phys. **8**, 2147 (1967).

[22] C. Fronsdal and L. E. Lundberg, Trieste Report No. IC/69/33 (unpublished).

[23] Compare G. Bisiacchi and G. Calucci, Phys. Rev. **181**, 185 (1969).

[24] These rules must be used with care. They are based on (A22), which holds if the scalar form factor is written in the form (A20). The differential operators do not act on the tensors ψ, which must be remembered when these tensors are eliminated by means of the completeness relation (A24).

The singularities of the form factors and the Compton scattering amplitude have been studied[12] and found to coincide with singularities of quantum field theory.

Amplitudes that involve scattering states in the external states can be evaluated in the same way as the Coulomb amplitude, but the easiest method is to use an algorithm that can be justified precisely as in the nonrelativistic theory.[22] We illustrate the procedure by calculating the photoeffect induced by a scalar photon incident on the ground state. See Fig. 1(b). The matrix element is

$$\psi(\mathbf{p},\mathbf{q})^\dagger[1/\hat{L}(p)]\psi(p',1). \tag{8.5}$$

Here $\psi(\mathbf{p},\mathbf{q})$—not to be confused with $\psi_q(p)$—is the wave function for an asymptotic scattering state: a distorted plane wave. Expanding the propagator as usual, we get a contribution from the pole of $\hat{L}_n(p^2)$ only:

$$1/\hat{L}(p)\to 1/\gamma(p^2)\Gamma(1-\nu)\Gamma(1+\nu) \times\sum_{lm}\psi(p,\nu lm)\psi'(p,\nu lm)^\dagger. \tag{8.6}$$

Inserting this into (8.5), we use

$$\psi(\mathbf{p},\mathbf{q})^\dagger\psi(p,\nu lm)=[\gamma(p^2)/\tilde{m}\sqrt{2}\pi][1/\Gamma(1+\nu)] \times V^{A_1}\cdots V^{A_{\nu-1}}\tilde{\psi}(p,\nu lm)_{A_1\cdots A_{\nu-1}}, \tag{8.7}$$

with

$$V_A=(1/\tilde{m})Q_A=(1/\tilde{m})(q_\mu,m_1), \tag{8.8}$$

which defines $\psi(\mathbf{p},\mathbf{q})$, and (A.20) for $n=1$:

$$\psi'(p,\nu lm)^\dagger\psi(p',1) =\sqrt{2}(1+\lambda'\lambda)^{-\nu}\tilde{\psi}(p,\nu lm)^{B_1\cdots B_{\nu-1}}\lambda_{B_1}'\cdots\lambda_{B_{\nu-1}}'. \tag{8.9}$$

Finally, since the tensor $V^{A_1}V^{A_2}\cdots$ is traceless, symmetric, and transverse to λ, the sum over l and m simplifies to

$$\sum_{lm}V^{A_1}V^{A_2}\cdots\tilde{\psi}(p,\nu lm)_{A_1A_2\cdots}\tilde{\psi}'(p,\nu lm)^{B_1B_2\cdots}\lambda_{B_1}'\lambda_{B_2}'\cdots =2^\nu(-V\lambda')^{\nu-1}. \tag{8.10}$$

Thus, using (8.6)–(8.10) we reduce (8.5) to

$$\frac{\Gamma(1-\nu)}{\tilde{m}\pi}(V\lambda')^{-1}\left(\frac{-2V\lambda'}{1+\lambda\lambda'}\right)^\nu =(p-p')^\mu(p+p'-2q)_\mu\frac{\Gamma(1-\nu)}{m_1\pi P'}\left(\frac{-2V\lambda'}{1+\lambda\lambda'}\right)^\nu. \tag{8.11}$$

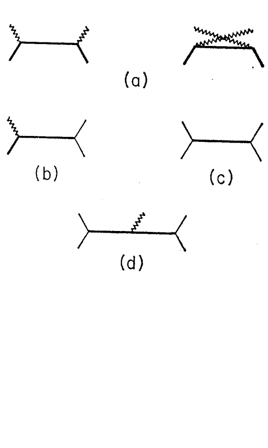

FIG. 1. Feynman diagrams; wavy lines represent external field quanta, thin lines constituent particles, and heavy lines the atom. (a) Compton scattering; (b) photoeffect; (c) Coulomb scattering; (d) bremsstrahlung.

