Slowly Rotating Radiating Sphere and a Kerr-Vaidya Metric

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The problem of a spherically symmetric radiating body was first considered by Vaidya, who obtained what is often referred to as the "radiating Schwarzschild metric." It is well known that this metric, if expressed in radiation coordinates, differs from the Schwarzschild metric only in that the parameter m has been replaced by a function of retarded time. In this paper, the parameter m of the Kerr metric is considered to be a function of the retarded time and an exact expression for the energy tensor is obtained. It is shown that if a/m is small then this energy tensor is appropriate for directed radiation. To this approximation [i.e., terms of order $(a/m)^2$ are neglected] the Landau-Lifschitz pseudotensor is used to show that the angular momentum radiated is $-m'a$. The metric is also used to describe the physical properties of a slowly rotating radiating spherical shell, and it is shown that (provided $2m/R\ll 1$) the radiation gives rise to surface pressures proportional to the momentum radiated.

I. INTRODUCTION

THE problem of spherically symmetric radiation has been considered by many authors. The original "radiation metric" was obtained by Vaidya,[1] and the various ramifications of this result have been discussed by Raychaudhuri,[2] Israel,[3] and Lindquist, Schwartz, and Misner.[4] A different approach, using a modified energy tensor, has been proposed by Kaufmann.[5] In the above papers it is assumed that the source is a nonrotating spherical body. In this paper we attempt to obtain a more general solution to the radiation problem, one which includes the possibility of a rotating source.

It is well known that the Schwarzschild metric may be put in the form

$$ds^2=(1-2m/r)du^2+2dudr-r^2(d\theta^2+\sin^2\theta\, d\phi^2). \quad (1.1)$$

Vaidya's "radiating Schwarzschild metric" is obtained from (1.1) by simply replacing the constant m by an arbitrary function $m(u)$. The resulting metric satisfies the field equations

$$R_{ij}=[2m'(u)/r^2]w_iw_j, \quad (1.2)$$

where w_i is the null vector defined by

$$w_i=(1,0,0,0). \quad (1.3)$$

[Throughout this paper we shall adopt the following convention: Latin indices take the values 0,1,2,3, while Greek indices take the values 1,2,3. For coordinates as in (1.1) we make the identification $x^0=u$, $x^1=r$, $x^2=\theta$, $x^3=\phi$.]

It is generally accepted that the exterior field for a rotating Schwarzschild mass is provided by the Kerr metric. Hence, it would be plausible to proceed as follows:

(i) Write the Kerr[6] metric in a form analogous to (1.1) and consider the parameters m and a to be functions of the retarded time u.

(ii) Compute the Ricci tensor and thereby determine the form of the energy tensor.

(iii) Examine the resulting energy tensor to see if it gives a plausible description of a rotating radiating body.

The original intent of the authors was to actually carry out steps (i) and (ii) with the hopes of obtaining an energy tensor similar to that used by Vaidya. However, the computations seemed so prohibitive we found it necessary to settle for something less.

The form (see Newman and Janis[7]) of the Kerr metric which seems most suitable for our purposes is the following:

$$g_{ij}=\begin{bmatrix} 1-2mr/\rho^2 & 1 & 0 & (2mar/\rho^2)\sin^2\theta \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -\rho^2 & 0 \\ (2mar/\rho^2)\sin^2\theta & -a\sin^2\theta & 0 & -\sin^2\theta[(2ma^2r/\rho^2)\sin^2\theta+r^2+a^2] \end{bmatrix}, \quad (1.4)$$

where

$$\rho^2\equiv r^2+a^2\cos^2\theta. \quad (1.5)$$

Preliminary computations show

$$R_{11}=0,\quad R_{12}=-(2aa'r^2\sin\theta\cos\theta)/\rho^4. \quad (1.6)$$

If a' is not zero then the above equations rule out the possibilities of having

$$T_{ij}=qw_iw_j, \quad (1.7)$$

[1] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951); Current Sci. (India) **21**, 96 (1952); Nature **171**, 260 (1953).
[2] A. K. Raychaudhuri, Z. Physik **135**, 225 (1953).
[3] W. Israel, Proc. Roy. Soc. (London) **A248**, 404 (1958).
[4] R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. **137**, B1364 (1965).
[5] W. J. Kaufmann, Astrophys. J. **153**, 849 (1968).
[6] R. Kerr, Phys. Rev. Letters **11**, 237 (1963).
[7] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).

