Equivalence Classes of Minimum Uncertainty Packets*

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We show that all the minimum uncertainty packets are unitarily equivalent to the coherent states and that coherence is in fact stationary minimality.

I. INTRODUCTION

WE consider the totality of wave packets in one dimension which minimize the uncertainty product $(\Delta x)(\Delta p)$, where x and p are the usual position and conjugate momentum operators which satisfy the commutation rule $[x,p]=i$ $(\hbar=1)$.

We develop an operator approach to classifying the minimal packets and show that the totality of minimal packets divides into equivalence classes.

One of the equivalence classes is the totality of the coherent states.[1] Besides demonstrating the unitary equivalence of the general minimum uncertainty packet to the coherent states, we show that coherence is essentially minimality.

II. MINIMALITY CONDITION

It is easily seen[2] that the most general one-dimensional wave packet that minimizes the uncertainty product of position and momentum is completely specified by the requirement that it be a normalizable eigenfunction of the operator $R=x+i\mu p$ with eigenvalue $\langle x\rangle+i\mu\langle p\rangle$. The only restriction on μ is that it be real and positive. The positivity requirement results from the requirement that the solution be normalizable. It is a trivial task to find the form of the most general minimum packet, and one can in fact find it in many textbooks on quantum theory.

What we do here is to approach the class of minimal packets in a slightly less pedestrian way and in so doing display all of the pertinent relationships in a lucid manner. First we define the operators

$$\hat{a}=(1/\sqrt{2})\left(\lambda x+\frac{i}{\lambda}p\right),$$
$$\hat{a}^\dagger=(1/\sqrt{2})\left(\lambda x-\frac{i}{\lambda}p\right), \tag{1}$$

where λ is any real, positive number. If λ takes on the value $(m\omega)^{1/2}$, then $\hat{a}$ and $\hat{a}^\dagger$ are the usual ladder operators for an harmonic oscillator of mass m and angular frequency ω. We can express the operator $\hat{R}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. For convenience, we consider $(2\mu)^{-1/2}\hat{R}$ instead of $\hat{R}$. This is quite permissible since the eigenvalue equation is homogeneous. Writing $\hat{S}=(2\mu)^{-1/2}\hat{R}$ we want to solve the eigenvalue equation

$$\hat{S}|\psi\rangle=s|\psi\rangle, \tag{2}$$

where $s=(2\mu)^{-1/2}[\langle x\rangle+i\mu\langle p\rangle]$.

In terms of $\hat{a}$ and $\hat{a}^\dagger$, we must solve the equation

$$(\hat{a}\cosh r+\hat{a}^\dagger\sinh r)|\psi\rangle=s|\psi\rangle, \tag{3}$$

where $\cosh r=(1+\mu\lambda^2)/(2\lambda\sqrt{\mu})$ and $\sinh r=(1-\mu\lambda^2)/(2\lambda\sqrt{\mu})$.

Rather than solve this problem in the conventional way, we note that if there exists an operator U which can, by means of a similarity transformation, transform $\hat{a}\cosh r+\hat{a}^\dagger\sinh r$ into $\hat{a}$, then the state $|\psi\rangle=U|\alpha\rangle$ for any coherent state[2] $|\alpha\rangle$ will be a solution. The proposed transformation preserves the commutation relation so we may hope to find a unitary operator which does the job.

III. UNITARY OPERATOR U_z

Consider the unitary operator,

$$U_z=\exp[\tfrac{1}{2}(z\hat{a}\hat{a}-z^*\hat{a}^\dagger\hat{a}^\dagger)], \tag{4}$$

where $z=re^{i\varphi}$ is an arbitrary complex number. Inspection of the above operator shows that $U_z^\dagger=U_z^{-1}=U_{-z}$.

A straightforward application of the formula[3]

$$e^A Be^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots \tag{5}$$

reveals the following relation:

$$U_z\hat{a}U_z^\dagger=\hat{a}\cosh r+\hat{a}^\dagger e^{-i\varphi}\sinh r,$$
$$U_z\hat{a}^\dagger U_z^\dagger=\hat{a}^\dagger\cosh r+\hat{a}e^{i\varphi}\sinh r. \tag{6}$$

So the operator U_z will more than suit our purpose, i.e., when z is purely real we have the transformation that we are seeking. Hence the most general minimal uncertainty packet is given by the expression

$$|\psi_{\mathrm{m.u.p.}}\rangle=|r;\alpha\rangle,\quad \alpha=\text{any complex number}$$
$$\equiv U_r|\alpha\rangle\quad (0\leq r<\infty). \tag{7}$$

There are three arbitrary real parameters here, i.e., r, $\mathrm{Re}\alpha$, and $\mathrm{Im}\alpha$, which is as it should be since the most general *minimum* packet is specified by the three

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[1] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[2] L. I. Schiff, *Quantum Mechanics*, 1st ed. (McGraw-Hill, New York).