In order to normalize to one initial atom per unit volume divide this by $\eta_1(p')^{1/2}$.

The calculation of the Coulomb amplitude [see Fig. 1(c)] now goes as follows:

$$\begin{aligned}T&=\psi^+(\mathbf{p},\mathbf{q}')[1/\hat{L}(p)]\psi(\mathbf{p},\mathbf{q})\\&=[1/\gamma(p^2)]\Gamma(1-\nu)\Gamma(1+\nu)\times\sum_{lm}\psi^\dagger(\mathbf{p},\mathbf{q}')\psi(p,\nu lm)\psi'(p,\nu lm)^\dagger\psi(\mathbf{p},\mathbf{q})\\&=[\gamma(p^2)/2\pi^2\tilde{m}^2][\Gamma(1-\nu)/\Gamma(1+\nu)]V'^A\cdots\times\sum_{lm}\tilde{\psi}_{A\cdots}\tilde{\psi}'^{B\cdots}V_B\cdots\\&=[\gamma(p^2)/\pi^2\tilde{m}^2][\Gamma(1-\nu)/\Gamma(1+\nu)](-2VV')^{\nu-1}.\end{aligned} \tag{8.12}$$

Similarly, the amplitude for bremsstrahlung of a scalar photon with four-momentum $p-p'$ is [see Fig. 1(d)]

$$\begin{aligned}\mathfrak{M}&=\psi^\dagger(\mathbf{p}',\mathbf{q}')[1/\hat{L}(p')][1/\hat{L}(p)]\psi(\mathbf{p},\mathbf{q})\\&=[1/\gamma(p^2)][1/\gamma(p'^2)]\\&\quad\times\Gamma(1-\nu)\Gamma(1-\nu')\Gamma(1+\nu)\Gamma(1+\nu')\\&\quad\times[\gamma(p^2)\gamma(p'^2)/2\pi^2\tilde{m}^2][1/\Gamma(1+\nu)][1/\Gamma(1+\nu')]\\&\quad\times V'^A\cdots\tilde{\psi}_{A\cdots}\{\psi'(p',\nu')^\dagger\psi(p,\nu)\}\tilde{\psi}'^{B\cdots}V_B\cdots.\end{aligned}$$

The quantity in { } is given by (A20); hence

$$\begin{aligned}\mathfrak{M}&=(1/\pi^2\tilde{m}^2)\Gamma(1-\nu)\Gamma(1-\nu')[(1+\lambda'\lambda)/2]^{1-\nu-\nu'}\\&\quad\times\sum_k\binom{\nu-1}{k}\binom{\nu'-1}{k}(V'\lambda)^{\nu'-1-k}(-V'\Delta V)^k(-\lambda'V)^{\nu-1-k}\\&=(1/\pi^2t)\Gamma(1-\nu)\Gamma(1-\nu')(2VV')^{\nu+\nu'}(\lambda'V)^{-\nu'}(\lambda V')^{-\nu}\\&\quad\times F_1\left(\nu,\nu';1;\frac{\omega(\lambda\lambda')-1}{(\lambda\lambda')-1}\right),\end{aligned} \tag{8.13}$$

where $t=(q-q')^2$. This is a Reggeized double-peripheral production amplitude, the "Toller variable" being defined by

$$\begin{aligned}\omega(\lambda\lambda')&=\lambda\lambda'+(VV')[1-(\lambda\lambda')^2]/(\lambda V')(\lambda'V)\\&=\epsilon^{AB}\epsilon'_{AB}/(\epsilon_{CD}\epsilon^{CD}\epsilon'_{AB}\epsilon'^{AB})^{1/2},\end{aligned} \tag{8.14}$$

where $\epsilon_{AB}=\epsilon_{ABCDE}V^C\lambda^D\lambda'^E$ and ϵ'_{AB} depends on V'. The Reggeized quantum number is n rather than l, but an expansion in terms of l poles is possible.

APPENDIX

Representation of $SO(4,1)$

Let[25] Z_A $(A=0,1,2,3,4)$ be a set of real coordinates subject to $Z_0>0$ and

$$Z^2\equiv Z_0^2-Z_1^2-Z_2^2-Z_3^2-Z_4^2=0. \tag{A1}$$

[25] Most of this Appendix is reproduced from Ref. 19. See also I. M. Gel-fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. V; and C. Itzykson and I. Todorov, in *Proceedings of the First Coral Gables Conference on Fundamental Interactions at High Energy* (Freeman, San Francisco, 1969).

