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# Scalar-Tensor Theory and Gravitational Waves*


ROBERT V. WAGONER†

*Center for Radiophysics and Space Research and Department of Astronomy, Cornell University, Ithaca, New York 14850*



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An analysis of general scalar-tensor gravitation theory, containing two arbitrary functions of the scalar field, is presented. The weak-field limit is considered in detail, and predictions for the classical tests of gravitation theory are derived. A definite relationship between the light propagation and perihelion shift effects is found to hold under very general conditions. The theory of the detection of gravitational waves is also investigated, and the observable differences between the scalar and tensor components are indicated. Finally, the relationship between the properties of the source and its radiation is considered in the weak-source limit, and expressions for the rate of energy loss are derived. It is shown that the existing observational data are consistent with the possibility that the scalar field represents a major component of gravitational radiation from astronomical sources.


## I. INTRODUCTION

OF the four fundamental interactions known to exist in nature, gravitation has the distinction of being the one with the longest history of observation, yet at the present time the one about which we have the least experimental information. Only in the weak-field regime are there a few actual numerical results which any theory must confront. However, many recent astronomical observations (microwave background radiation,[1] quasistellar objects,[2] pulsars,[3] etc.) indicate that we may now have indirect evidence of the properties of strong gravitational fields as well. In addition, solar system experiments of increased precision will soon be providing more information about the behavior of weak fields.[4]

Thus as observational evidence is being brought to bear more heavily upon gravitation theory, it would appear that we have reached a stage where it has become of primary importance to gain more information concerning the correct theory of gravitation, rather than proceeding with those detailed calculations within any specific theory that are not directly related to the observations. By interpreting the observational results within the framework of a broad class of theories, one also allows for a wider range of qualitatively different gravitational effects than predicted by any single theory.

In addition to considering the classical tests of gravitation theory, we shall be particularly concerned in this paper with the interpretation of Weber's apparent detection of gravitational waves.[5] Among the many implications of the detection, if it is real, is a new outlook on many aspects of astrophysics and cosmology.[6] The interpretation of Weber's results depends critically, however, on the assumption that the energy flux in the wave is related to the components of the gravitational field which are actually measured in roughly the way predicted by general relativity.[7]

In the general theory which we shall investigate, the gravitational field can be thought of as consisting of both spin-zero and spin-2 particles. Thus there is a place in this theory for nature's simplest field, a scalar field. It has so far proven impossible to find a theoretical reason for the nonexistence of such a long-range field.[8]

The details of the derivation of some of the results which we shall present have been omitted, since the calculations are similar to those employed in general relativity or the Brans-Dicke theory.[9] The notation and


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† Alfred P. Sloan Foundation Research Fellow.


[1] R. B. Partridge, Am. Scientist **57**, 37 (1969).

[2] E. M. Burbidge, Ann. Rev. Astronomy Astrophys. **5**, 399 (1967); M. Schmidt, *ibid.* **7**, 527 (1969).

[3] F. G. Smith and A. Hewish, *Pulsating Stars* (MacMillan, London, 1968).

[4] K. S. Thorne and C. M. Will, Comments Astrophys. Space Phys. **2**, 35 (1970).

[5] J. Weber, Phys. Rev. Letters **22**, 1320 (1969); J. Weber, *ibid.* **24**, 276 (1970).

[6] G. B. Field, M. J. Rees, and D. W. Sciama, Comments Astrophys. Space Phys. **1**, 187 (1969); D. W. Sciama, G. B. Field, and M. J. Rees, Phys. Rev. Letters **23**, 1514 (1969).

[7] J. Weber, *General Relativity and Gravitational Waves* (Interscience, New York, 1961).

[8] R. H. Dicke, Phys. Rev. **126**, 1875 (1962).

[9] R. H. Dicke, *The Theoretical Significance of Experimental Relativity* (Gordon and Breach, New York, 1964).

conventions are as given in Ref. 10, except that the tensor field $g_{ij}$ is of signature $-2$, so that the Minkowski tensor $\eta_{ij}=\mathrm{diag}(-1,-1,-1,+1)$. Greek indices run from 1 to 3, with $x^4=ct$. Also, the subscripts $,jk\ldots$ and $;jk\ldots$ indicate repeated partial and covariant differentiation, respectively.

## II. GENERAL THEORY

The class of classical gravitation theories which we wish to consider is characterized by the following five properties.

