this was an unnecessary precaution. That is, they could be extended quite naturally into this region.

Mention might also be made at this point of the possibility of utilizing the techniques presented in the solution of this problem in finding other solutions to Einstein's time-dependent equations. A subsequent paper will follow this format, with the initial conditions corresponding to a uniform Maxwellian magnetic field as is found in the interior of a long ideal solenoid. Thus the behavior of an initial radially limited bundle of flux will be presented which possibly has relevance to the question of the ability of an astrophysical magnetic field to affect the process of "gravitational collapse."

In conclusion, the authors believe this work shows that imposing "boundary conditions at infinity" is not always the most appropriate way to start on a general-relativistic problem, and that more progress in finding time-dependent solutions may come by making other assumptions and investigating what happens "at infinity."

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Noncausality and Instability in Ultradense Matter

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In a classical system of particles interacting via neutral vector-meson exchange, at very high densities the pressure exceeds the relativistic energy density, and the speed of low-frequency sound waves exceeds the speed of light in vacuum. A quantum version of the same model, if it is stable against spontaneous pair production, can be neither ultrabaric nor superluminal, if, at high density, the correlation energy increases faster than the number of particles. Real matter, if it is stable at very high densities, is not expected to show noncausal sound propagation.

I. NONCAUSALITY AND INSTABILITY

IN two earlier papers,[1,2] we considered the possibility of pressure exceeding energy density, $p>\epsilon$, in unquantized relativistic matter. If, at very high densities, real matter could be ultrabaric ($p>\epsilon$), this would be important, in principle, because the speed of low-frequency compressional waves must then exceed that of light in vacuum, $c_s=c(dp/d\epsilon)^{1/2}>c$. The possibility $p>\epsilon$ would also be important practically because the upper limit on the mass of neutron stars could then be larger than heretofore thought possible.

A superluminal group velocity ($c_s>c$) for a particular frequency need not violate causality. It is sufficient for the causal propagation of stable signals that the index of refraction, $n(\omega)=ck/\omega$, obey the Kramers-Kronig dispersion relation

$$n(\omega)-1=\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}n(\omega')2\omega' d\omega'}{\omega'^2-\omega^2}. \qquad (1.1)$$

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[1] S. A. Bludman and M. A. Ruderman, Phys. Rev. **170**, 1176 (1968).

[2] M. A. Ruderman, Phys. Rev. **172**, 1286 (1968).

A system can have $(\omega/k)_{\omega=0}>c$ and still be causal, provided

$$n(0)=1+\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}n(\omega')d\omega'}{\omega'}<1,$$

i.e., provided the system is amplifying enough at high frequencies [$\mathrm{Im}n(\omega)<0$] or, in other words, provided the system is not in its lowest-energy state. In our examples, however, a dynamic calculation of the k-ω eigenvalue equation showed that $k(\omega)$ [and, therefore, $n(\omega)$] had branch singularities[2] in the upper half of the complex ω plane, so that the Kramers-Kronig dispersion relation was indeed not satisfied.

The significance of singularities in the upper half of the complex ω plane is that some normal modes grow in time so that for arbitrary initial or asymptotic conditions a wave packet is generally unstable. Wave solutions that remain bounded in the future are possible if suitable initial or asymptotic conditions are imposed, but then the system preaccelerates before the imposition of any external force. Upper-half-plane singularities thus mean that the system is *unstable* or *noncausal*, or in electrical-engineering terminology, that the system is *active*. In either case, the speed of zero-frequency waves $c_s>c$, and the time development of the system is not fixed by Cauchy conditions.

In an active system, the choice between instability and noncausality is made by boundary conditions. If, in the complex $\omega(k)$ plane, one chooses a contour along the real ω axis, then the upper-half-plane singularities give a nonvanishing Green's function for $t<0$. For $t>0$, on the other hand, these upper-half-plane singularities do not contribute and the Green's function is bounded in time. For such a contour, C_{stable}, the system's development is stable but noncausal and, because some $k(\omega)$ are excluded for $t>0$, also nonlocal in space. Alternatively, one may choose a contour above all singularities in the upper half of the complex ω plane. For this contour, C_{causal}, the Green's function vanishes for $t<0$. For $t>0$, however, it contains normal modes that grow exponentially in time.