Let $f(Z)$ be one of a set of continuous functions of Z_A and let $Z \to \Lambda Z$ be a proper $(4+1)$-dimensional Lorentz transformation, the operators

$$T(\Lambda): f(Z) \to f(\Lambda^{-1}Z) \quad \text{(A2)}$$

form a linear representation of $SO(4,1)$. If $f(Z)$ is homogeneous of degree N in Z_A, then so is $f(\Lambda^{-1}Z)$. We may therefore fix N, and we choose $N=-2$ because this is the only choice that allows for the existence of Majorana matrices. When we try to define a Hilbert space by introducing an inner product of the type

$$\int f^*(Z) K(Z,Z') g(Z') (dZ)(dZ'), \quad \text{(A3)}$$

$$(dZ) = d^5 Z\, \theta(Z_0) \delta(Z^2), \quad \text{(A4)}$$

we discover that the requirements of invariance under (A2) together with homogeneity of the functions f and g always lead to divergent integrals. This must be remedied by allowing a finite range of N. The result of a rigorous treatment is that integrals like (A3) give sensible results if and only if the degree of homogeneity of the integrand (including the differentials) is fixed and equal to zero. Thus $K(Z,Z')$ must be invariant and homogeneous of degree $-N-3$ in both Z and Z'. In other words, the inner product is

$$(f,g) \sim \int (dZ)(dZ') f^*(Z) (ZZ')^{-N-3} g(Z'). \quad \text{(A5)}$$

The rule for converting this formal divergent integral to a meaningful expression is to introduce homogeneous coordinates and drop one of the integrations.[25] Define

$$q_\mu \equiv m_1 Z_4^{-1} Z_\mu, \qquad \mu = 0, 1, 2, 3 \quad \text{(A6)}$$

$$\psi_q \equiv Z_4^{-N} f(Z). \quad \text{(A7)}$$

Substituting this into (A5), dropping $\int d(\ln Z_4) d(\ln Z_4')$, and setting $N=-2$, one gets Eq. (3.6). The constant factor is arbitrary except for the sign which is arranged to make the norm positive.

The expressions (3.3) and (3.4) for the infinitesimal generators are easily obtained from (A2) and (A7). The Hermiticity of these operators in the metric (3.6) can easily be verified directly.

Majorana Matrices

As remarked above, it is possible to make sense out of the integral

$$F(Z) = \int f(Z') (ZZ')^{-N-3} (dZ'). \quad \text{(A8)}$$

This function is homogeneous of degree $-N-3$; consequently, if the functions $f(Z)$ of degree $N=-2$ are the basis for an irreducible representation of $SO(4,1)$, then the functions $F(Z)$ of degree $N=-1$ are the basis for an equivalent representation. There follows that multiplication of $f(Z)$ by Z_A can be interpreted as the action of operators Γ_A in Hilbert space. In fact, we may define[26]

$$\Gamma_A f(Z) \sim \int Z_A' f(Z') (ZZ')^{-N-4} (dZ'), \quad \text{(A9)}$$

$$\Gamma_A^{-1} f(Z) \sim Z_A^{-1} \int f(Z') (ZZ')^{-N-3} (dZ'). \quad \text{(A10)}$$

Both integrals make sense and both operators preserve the degree of homogeneity. If we use (A6) and (A7) and drop the integral $\int d(\ln Z_4')$ as before, we obtain Eqs. (3.2) and (3.5), except for the constant factor in (3.2) which will be calculated presently.

Discrete Eigenstates

It turns out that the representation is irreducible. Reduction according to the chain $SO(4,1) \to SO(4) \to SO(3) \to SO(2)$ gives a complete orthogonal basis consisting of the functions

$$Z_0^{-2} Y_{nlm}(Z_a/Z_0), \quad \text{(A11)}$$

where Y_{nlm} are four-dimensional spherical harmonics and $a=1, 2, 3, 4$. The parameter values are

$$n = 1, 2, \ldots, \quad \text{(A12)}$$

$$l = 0, 1, \ldots, n-1, \quad \text{(A13)}$$

and, of course, $m=-l, -l+1, \ldots, l$. There are n^2 basis vectors for each value of n; these form an irreducible representation of $SO(4)$ with the Casimir operator ($a, b = 1, 2, 3, 4$)