where w_i is some null vector. Although it is unlikely that T_{ij} would be in a form identical to (1.7), it is not unlikely that the energy tensor should be in some sense asymptotic to that form. On the other hand, if the physical situation were such that terms of the form aa'/r^2 could be neglected, then (1.6) and (1.7) would be less incompatible. The authors felt that some exact expressions would be of more interest so it was decided to make the *ad hoc* assumption $a'\equiv 0$. However, the question arises as to whether this assumption leads to anything physically realistic. The identification of ma with the angular momentum of a rotating sphere together with the assumption $a'=0$ would imply that

$$\omega R^2=\text{const}, \tag{1.8}$$

where ω is the angular velocity of the sphere with radius R. This situation could occur if both ω and R were not significantly affected by the radiation. In order to have a' identically zero, it is evident that the photons would have to be emitted in a particular fashion. However, it is not entirely unreasonable that terms of orders a' and ma'' would be negligible in the first-order analysis of a slowly rotating massive star. Since terms involving m' are to be retained, they must be large relative to a'. A more likely physical model would be a uniformly rotating spherical shell with a decreasing surface density. [In fact, if one initially ignored terms of orders $(a/m)^2$, a', and ma'' the result would be Eq. (3.1).]

II. RICCI EXPRESSIONS FOR $a'=0$

The computations which take one from (1.4) to the Ricci components will not be presented here. However, the procedure was as follows: The Christoffel symbols were written in the form

$$\left\{ {i \atop jk} \right\} = \left\{ {i \atop jk} \right\}^* + \Gamma^i{}_{jk}, \tag{2.1}$$

where $\{{}_j{}^i{}_k\}^*$ are the Christoffel symbols for the usual Kerr metric in which m is constant. The $\Gamma^i{}_{jk}$ involved the terms depending on m'. Obviously

$$R_{ij}=R_{ij}^*+\Gamma_{ij}, \tag{2.2}$$

where R_{ij}^* is the Ricci tensor formed from the $\{{}_j{}^i{}_k\}^*$ while Γ_{ij} involves the "perturbation terms." Since $R_{ij}^*\equiv 0$ it follows that

$$R_{ij}=\Gamma_{ij}. \tag{2.3}$$

Hence, in making the computations one can ignore terms involving strictly "starred quantities."

The above procedure led to the following:

$$R_{00}=2m'r^2(r^2+a^2)/\rho^6+(m''ra^2\sin^2\theta)/\rho^4, \tag{2.4}$$

$$R_{03}=-a\sin^2\theta[R_{00}+m'(r^2-a^2\cos^2\theta)/\rho^4], \tag{2.5}$$

$$R_{33}=a^2\sin^4\theta[R_{00}+2m'(r^2-a^2\cos^2\theta)/\rho^4], \tag{2.6}$$

$$R_{02}=(2m'ra^2\sin\theta\cos\theta)/\rho^4, \tag{2.7}$$

$$R_{32}=-a\sin^2\theta R_{02}. \tag{2.8}$$

All other R_{ij} are identically zero.