(1) The principle of general covariance is imposed, leading to tensor equations.

(2) The field equations are derived from the usual invariant variational principle

$$\delta\int(\mathfrak{L}_F+\mathfrak{L}_I)d^4x=0\,, \tag{1}$$

where $\mathfrak{L}_F$ represents the contribution from the gravitational fields alone, and $\mathfrak{L}_I$ represents their interaction with all other forms of matter. The variations of the fields and their derivatives are assumed to vanish on the boundary of the region of integration.

(3) The long-range fields of nature are composed of the three lowest spin bosons, leading to scalar, vector, and second-rank symmetric tensor fields. We assume that electromagnetism represents the only vector field, leaving a tensor $g_{ij}$ and a scalar $\phi$ to the gravitational field, which we take to be real. The observational evidence indicates that the tensor field must be present, while offering no reason why the scalar field should not also exist.[9] Indeed, there is reason to believe that the scalar field should be of roughly the same strength as the tensor field,[8] although its effects must be somewhat smaller in the solar system.

(4) The field equations are of at most second differential order. This leads to the general form

$$\mathfrak{L}_F=(-g)^{1/2}[h(\phi)R+l(\phi)g^{ij}\phi_{,i}\phi_{,j}+2\lambda(\phi)] \tag{2}$$

for the field Lagrangian density, within a divergence.[11] It involves the arbitrary functions $h$, $l$, and $\lambda$.

(5) We postulate a principle of mutual coupling, in which the interaction Lagrangian density depends upon the gravitational fields according to

$$\mathfrak{L}_I=\mathfrak{L}_I[\psi^2(\phi)g_{ij},\ldots]\,, \tag{3}$$

where $\psi(\phi)$ is another arbitrary function of the scalar field. This implies that $g^{ij}$ occurs in the form $\psi^{-2}g^{ij}$. Such a functional dependence guarantees that a local inertial system in which the laws of special relativity and electromagnetism hold can always be constructed. Proper time intervals $d\tau$ and distance elements $dl$ (as measured by the radar method, for instance) can then be shown to be related to the corresponding coordinate separations by

$$cd\tau=\psi(\phi)[g_{ij}dx^idx^j]^{1/2}\,, \tag{4}$$

$$dl=\psi(\phi)[(g_{4\alpha}g_{4\beta}g_{44}^{-1}-g_{\alpha\beta})dx^\alpha dx^\beta]^{1/2}. \tag{5}$$

Thus all space-time measurements are affected by the scalar field with the factor $\psi(\phi)$.

By employing the representation transformations

$$\bar{g}_{ij}=hg_{ij}\,, \tag{6}$$

$$\frac{d\phi}{d\bar{\phi}}=h\left|hl-\frac{3}{2}\left(\frac{dh}{d\phi}\right)^2\right|^{-1/2}, \tag{7}$$

we may put the theory in its final form (suppressing the bars)

$$\delta\int[(-g)^{1/2}(R-ng^{ij}\phi_{,i}\phi_{,j}+2\lambda)+\mathfrak{L}_I(\psi^2g_{ij},\ldots)]d^4x=0\,, \quad (n=\pm1) \tag{8}$$

over the range of $\phi$ where Eqs. (6) and (7) remain nonsingular. Thus the theory contains two undetermined functions, a "cosmological function" $\lambda(\phi)$ and a "mass function" $\psi(\phi)$. The Brans-Dicke theory[12,13] is the member of this class characterized by

$$\psi^2\propto\exp[-\phi(\omega+\tfrac{3}{2})^{-1/2}],\quad \omega=\mathrm{const};\qquad \lambda=0;\quad n=+1. \tag{9}$$

Of course, Einstein's theory is obtained by setting $\phi=\mathrm{const}$.