We want to emphasize that for an active system (one violating the Kramers-Kronig relation), a stable Cauchy problem does not exist. One can insist on causal propagation at the price of instability,[3] or one can achieve stability at the price of some spatial nonlocality and temporal noncausality. For example, the motion of a single relativistic classical Lorentz-Dirac electron[4] could be either "runaway" (self-accelerated) or noncausal (preaccelerated). Because, under the action of finite forces, the motion of a single electron is expected to be bounded, an asymptotic condition is imposed which rejects the runaway solution and makes the electron preaccelerate at times $2e^2/3Mc^3$ before the imposition of any external force. This nonvanishing preacceleration is clearly associated with the use of a finite renormalized mass M which assures the electron's stability by compensating its internal stress.

We considered[1,2] a lattice of N classical particles interacting among themselves through retarded neutral-vector-meson fields and found that, when sufficiently compressed, $p>\epsilon$ and $c_s>c$. In this case, contrary to the electromagnetic-interaction case,[5] the finite range of interaction between particles permitted the microscopic noncausality-instability to become macroscopic. This classical lattice is fundamentally unstable against self-accelerating runaway because the stationary lattice is not a system with a true energy minimum. If, as is done for a single particle, these runaway solutions are removed by fiat, so that the stationary lattice is a true ground state, then the lattice becomes noncausal. In the instability case, the Green's functions vanish outside the light cone and signals do not propagate faster than light. In the second case, the Green's function is nonvanishing outside the light cone and signal propagation is noncausal.

[3] This appears to be the point of view of Bers, Fox, Kuper, and Lipson [reports from Technion-Israel Institute of Technology (unpublished)] towards tachyons and towards our classical nonlinear-field model. Note that even if the attitude of Fox, Kuper, and Lipson is imposed to make the acoustic branch of our model causal, the optic branch of our model remains noncausal.

[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938); see also F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[5] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

II. CONDITIONS FOR MATTER TO BE ULTRABARIC OR SUPERLUMINAL

In this paper, we describe the quantized version of this same lattice and deduce the equation of state in the high-density limit, continuing to assume the existence of a lowest-energy state. The energy of an N-particle system can be written $E=E_0+E_H+E_c$ in terms of $E_0=NMc^2$, the renormalized rest energy of the free sources; E_H, the Hartree energy due to a source's interaction with the uniformly smeared continuum of uncorrelated sources; and E_c, the correlation energy correcting for the anticorrelation in position between near neighbors. For fixed volume V, E_H is always proportional to $\frac{1}{2}N^2$, the number of particle pairs. Since correlation lowers the energy, E_c is always negative, vanishing when the number density $n=0$ and dominated by the Hartree energy at $n=\infty$. In terms of the energy density $\epsilon=E/V$,

$$\epsilon=\epsilon_0+\epsilon_H+\epsilon_C=nMc^2+Bn^2+\epsilon_c\,, \tag{2.1}$$

where B is a positive constant and $\epsilon_c<0$. The pressure

$$p=-\frac{dE}{dV}=n^2\frac{d(\epsilon/n)}{dn}=n\frac{d\epsilon}{dn}-\epsilon\equiv n\mu-\epsilon\,, \tag{2.2}$$

and

$$(c_s/c)^2=nd\mu/\mu dn\,. \tag{2.3}$$

Here $\mu=d\epsilon/dn$ is the chemical potential, the energy necessary to add one more particle to the system.

The condition for matter to be ultrabaric is

$$0<n\mu-2\epsilon \tag{2.4}$$

and, to be superluminal, is

$$0<nd\mu/dn-\mu\,. \tag{2.5}$$

In terms of the correlation energy and the correlation pressure,

$$p_c=n\mu_c-\epsilon_c\,, \tag{2.6}$$

these conditions are

$$\begin{aligned} Mc^2&<(p_c-\epsilon_c)/n\,, &&\text{ultrabaric}\\ &<\frac{d}{dn}(p_c-\epsilon_c)\,, &&\text{superluminal.} \end{aligned} \tag{2.7}$$

Since, at low density, ϵ_c and p_c vanish at least as fast as n, matter can become ultrabaric at some density n_U only if $p_c-\epsilon_c$ increases faster than $2Mc^2n$ at some lower density $n_s<n_U$.