$$\tfrac{1}{2} \sum_{a,b} s_{ab}^2 = n^2 - 1. \quad \text{(A14)}$$

It turns out that Γ_0 is a multiple of n; we choose the normalization of Γ_0 so that $\Gamma_0 = n$; then it may easily be verified that

$$[\Gamma_A, \Gamma_B] = -i s_{AB}, \quad \text{(A15)}$$

which shows that the operators Γ_A and s_{AB} satisfy the commutation relations of $SO(4,2)$.[27] The operators Γ_A are Hermitian, so this representation is unitary. From this we can also conclude that Γ_1, Γ_2, Γ_3 and Γ_4 have continuous spectra covering the whole real axis. These results concerning the spectra of Γ_A lead easily to (4.6).

If $\lambda_0 > |\lambda_4|$, then the eigenstates of the operator $\Gamma_0' = \lambda_0 \Gamma_0 - \lambda_4 \Gamma_4$ are $(Z_0')^{-2} Y_{nlm}(Z_a'/Z_0')$, where Z_A' are defined just like Γ_A'—Eqs. (4.10) and (4.11)—and the arguments Z_a'/Z_0' are the u_a defined by (4.12). Using (A7) we find the corresponding q-space wave functions (4.15).

The normalization constant in the expression (3.2) for Γ_4^{-1} can now be calculated. The eigenvector of Γ_0 with eigenvalue 1 is q_0^{-2} (up to a constant). Thus

$$\Gamma_4^{-1} q_0^{-2} = \Gamma_4^{-1} \Gamma_0 q_0^{-2} = m_1^{-1} q_0^{-1}. \quad \text{(A16)}$$

[26] The fact that the two operators defined by (A9) and (A10) are inverses of each other follows easily from the fact that their product is an $SO(4,1)$ invariant of an irreducible representation.

[27] The matrix elements of all the $SO(4,2)$ generators, including the Γ matrices, were worked out in Ref. 19.

But

$$\Gamma_4^{-1}q_0^{-2}=c\int[(dq')/(q-q')^2]q_0'^{-1}$$
$$=-\pi^2 c/m_1 q_0, \quad \text{(A17)}$$

so that c must be $-\pi^{-2}$.

Form Factors

The inner products

$$\psi(p,nlm)^\dagger\psi(p',n'l'm')=(\psi(\cdot),\Gamma_4^{-1}\psi(\cdot))$$

can be found by evaluating the integrals

$$-\pi^{-2}\int(dq)(dq')(q-q')^{-2}\psi_q^*(\cdot)\psi_{q'}(\cdot)$$
$$=m_1^{-1}\int(dq)(\lambda' Q)\psi_q^*(\cdot)\psi_q(\cdot).$$

The integrations can actually be carried out if one recognizes that the wave functions are five-dimensional spherical harmonics on the cone (A1), with the help of the addition formulas for Gegenbauer functions. The desired results can also be obtained with less effort by tensor methods. Let us expand the polynomial $Y_{nlm}(\lambda,x)$ by writing

$$2\pi n^{-1/2}Y_{nlm}(\lambda,x)=x^{A_1}\cdots x^{A_{n-1}}\tilde{\psi}(p,nlm)_{A_1\cdots A_{n-1}}. \quad \text{(A18)}$$

Here λ is the quantization axis for n, x is any unit vector normal to λ, and the tensor $\tilde{\psi}$ is traceless, symmetric, and transverse to λ. The wave functions are thus given by

$$\psi_q(p,nlm)=(m_1 n/\pi\sqrt{2})(\lambda Q)^{-n-1}$$
$$\times Q^{A_1}\cdots Q^{A_{n-1}}\tilde{\psi}(p,nlm)_{A_1\cdots A_{n-1}}. \quad \text{(A19)}$$

We may sometimes suppress the arguments nlm in $\tilde{\psi}$, since n is already given by the number of indices and the angular momentum decomposition is usually not needed.