Now define the vectors w_i and a_i by

$$w_i=(1,0,0,-a\sin^2\theta), \tag{2.9}$$

$$a_i=(0,0,R_{02},-m'a\sin^2\theta(r^2-a^2\cos^2\theta)/\rho^4). \tag{2.10}$$

It follows from (2.3)–(2.9) that

$$R_{ij}=qw_iw_j+w_ia_j+w_ja_i, \tag{2.11}$$

where

$$q\equiv R_{00}.$$

From (1.4) it is seen that

$$w^i=(0,1,0,0), \tag{2.12}$$

and we conclude

$$w_iw^i=0,\quad w_ia^i=0,\quad R=0. \tag{2.13}$$

If we assume that our metric is a solution to the field equations

$$R_{ij}-\tfrac{1}{2}g_{ij}R=-8\pi T_{ij}, \tag{2.14}$$

then if follows that

$$T_{ij}=-(1/8\pi)(qw_iw_j+w_ia_j+w_ja_i). \tag{2.15}$$

III. THE CASE $(a/m)^2\ll 1$

The metric tensor and the field equations would be simplified if we could ignore terms of order $(a/r)^2$. On the other hand, if one is to avoid coordinate singularities we must have $r>2m$. Hence, it seems appropriate to assume that $(a/m)^2\ll 1$, an assumption that would be applicable to slowly rotating massive stars. If such is the case, the nonvanishing components of R_{ij} are

$$R_{00}=2m'/r^2,\quad R_{03}=(-3m'a\sin^2\theta)/r^2. \tag{3.1}$$

It follows that

$$T_{ij}=-(m'/4\pi r^2)\nu_i\nu_j, \tag{3.2}$$

where

$$\nu_i=(1,0,0,-\tfrac{3}{2}a\sin^2\theta). \tag{3.3}$$

Evidently

$$\nu^i=(0,1,0,\tfrac{1}{2}a/r^2), \tag{3.4}$$

so ν_i is the propagation null vector for the directed radiation. The solutions to the differential equation

$$\frac{dx^i}{ds}=\nu^i \tag{3.5}$$

are null geodesics which presumably give the path of the radiated photons. The solution to (3.5) may be put in the form

$$\phi=K_1-a/2r, \tag{3.6}$$

$$\theta=K_2,\quad u=K_3, \tag{3.7}$$

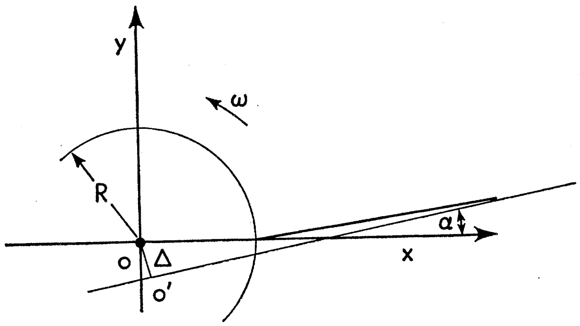

FIG. 1. Path of a photon.

the K_α being arbitrary constants. A photon emitted from the point $(u, R, \frac{1}{2}\pi, 0)$ travels in the equatorial plane along the curve

$$r=R\alpha/(\alpha-\phi), \quad \alpha\equiv a/2R. \tag{3.8}$$

(See Fig. 1.) The trajectory is asymptotic to the line

$$y=x\tan\alpha-\tfrac{1}{2}a\sec\alpha. \tag{3.9}$$

To a distant observer on this line the star would appear to be at 0′. However, if the star was not rotating, the radiation would be along radial lines and the same observer would see the star at 0. Hence, the rotation appears to displace the star by an amount

$$\Delta=\tfrac{1}{2}a. \tag{3.10}$$

The question arises as to whether the situation in Fig. 1 occurs when $a'\equiv 0$ or when a' is merely negligible. If a' were identically zero, then the photons would have to be emitted in a direction such that the back stress would tend to increase the angular velocity. As this does not seem to be the case, it would appear that the latter alternative is more tenable. In that case a random radiation on the surface could lead to an average photon path resembling that in Fig. 1. This does not seem inconsistent with the fact that a rotating object tends to drag local inertial frames in the direction of the rotation.