We have already mentioned that $-\epsilon_c$ does not, at high n, increase faster than n^2 (which is proportional to $\frac{1}{2}N^2$ the number of pairings between *all* N particles). Because the correlation feels the effects of pairings with *some* of the N particles, one would expect, at high n, that $-\epsilon_c$ would increase faster than linearly in n, unless a phase transition takes place. If this expected behavior takes place, then at high n,

$$\gamma(n)\equiv\frac{nd\epsilon_c/dn}{\epsilon_c}>1\,,$$

and $p_c=(\gamma-1)\epsilon_c<0$. Indeed, whenever in any density region the correlation energy obeys a power law $\epsilon_c=-A_n{}^\gamma$ with constant exponent $\gamma>1$, then p_c is negative in that density region.

III. CLASSICAL BARYONIC LATTICE

We first recapitulate the result obtained earlier[1] for a classical lattice of point sources which repel each other because they are coupled with coupling constant g to a neutral-vector-meson field of finite range μ^{-1}. At fixed average density $n=N/V=1/a^3$ and zero temperature, N such classical sources form a lattice, with average separation a, whose energy per particle is

$$\begin{aligned}\epsilon/n=E/N&=(E_0+E_H+E_c)/N\\&=Mc^2+2\pi g^2/\mu^2a^3-\beta 2\pi g/a+O(1),\end{aligned}\quad(3.1)$$

in the high-density limit $\mu a\ll 1$. In this case,[1] the correlation energy satisfies a $\frac{4}{3}$ power law,

$$\epsilon=nMc^2+Bn^2-An^{4/3},\quad(3.2)$$

with $B=2\pi g^2/\mu^2$, $A=2\pi g^2\beta$, β being a dimensionless constant which equals 1/12 for a simple cubic lattice. The superluminality condition (2.7) is satisfied when

$$Mc^2<\frac{8}{9}(2\pi\beta)\frac{g^2}{a}.\quad(3.3)$$

This exact result shows that, for any repulsive coupling strength, such classical matter must become superluminal and ultrabaric when that fraction of a baryon's self-energy outside a exceeds the phenomenological mass Mc^2. Mass renormalization, which reduces the energy density while leaving the dynamic pressure unaltered, makes matter superluminal and ultrabaric at sufficiently high particle densities. This static result is confirmed[1] in a dynamic calculation of all the lattice normal modes using the usual relativistic and retarded interaction between particles. The wave number $k(\omega)$ has an infinite number of branch points in the upper half of the complex ω plane,[2] thus violating Kramers-Kronig causality.

IV. QUANTUM BARYONIC MATTER

The quantum treatment of this same lattice model would involve the additional consideration of zero-point energy, of real particle-antiparticle production, and of radiative corrections. In considering quantum matter we will now do two things: (1) We prove that the Hartree energy, which neglects correlations including that due to the identity of particles, in quantum theory retains the same form (3.1) as in classical theory, i.e., we will show that the Hartree energy increases with N^2 and is unaffected by vacuum polarization and other radiative corrections. (2) We show that, if at high density $-\epsilon_c(n)$ increases with n faster than linearly so that $\gamma(n)>1$, then being ultrabaric is incompatible with stability against baryon-antibaryon production.

A. Hartree Energy

In a box of volume $V=Na^3$, consider N physical (dressed) spinless baryons which are the conserved sources of a neutral-vector-meson field of bare mass μ_0 and bare coupling g_0. Let ψ_0 be a product of N independent dressed-particle wave functions, each of zero momentum. ψ_0 neglects correlations between baryon sources and bosons and is a state of zero total four-momentum. We prove the following:

Theorem: In the approximate ground state ψ_0, the exact Hamiltonian has the expectation

$$\langle\psi_0|H|\psi_0\rangle=NMc^2+2\pi g_0{}^2N^2/\mu_0{}^2V.\quad(4.1)$$

Proof: All of the self-interactions of the N baryons at rest are contained in the phenomenological mass term NMc^2. The second term follows from current conservation which makes the exact unrenormalized vector-meson propagator $\Delta_F'(k^2)\delta_{\mu\nu}$ reduce to the bare propagator $(k^2-\mu_0{}^2)^{-1}\delta_{\mu\nu}$ in the limit $k\to 0$.[6] For two fixed dressed baryons, the space-integrated potential energy is

$$\int d^3x\, g_0{}^2\Delta_F'(x)=g_0{}^2 4\pi/\mu_0{}^2,$$

so that the average potential energy of two baryons is $4\pi g_0{}^2/\mu_0{}^2V$. Since there are $\frac{1}{2}N^2$ pairings of two particles, the total potential energy is given by the last term in Eq. (3).

Because all physical states in ψ_0 have zero four-momentum, only the exchange of vector mesons of zero four-momentum between dressed sources can contribute to the ground-state energy. (Other meson contributions are included in the phenomenological rest masses.) For these, by Ward's identity, radiative corrections vanish, as do the effects of vacuum polarization. This is how current-conserving interactions maintain the Hartree energy (4.1), so long as correlations and the identity of source bosons and of vector mesons are omitted from ψ_0.

B. Correlation Energy and Instability

Now, the symmetry of the Hamiltonian will make the true ground state ψ_0' automatically symmetric under particle exchange. The ground-state energy calculated with ψ_0' must be less than (4.1), i.e., that computed with the simple-product wave function ψ_0. Therefore, it remains to consider only the effects of nonstatistical correlation, which also lower the energy. In the classical lattice, the correlation energy $E_c=-\beta 2\pi g^2N/a$ increased faster than N as the particle spacing $a\to 0$, i.e., $\epsilon_c\sim n^{4/3}$. This led to superluminal and ultrabaric behavior.

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[6] K. Johnson, Nucl. Phys. 25, 431 (1961).

1. *Stability of Vacuum*

We now show that if the vacuum is to be stable against baryon-antibaryon production, $-\epsilon_c$ cannot increase faster than nMc^2.[7]

Consider first a volume V containing N_+ baryons described by their exact wave functions $\psi_0'(N_+)$, including the baryon-baryon correlation $\epsilon_c^{(NN)}$. Consider also an identical volume V containing N_- antibaryons and let $N_-=N_+$ so that the energies of these two separated systems are equal. Now let us put this baryon system and this antibaryon system together in one box of volume V and describe the ground state of the combined system by the approximate product wave function $\psi_0'=\psi_0'(N_+)\psi_0'(N_-)$. The combined system is over-all neutral so the Hartree energy is zero. The true correlation energy E_c consists of baryon-baryon and antibaryon-antibaryon correlations, which are equal, and of a baryon-antibaryon correlation, which is negative. Thus

$$E_c=E_c^{(NN)}+E_c^{(\bar{N}\bar{N})}+E_c^{(N\bar{N})}<2E_c^{(NN)}.$$

Now, for the product wave function,

$$\langle\psi_0'|H|\psi_0'\rangle=(N_++N_-)Mc^2+2E_c^{(NN)},$$

while for the exact wave function including correlations, Φ_0',

$$E=\langle\Phi_0'|H|\Phi_0'\rangle\leqslant\langle\psi_0'|H|\psi_0'\rangle.$$

Thus for the true energy E of the system of N_+ real baryons and N_- real antibaryons in the box V, we have

$$E<(N_++N_-)Mc^2+2E_c^{(NN)}. \tag{4.2}$$

But if the vacuum is to be stable against the production of such $N_+=N_-$ real baryons and antibaryons, then $E\geqslant 0$ is necessary. This requires

$$-\epsilon_c<nMc^2, \tag{4.3}$$

where $n=N_+/V=N_-/V$ is the number density of baryons or antibaryons in a separated system.