The inner products can be evaluated very easily by extending the $SO(4)$ tensor notation to $SO(4,1)$. The result is[22]

$$\psi(p,n)^\dagger\psi(p',n')$$
$$=(-)^{n+n'}(1+\lambda\lambda')^{1-n-n'}\sum_{k=0,1,\cdots}\binom{n-1}{k}\binom{n'-1}{k}$$
$$\times[\tilde{\psi}(p)^{\dagger A_1\cdots A_{n-1}}\lambda'_{A_{k+1}}\cdots\lambda'_{A_{n-1}}]\Delta_{A_1}{}^{B_1}\cdots\Delta_{A_k}{}^{B_k}$$
$$\times[\lambda^{B_{k+1}}\cdots\lambda^{B_{n'-1}}\tilde{\psi}(p')_{A_1\cdots A_{n'-1}}], \quad \text{(A20)}$$

where

$$\Delta_A{}^B=\lambda_A'\lambda^B-(1+\lambda\lambda')\delta_A{}^B. \quad \text{(A21)}$$

In the case of the discrete states, when n and/or n' is an integer, the sum over k is cut off by the binomial coefficients. The matrix elements of Γ_A are given by[22,25]

$$\psi(p,n)^\dagger\Gamma_A\psi(p',n')$$
$$=-(\partial/\partial\lambda^A+\partial/\partial\lambda'^A)\psi(p,n)^\dagger\psi(p',n'). \quad \text{(A22)}$$

In order for this formula to be applicable, it is necessary to represent the inner product by (A20); replacement of 1 by λ^2 anywhere in (A20) invalidates (A22).

Addition Formulas

The sum over l and m in (5.3) may be carried out by using (4.20) and

$$\sum_{lm}Y_{nlm}(\lambda,\lambda'')Y_{nl-m}(\lambda,\lambda')=(n/2\pi^2)P_{n,4}(\phi) \quad \text{(A23)}$$

or more directly by using (A19) and

$$\lambda''^{A_1}\cdots\lambda''^{A_{n-1}}\sum_{lm}\tilde{\psi}(p)_{A_1\cdots A_{n-1}}\tilde{\psi}(p)^{\dagger B_1\cdots B_{n-1}}\lambda_{B_1}'\cdots\lambda_{B_{n-1}}'$$
$$=2[(\lambda''\lambda)^2-1]^{(n-1)/2}[(\lambda'\lambda)^2-1]^{(n-1)/2}P_{n,4}(\phi). \quad \text{(A24)}$$

The argument ϕ is the angle between the projections of λ' and λ'' on the plane normal to λ, and

$$P_{n,4}(\phi)=\sin(n\phi)/\sin\phi. \quad \text{(A25)}$$

When n is pure imaginary the addition formula (A23) remains valid, with l running over all non-negative integers.[28] The precise expressions for Y_{nlm} in the two cases are

$$Y_{nlm}=\frac{(2/\pi)^{1/2}}{(2l+1)!!}\left[n\frac{(n+l)!}{(n-1-l)!}\right]^{1/2}\sin^l\phi$$
$$\times F(1+l+n;1+l-n;\tfrac{3}{2}+l;\tfrac{1}{2}-\tfrac{1}{2}\mu)Y_{lm} \quad \text{(A26)}$$

when ϕ is real and n is integer, and

$$Y_{nlm}=\frac{i(2/\pi)^{1/2}}{(2l+1)!!}\left[\frac{(n+l)!(l-n)!}{(n-1)!(-n-1)!}\right]^{1/2}\sinh^l(i\phi)$$
$$\times F(1+l+n,1+l-n;\tfrac{3}{2}+l;\tfrac{1}{2}-\tfrac{1}{2}\mu)Y_{lm} \quad \text{(A27)}$$

when ϕ and n are imaginary.

Symmetry of T Matrix

The T matrix is a matrix element:

$$T=\psi^\dagger\hat{L}_0\hat{L}^{-1}\hat{L}_0\psi,$$

with $\hat{L}=\Gamma_4 L$ and $\hat{L}_0=\Gamma_4(p_2^2-m_2^2)$. If a prime indicates the result of a transformation that is unitary in the group metric $\psi^\dagger\psi$ of $\mathcal{H}_R$, then

$$T=\psi'^\dagger\hat{L}_0'\hat{L}'^{-1}\hat{L}_0'\psi'.$$

One such transformation is an $SO(4,1)$ rotation in the plane formed by p_μ and the fourth axis. In the center-of-mass system, consider the transformation

$$\Gamma_4'=\cosh\theta\,\Gamma_4+\sinh\theta\,\Gamma_0,$$
$$\Gamma_0'=\cosh\theta\,\Gamma_0+\sinh\theta\,\Gamma_4,$$
$$\mathbf{\Gamma}'=-\mathbf{\Gamma},\qquad \sinh\theta=-2p_0m_-(p^2-m_-^2)^{-1}.$$

[28] A more general addition formula of this type was derived in C. Fronsdal, Trieste Internal Report No. 15, 1967 (unpublished). The special case needed here was given by M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 346 (1966).