[3] A. Messiah, *Quantum Mechanics* (Interscience, New York, 1961), p. 339.

numbers $\langle x\rangle$, $\langle p\rangle$, and Δx or Δp. For $r=0$ we have all the coherent states.

We might just as well consider the more general states $|z;\alpha\rangle$ when we consider detailed properties. We shall see that in general these are not *minimal* packets.

IV. PROPERTIES OF STATES $|z;\alpha\rangle$

A. Completeness (Overcompleteness)

This follows directly from the completeness or rather overcompleteness of the $|\alpha\rangle$ in a trivial fashion, i.e.,

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|=1. \tag{8}$$

Multiplying this equation by U_z on the left and by $U_z^\dagger$ on the right, we have[4]

$$\frac{1}{\pi}\int d^2\alpha|z;\alpha\rangle\langle z;\alpha|=1. \tag{9}$$

B. Connection with Other Bases

In this regard, all we need do is calculate the transformation function $\langle\alpha|z;\beta\rangle=\langle\alpha|U_z|\beta\rangle$ which can be done very simply as follows.

If $|\alpha\rangle$ and $|\beta\rangle$ are coherent states, i.e., eigenstates of the operator a, then we have the following relation:

$$\langle\alpha|U_z|\beta\rangle=\beta^{-1}\langle\alpha|U_z\hat{a}|\beta\rangle=\frac{1}{\beta}\langle\alpha|U_z\hat{a}U_z^\dagger U_z|\beta\rangle$$
$$=\beta^{-1}\langle\alpha|(c_r\hat{a}+s_re^{-i\varphi}\hat{a}^\dagger)U_z|\beta\rangle. \tag{10}$$

In arriving at the last results we have used Eq. (6) and written c_r for $\cosh r$ and s_r for $\sinh r$. In Eq. (10) we make use of two properties of the coherent states, namely, $\langle\alpha|\hat{a}^\dagger=\alpha^*\langle\alpha|$; $\langle\alpha|\hat{a}=(\partial/\partial\alpha^*+\frac{1}{2}\alpha)\langle\alpha|$, which gives us the following differential equation for $\langle\alpha|U_z|\beta\rangle$:

$$\left[c_r\frac{\partial}{\partial\alpha^*}+s_re^{-i\varphi}\alpha^*+(\tfrac{1}{2}\alpha c_r-\beta)\right]\langle\alpha|U_z|\beta\rangle=0. \tag{11}$$

The solution to this is

$$\langle\alpha|U_z|\beta\rangle=K\exp\left(-\tfrac{1}{2}|\alpha|^2+\frac{\alpha^*\beta}{c_r}-\frac{s_re^{-i\varphi}\alpha^{*2}}{2c_r}\right), \tag{12}$$

where K may depend on α, β, β^*, r, and φ, but not on α^*. We can use the unitarity of U_z to determine the functional form of K. We have

$$\langle\alpha|U_z|\beta\rangle^*=\langle\beta|U_z^{-1}|\alpha\rangle=\langle\beta|U_{-z}|\alpha\rangle. \tag{13}$$

Using Eq. (12) in this relation, we have

$$K^*(\alpha,\beta,\beta^*,r,\varphi)\exp\left(-\tfrac{1}{2}|\alpha|^2-\frac{s_re^{i\varphi}\alpha^2}{2c_r}\right)$$
$$=K(\beta,\alpha,\alpha^*,r,\varphi+\pi)\exp\left(-\tfrac{1}{2}|\beta|^2+\frac{s_re^{i\varphi}\beta^{*2}}{2c_r}\right). \tag{14}$$

The solution to this functional equation for K is

$$K(\alpha,\beta,\beta^*,r,\varphi)=\exp\left(-\tfrac{1}{2}|\beta|^2+\frac{s_re^{i\varphi}\beta^2}{2c_r}\right).$$

So we have, finally,

$$\langle\alpha|U_z|\beta\rangle=\exp\left(\frac{\alpha^*\beta}{c_r}-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2+\frac{s_r}{2c_r}(\beta^2e^{i\varphi}-\alpha^{*2}e^{-i\varphi})\right). \tag{15}$$

Note that $\langle 0|U_z|0\rangle=1$ so we are not dealing with an inequivalent representation.