2. *Stability of N-Baryon System*

A second stability condition follows from the requirements that the N-baryon system be stable against the production of an additional baryon-antibaryon pair. Since $\mu_c(n)$ is the energy necessary to add one more baryon or antibaryon, with such a baryon-antibaryon pair the correlation energy

$$E_c^{(\mathrm{pair})}<2\mu_c(n),$$

since $E_c^{(\mathrm{pair})}$ includes the negative energy of the baryon-antibaryon interaction. The creation of such a baryon-antibaryon pair requires energy

$$E^{(\mathrm{pair})}=2Mc^2+E_c^{(\mathrm{pair})}<2Mc^2+2\mu_c. \tag{4.4}$$

Stability requires that $E^{(\mathrm{pair})}>0$, or

$$-\mu_c<Mc^2. \tag{4.5}$$

Note that if $p_c=n\mu_c-\epsilon_c>0$, vacuum stability (4.3) implies N-baryon stability (4.5), while if $p_c<0$ the reverse is the case.

Combining the results (4.3) and (4.5), we have:

$$\begin{aligned}&\text{for vacuum stability,} && -\epsilon_c/n<Mc^2;\\ &\text{for } N\text{-particle stability,} && -\mu_c<Mc^2.\end{aligned} \tag{4.6}$$

Comparing with the conditions (2.7), we find the following:

(i) For matter to be ultrabaric and stable against baryon-antibaryon production in (1) the vacuum or (2) the N-baryon state, it is necessary that $p_c\geq 0$.

(ii) For matter to be superluminal the weaker requirements

$$dp_c/dn-p_c/n\geq 0, \quad dp_c/dn\geq 0$$

are necessary for stabilities (1) and (2), respectively.

Our conclusion is that, if at high density the correlation pressure

$$p_c<0$$

as expected, then the stability of the vacuum or N-baryon state against baryon-antibaryon production prevents matter from being ultrabaric or superluminal. Indeed, if, as in the classical lattice or quantum systems in the approximations considered in Ref. 7, ϵ_c obeys a power law with exponent $\gamma>1$, then stable matter can be neither superluminal nor ultrabaric at any density where the power law obtains.

If baryon-baryon interactions were repulsive enough to make matter highly incompressible, then baryon-antibaryon interactions would be attractive enough to make the vacuum unstable and to give matter no lowest energy.[8]

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To the Physics Department, Imperial College, where this work was begun, and to the Aspen Center for Physics, where it was completed, we are indebted for kind hospitality.

[7] When particle production is neglected, E_c can increase faster than N. For a nonrelativistic high-density Bose gas with repulsive Coulombic interactions, the correlation energy per unit volume is $\epsilon_c=-S(\hbar^2/ML^2)L^{-3}$, where S is a positive constant of order unity and $L=(\hbar^2/4\pi me^2M)^{1/4}$; thus $\epsilon_c/n\sim n^{1/4}=a^{-3/4}$. If the same static interaction is considered for a relativistic Bose gas, $\epsilon_c=-S'\times(\hbar c/4\pi me^2)^{1/3}$; thus relativistically $\epsilon_c/n\sim n^{1/3}=a^{-1}$. For fermion systems $n=a^{-3}=(8\pi/3)(p_F/2\pi\hbar)^3$ and $E_0/N=\frac{3}{4}(p_Fc)$ relativistically. The antisymmetrization introduces a negative Fock or exchange energy $-(3/2\pi)g_0^2(p_F/\hbar)$, in addition to the Hartree energy. Thus $E/N=(9\pi/8)^{2/3}\hbar c/a+4\pi g_0^2/\mu_0^2a^3-(81/8\pi)^{1/3}g_0^2/a$ in the Hartree-Fock approximation. Thus statistical anticorrelation suffices to make $p>\epsilon$ or $dp/d\epsilon>1$ for small enough a, provided $g_0^2/\hbar_c>\frac{1}{2}\pi$. The bare coupling constant $g_0^2/\hbar c=Z_3^{-1}(g^2/\hbar c)$, where the physical coupling constant (ρ mesons to nucleons) $g^2/\hbar c\sim 3$ and $Z_3^{-1}>1$ and, in perturbation theory, $Z_3^{-1}=\infty$.

[8] Indeed, this high-density instability develops for weakly interacting charged bosons [F. J. Dyson, J. Math. Phys. **8**, 1538 (1967)] and for sufficiently compressed gravitating matter.