IV. LUMINOSITY AND ANGULAR MOMENTUM

The results on luminosity and momentum which were presented by Lindquist, Schwartz, and Misner[4] for the Vaidya case can readily be obtained for the "linearized" Kerr-Vaidya metric. In order to use the Landau-Lifshitz[8] pseudotensor, the coordinates must be such that the g^{ij} are asymptotic to the Minkowski metric. In these coordinates the g^{ij} have the form

$$\begin{aligned} g^{00}&=1+2m/r, \quad g^{0\alpha}=2mv^\alpha/r^2, \\ g^{\alpha\beta}&=-\delta^{\alpha\beta}+2mv^\alpha v^\beta/r^3, \end{aligned} \tag{4.1}$$

where

$$v^\alpha=x^\alpha+au^\alpha, \quad u^\alpha=(-y/r, x/r, 0). \tag{4.2}$$

As in the Vaidya case, it follows that the momentum four-vector and total luminosity are given by

$$p^i=(m;0,0,0), \quad L=-m'. \tag{4.3}$$

The angular momentum three-tensor is defined by

$$M^{\alpha\beta}=\int_s (x^\alpha h^{\beta0\gamma}-x^\beta h^{\alpha0\gamma}+\lambda^{\alpha0\gamma\beta})df_\gamma, \tag{4.4}$$

where

$$\begin{aligned} h^{\alpha0\beta}&\equiv(1/16\pi)(g^{\alpha0}g^{\beta l}-g^{l0}g^{\alpha\beta})_{,l} \\ &=\lambda^{\alpha0\beta l}{}_{,l} \end{aligned} \tag{4.5}$$

and the integration is over the two-dimensional surface enclosing the volume under consideration. In our case the surface will be a large sphere with center at the origin. Neglecting terms of orders $1/r^3$ and a^2/r^2, Eq. (4.4) reduces to

$$M^{\alpha\beta}=\frac{1}{16\pi}\int_s\left[\frac{2ma}{r^3}(\delta^{\alpha\gamma}u^\beta-\delta^{\beta\gamma}u^\alpha) +\frac{4ma}{r^5}x^\gamma(x^\alpha u^\beta-x^\beta u^\alpha) -\frac{2m}{r^2}(x^\alpha v^{\beta\gamma}-x^\beta v^{\alpha\gamma})\right]df_\gamma. \tag{4.6}$$

On performing the integration, one finds that all the $M^{\alpha\beta}$ vanish except for M^{12} which is given by[9]

$$M^{12}=ma, \tag{4.7}$$

where m is now a function of $t-r$. It follows that the angular momentum radiated per unit time is $-m'a$.

V. SLOWLY ROTATING SPHERICAL SHELL

In a paper by De La Cruz and Israel[10] it is shown that the intrinsic metric which the Kerr geometry induces on the "sphere" $r=R$ can be joined continuously to a flat interior (that is, to first order in a). The Lanczos jump conditions[11] were then used to provide the surface energy tensor. The transformation given there can be modified to include the radiating case. If u, r, θ, and ϕ are taken as the exterior coordinates, then the metric on the cylindrical hypersurface $r=R$ is given by

$$ds^2=(1-2m/R)du^2+(4ma/R)\sin^2\theta\, d\phi du -R^2(d\theta^2+\sin^2\theta\, d\phi^2). \tag{5.1}$$

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[8] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), Sec. 100.

[9] The a used in (1.4) is the negative of the a used by various other authors. For example, see Ref. 10.

[10] V. De La Cruz and W. Israel, Phys. Rev. **170**, 1187 (1968).

[11] C. Lanczos, Z. Physik **23**, 529 (1922); Ann. Physik **74**, 518 (1924).

The "surface" transformation

$$\theta=\theta\,,\quad t=\int\left(1-\frac{2m}{R}\right)^{1/2}du\,,\quad \psi=\phi-\frac{2a}{R^3}\int m du\,,\quad (5.2)$$

yields

$$ds^2=dt^2-R^2(d\theta^2+\sin^2\theta\, d\psi^2)\,, \quad (5.3)$$

which is compatible with the flat-space interior coordinates t, r, θ, and ψ. If we denote the hypersurface $r=R$ by Σ, the interior space-time by V^-, the exterior space-time by V^+, then the extrinsic curvatures of Σ in V^+ and V^- are given by