We can use Eq. (15) to find the transformation to the number state basis, i.e., the eigenstates of the operator $\hat{a}^\dagger\hat{a}$. This is done quite easily by expanding $\langle\alpha|$ in terms of the $\langle n|$ states in $\langle\alpha|U_z|\beta\rangle$ and expanding the right-hand side of (15) in powers of α^*. The last-mentioned expansion is done by means of the generating function for the Hermite polynomials. The result is

$$\langle n|z;\beta\rangle=(n!)^{-1/2}t^n\exp(-\tfrac{1}{2}|\beta|^2+\beta^2t^{*2})H_n(\beta/2c_rt), \tag{15$'$}$$

where $t=(s_re^{-i\varphi}/2c_r)^{1/2}$ and H_n is the nth order Hermite polynomial.

C. Expectation Values

The simplest way to compute expectation values in this approach is by means of the characteristic function. The expectation value of any positive integral power of x, p, or a linear combination of these may be obtained from the characteristic function of the operator $\hat{A}=\gamma\hat{a}+\gamma^*\hat{a}^\dagger$, γ being an arbitrary complex number. By a straightforward application of Eq. (15), we find the characteristic function $C_A(\xi)$ of $\hat{A}$ in the state $|z;\alpha\rangle$ to be

$$C_A(\xi)\equiv\langle z;\alpha|\exp[i\xi(\gamma\hat{a}+\gamma^*\hat{a}^\dagger)]|z;\alpha\rangle$$
$$=\exp[\tfrac{1}{2}(i\xi)^2|k|^2+i\xi(k^*\alpha^*+k\alpha)], \tag{16}$$

where $k=\gamma c_r-\gamma^*s_re^{i\varphi}$.

The expectation value of A^n is given by

$$\langle A^n\rangle=\left.\frac{\partial^nC_A(\xi)}{\partial(i\xi)^n}\right|_{\xi=0}. \tag{17}$$

[4] In exactly the same fashion one can demonstrate that all of the properties of the states $|z;\alpha\rangle$ as regards the expansion of operators, diagonal representation, quasiprobability distributions, etc., follow from those of the $|\alpha\rangle$ states automatically. Equation (9) is a special case of the identity for overcomplete states which may be found in J. R. Klauder, J. Math. Phys. 5, 177 (1964).

If we let $\gamma=1/\lambda\sqrt{2}$, then $\hat{A}=x$ and we find

$$\langle x\rangle_{z;\alpha}=\frac{\sqrt{2}}{\lambda}\,\mathrm{Re}[\alpha(c_r-s_re^{i\varphi})]. \tag{18a}$$

In a similar manner, we also find

$$\langle p\rangle_{z;\alpha}=\lambda\sqrt{2}\,\mathrm{Im}[\alpha(c_r+s_re^{i\varphi})], \tag{18b}$$

$$[(\Delta x)^2]_{z;\alpha}=(1/2\lambda^2)[c_r^2+s_r^2-2s_rc_r\cos\varphi], \tag{18c}$$

$$[(\Delta p)^2]_{z;\alpha}=\tfrac{1}{2}\lambda^2[c_r^2+s_r^2+2c_rs_r\cos\varphi], \tag{18d}$$

$$[(\Delta x)^2(\Delta p)^2]_{z;\alpha}=\tfrac{1}{4}(1+4s_r^2c_r^2\sin^2\varphi). \tag{18e}$$

The last of these results shows that the state $|z;\alpha\rangle$ is a minimal uncertainty packet only if z is real. Notice also that the variances are independent of α.

V. z CLASSES

We can group the states $|z;\alpha\rangle$ into equivalence classes, each of which is specified by its own particular z. Since the width of these packets is independent of α, we may regard the equivalence relation as being "having the same width as."

The Weyl operator $D(\beta)=\exp(\beta\hat{a}^\dagger-\beta^*\hat{a})$ acting on a given state $|z;\alpha\rangle$ takes it into a width-equivalent one. This can be seen as follows:

$$D(\beta)|z;\alpha\rangle=D(\beta)U_z|\alpha\rangle=U_{-z}^\dagger(U_{-z}D(\beta)U_{-z}^\dagger)|\alpha\rangle. \tag{19}$$

From Eq. (6) we can also show that $U_{-z}D(\beta)U_{-z}^\dagger$ is equal to $D(\beta c_r+\beta^*s_re^{-i\varphi})$.