The parameter was chosen so that the effect on L is given by

$$\hat{L}'=(p^2-m_1^2+m_2^2)\Gamma_4'-2m_2\Gamma_0'p_0-\gamma.$$

This is the same form as L, with m_1 and m_2 interchanged, which reveals a symmetry of the wave operator, of the solutions of the wave equation, and ultimately of T. To express this symmetry in terms of the momentum variables we must define q_μ' symmetrically; thus, while $q_\mu=m_1\Gamma_4^{-1}\Gamma_\mu$, we must define $q_\mu'=m_2\Gamma_4'^{-1}\Gamma_\mu'$. Then

$$T(p,q,m_1,m_2)=T(p,q',m_2,m_1).$$

Here a single set of momentum variables represents both initial and final values, to avoid a cluttered notation.

The expression for q' in terms of q is

$$q_0'=m_2\frac{q_0\cosh\theta+m_1\sinh\theta}{m_1\cosh\theta+q_0\sinh\theta},$$

$$\mathbf{q}'=\frac{-m_2\mathbf{q}}{m_1\cosh\theta+q_0\sinh\theta}.$$

On the mass shell

$$(p-q)^2=m_2^2$$

or

$$q_0=(p^2+m_1^2-m_2^2)/2p_0,$$

which gives

$$q_0'=(p^2-m_1^2+m_2^2)/2p_0=p_0-q_0,$$

and $\mathbf{q}'=-\mathbf{q}$. Thus, on the mass shell $q'=p-q=p_2$, and the symmetry of the T matrix takes the form

$$T(p,p_1,m_1,m_2)=T(p,p_2,m_2,m_1),$$

or, briefly, since $p=p_1+p_2$ and $p_1^2=m_1^2$, $p_2^2=m_2^2$,

$$T(p_1,p_2)=T(p_2,p_1),$$

which is the desired result.

So far we have treated $\gamma=\Gamma_4V$ as a constant. A sufficient condition for the symmetry of the scattering matrix is that γ be symmetric (invariant) under the above transformation, including the interchange of m_1 and m_2. This condition is not necessary, since V is not completely determined by the on-shell T matrix. A necessary condition is that V be symmetric on shell.

PHYSICAL REVIEW D VOLUME 1, NUMBER 12 15 JUNE 1970

Feynman-Like Diagrams Compatible with Duality. II. General Discussion Including Nonplanar Diagrams*

K. Kikkawa†
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706
AND
S. A. Klein
Joint Science Department, Claremont Colleges, Claremont, California 91711
AND
B. Sakita and M. A. Virasoro
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706

(Received 2 July 1969)

A general formulation of duality theory is presented that includes nonplanar Feynman-like diagrams. All diagrams, planar as well as nonplanar, are so classified that the diagrams in a given class are mutually connected by duality. A prescription is given for constructing an integral representation of the scattering amplitude for each class. Some fundamental properties of the duality relations are discussed.

I. INTRODUCTION

In Paper I[1] we have discussed planar Feynman-like diagrams (FLD), the corresponding duality amplitudes, and their high-energy behavior. In this paper we continue the program to include nonplanar diagrams.

As was shown in I for the planar diagrams, FLD's and dual diagrams have a one-to-one correspondence. Once the dual diagram of a given FLD is drawn, by erasing its internal lines we obtain the duality diagram. By performing various triangulations of the duality diagram, we obtain a class of dual diagrams to which all the FLD's connected by duality correspond. Since a dual diagram is composed of a set of triangles which are connected by the common sides, it is a surface. Therefore it is possible to study the classification problem of dual diagrams, accordingly of FLD's, based on the topology of two-dimensional surfaces.

* Work supported in part by the University of Wisconsin Research Committee, with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U.S. Atomic Energy Commission under Contract No. AT(11-1)-881, C00-248.

† Present address: Department of Physics, University of Tokyo, Tokyo, Japan.

[1] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969).