$$K_{AB}{}^{\pm}=n_{i;j}\frac{\partial x^i}{\partial\xi^A}\frac{\partial x^j}{\partial\xi^B}\bigg|^{\pm}\,, \quad (5.4)$$

the ξ^A being the intrinsic coordinates of Σ. The unit vector n_i is the outward normal to Σ, and the covariant differentiation is with respect to the metrics of V^+ and V^-. We shall adopt the convention that capitalized indices take values 0, 2, 3 and $\xi^0=u$, $\xi^2=\theta$, $\xi^3=\phi$.

The surface energy tensor can be computed from the relations

$$-8\pi S_B{}^A=\gamma_B{}^A-\delta_B{}^A\gamma\,, \quad (5.5)$$

where

$$\gamma_B{}^A\equiv K_B{}^{A+}-K_B{}^{A-}. \quad (5.6)$$

The nonzero components of $S_B{}^A$ are

$$S_\theta{}^\theta=S_\phi{}^\phi=\frac{m-R(1-V)}{8\pi R^2V}+\frac{m'}{8\pi RV^3}\,, \quad (5.7)$$

$$S_u{}^u=\frac{1-V}{4\pi R}\,, \quad (5.8)$$

$$S_u{}^\phi=\frac{ma}{8\pi R^4V}\left(1+2V-\frac{2m'}{V^2}\right), \quad (5.9)$$

$$S_\phi{}^u=\frac{-3ma\sin^2\theta}{8\pi R^2V}\,, \quad (5.10)$$

where

$$V^2=1-2m/R. \quad (5.11)$$

The angular velocity ω (as measured by an observer at infinity) and surface density σ can be obtained from the equations

$$S_A{}^Bu^A=\sigma u^B\,,\quad u_Au^A=1\,,\quad u^\phi=\omega u^u. \quad (5.12)$$

Neglecting terms of order a^2, these equations reduce to

$$\omega S_\phi{}^\phi+S_u{}^\phi=\omega\sigma\,,\quad S_u{}^u=\sigma\,,\quad S_\theta{}^\theta u^\theta=u^\theta\,, \quad (5.13)$$

and the conclusion is

$$u^\theta=0\,,\quad \sigma=\frac{1-V}{4\pi R}\,,$$
$$\omega=\frac{2ma[1+2V-2m'/V^2]}{R^3[(1-V)(1+3V)+2m'/V^2]}. \quad (5.14)$$

If the quantities $2m/R$ and m' are considered to be small relative to unity, then the above expressions lead to a variable density and constant angular velocity which are given by

$$\sigma=m(u)/4\pi R\,,\quad \omega=3a/2R^2. \quad (5.15)$$

Equations (5.9) and (5.10) give rise to the classical expressions for momentum density, but (5.7) implies the existence of surface "pressures" given by

$$S_\theta{}^\theta=S_\phi{}^\phi=m'/8\pi R. \quad (5.16)$$

Evidently these stresses are needed to balance the back-pressure of the radiation. Since the radiation intensity is $q=-m'/4\pi R^2$, a unit area of the surface imparts a momentum of $q/c=-m'/4\pi R^2$ to the photons in unit time. The surface pressure p required to accomplish this would be given by the expression

$$2p/R=q/c=-m'/4\pi R^2\,, \quad (5.17)$$

which is consistent with (5.16).

On the other hand, if $2m/R\cong1$ then it follows that $V\rightarrow0$ and the stresses in (5.7), (5.9), and (5.10) become infinite. These difficulties which also occurred in the nonradiating model have been discussed by Brill and Cohen[12,13] and by De La Cruz and Israel.[10] The appearance of the m' terms in (5.7) would imply that the radiation forces become infinite as $R\rightarrow2m$.

ACKNOWLEDGMENTS

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[12] D. R. Brill and J. M. Cohen, Phys. Rev. **143**, 1011 (1966).
[13] J. M. Cohen, J. Math. Phys. **8**, 1477 (1967).