Since $D(\alpha)|0\rangle=|\alpha\rangle$ and

$$D(\alpha)D(\beta)=\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]D(\alpha+\beta),$$

we have

$$D(\beta)|z;\alpha\rangle=e^{i\eta}|z;\alpha+(c_r\beta+s_r\beta^*e^{-i\varphi})\rangle, \tag{20}$$

where $\eta=\mathrm{Im}[c_r\beta\alpha+s_r(\alpha\beta^*e^{-i\varphi})]$. Hence $D(\beta)$ does not alter the value of z and is thus reduced by z classes. For *real* z, we have $U_zU_{z'}=U_{z+z'}$, so we can transform a minimal packet from one class to another by means of U_z.

VI. TIME EVOLUTION OF $|z;\alpha\rangle$

In Schrödinger representation, the state at time t which evolves from $|z;\alpha\rangle$ at $t=0$ is given by $|z;\alpha\rangle_t=\exp(-iHt)|z;\alpha\rangle$. H is the Hamiltonian operator governing the system involved.

Let us consider the case of an harmonic oscillator initially in the state $|z;\alpha\rangle$. Then the Hamiltonian is $H=\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$. A simple calculation then shows that under these circumstances we have $|z;\alpha\rangle_t=|ze^{2i\omega t};\alpha e^{-i\omega t}\rangle$. So we see that the free oscillator Hamiltonian will take a state out of its z class but periodically return it. In particular, if z is real at $t=0$ (i.e., $\varphi=0$), then at a later time the state is in general no longer a minimal packet.

We can ask what is the most general Hamiltonian that keeps a minimal packet minimal. This question has been answered for the special case of coherent states by Sudarshan and Mehta.[5] They show that the most general Hamiltonian that keeps an α state an α state is given (in Heisenberg representation) by

$$H(t)=\omega(t)\hat{a}^\dagger(t)\hat{a}(t)+f(t)\hat{a}^\dagger(t)+f^*(t)\hat{a}(t)+\beta(t).$$

The functions $\omega(t)$ and $\beta(t)$ must be real to make $H(t)$ Hermitian.

We can use this result and the unitary equivalence of general minimal packets to coherent states to arrive at the form of the most general minimality-preserving Hamiltonian. The result is

$$H(t)=\omega(t)c_r(t)\hat{a}^\dagger(t)\hat{a}(t)+\tfrac{1}{2}\omega(t)s_r(t)[\hat{a}^2(t)+\hat{a}^{\dagger 2}(t)]+g(t)\hat{a}^\dagger(t)+g^*(t)\hat{a}(t)+b(t), \tag{21}$$

where $c_r(t)=\cosh r(t)$, $s_r(t)=\sinh r(t)$, $r(t)=$ any real positive function of t; and $\omega(t)$, $b(t)$ are real but otherwise arbitrary; $g(t)$ is also arbitrary.

VII. DISCUSSION

We have demonstrated the unitary equivalence of all minimum-uncertainty packets to the coherent states. It is clear that these results are easily extended to the case of more than one degree of freedom. The technique employed here serves to reveal in a simple way the structure of minimal-uncertainty packets. The coherent states have found a very wide range of usefulness in a variety of contexts and the structure discussed here seems to be relevant to that usefulness. For example, I have been able to generate exact solutions to some nonlinear oscillator problems and also arrive at known solutions to linear parametric problems in a simple way. In a future paper the details of this and a number of other applications will be given.

It is of interest to note that in a state of minimum uncertainty product, the position and momentum variables are uncorrelated. This is because a correlation between x and p would serve as a constraint on the minimization of the uncertainty product and prevent it from attaining its lowest possible value.

The results reported here have a very interesting bearing on quantum optics. First note that of all the z classes it is the $z=0$ class that contains the state $|0\rangle$ which for quantum optics is the no-photon state. Furthermore it is only the $z=0$ class that *remains minimal* at all times under the action of the free-field Hamiltonian. Hence propagation of minimality (in time) alone serves to set the coherent states apart from all other minimum-uncertainty states. In this sense we may identify coherence with minimality.

[5] E. C. G. Sudarshan and C. L. Mehta, Phys. Letters **22**, 574 (1966).