The matter stress-energy tensor $T^{ij}$ is defined through the relation

$$\frac{\partial\mathfrak{L}_I}{\partial g_{ij}}=\frac{8\pi G^*}{c^4}(-g)^{1/2}T^{ij}\,, \tag{10}$$

with the constant $G^*$ to be determined. It then follows that

$$\frac{\partial\mathfrak{L}_I}{\partial\phi}=\frac{2}{\psi}\frac{d\psi}{d\phi}g_{ij}\frac{\partial\mathfrak{L}_I}{\partial g_{ij}}=\frac{16\pi G^*}{c^4}\frac{1}{\psi}\frac{d\psi}{d\phi}(-g)^{1/2}T. \tag{11}$$

Employing the variational principle indicated by Eq. (8) then leads to the Euler-Lagrange field equations

$$R^i{}_j-\tfrac{1}{2}\delta^i{}_jR-n(\delta^k{}_jg^{im}-\tfrac{1}{2}\delta^i{}_jg^{km})\phi_{,k}\phi_{,m}-\lambda\delta^i{}_j=8\pi G^*c^{-4}T^i{}_j\,, \tag{12}$$

$$(-g)^{-1/2}[(-g)^{1/2}g^{ij}\phi_{,i}]_{,j}+n\frac{d\lambda}{d\phi}=-n\frac{8\pi G^*}{c^4}\frac{1}{\psi}\frac{d\psi}{d\phi}T. \tag{13}$$

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[10] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, revised 2nd ed. (Addison-Wesley, Reading, Mass., 1965), Chaps. 10 and 11.

[11] P. G. Bergmann, International J. Theoret. Phys. 1, 25 (1968).

[12] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[13] R. H. Dicke, Phys. Rev. **125**, 2163 (1962).

As in general relativity, the equations of motion of a free neutral test particle follow from the field equations, which imply the generalized Bianchi identities

$$(T^m{}_i)_{;m}-(d\ln\psi/d\phi)T\phi_{,i}=0. \quad (14)$$

The form of the interaction Lagrangian is uniquely determined from our assumptions to be

$$\mathfrak{L}_I\to 16\pi G^*c^{-2}m\int\psi(\phi)(g_{ij}\dot{x}^i\dot{x}^j)^{1/2}\delta^4[x^k-x^k(\chi)]d\chi \quad (15)$$

in this limit. Here $m$ is the mass of the particle, $\chi$ an arbitrary path-length parameter, and $\dot{x}^i\equiv dx^i/d\chi$. Using the expression for the proper time interval, Eq. (4), the equations of motion can be put in the form

$$\frac{d}{d\tau}(\psi^2 g_{ij}U^j)=\tfrac{1}{2}\psi^2U^jU^kg_{jk,i}+\frac{c^2}{\psi}\frac{d\psi}{d\phi}\phi_{,i}, \quad (16)$$

where the four-velocity $U^i=dx^i/d\tau$, with $\psi^2U^jU_j=c^2$. Thus $P_i=m\psi^2g_{ij}U^i$ represents the four-momentum of the particle, which responds to both the scalar and tensor fields in this representation of the theory. As usual, photons are assumed to obey these equations of motion in the limit $d\tau\to 0$, which leads to the result that they do not respond to the scalar field.

Within a macroscopic body which can be represented as a perfect fluid, the use of Eq. (15) leads to the form

$$T^i{}_j=\psi^4[(\rho+p/c^2)\psi^2U^iU_j-p\delta^i{}_j] \quad (17)$$

for the stress-energy tensor, where $\rho$ and $p$ are the total mass density and pressure as measured by observers moving with the fluid. The additional factor of $\psi^3$ comes from the use of proper volume elements.

In this form of the theory, characterized by the action principle (8), one may consider the scalar field as an additional source of the tensor field.[13] It is thus possible to associate with the scalar field a stress-energy tensor $S^{ij}$, defined as in Eq. (10) with $\mathfrak{L}_I$ replaced by $\mathfrak{L}_\phi=\mathfrak{L}_F-(-g)^{1/2}R=(-g)^{1/2}(-ng^{ij}\phi_{,i}\phi_{,j}+2\lambda)$. One thus obtains

$$S^{ij}=(c^4/8\pi G^*)[(-\tfrac{1}{2}ng^{km}\phi_{,k}\phi_{,m}+\lambda)g^{ij}+ng^{ik}g^{jm}\phi_{,k}\phi_{,m}]. \quad (18)$$

Of course, it then follows that $(T^m{}_i+S^m{}_i)_{;m}=0$.

### III. WEAK-FIELD LIMIT

We now consider the properties of a gravitational field at distances far enough from its source so that it may be expanded in terms of some small parameter $\epsilon$ about the values $\phi=\phi_0=\text{const}$, $g_{ij}=\eta_{ij}$ corresponding to no source. The magnitude of $\phi_0$ is determined by the distribution of matter at cosmological distances,[8] and so will not be strictly constant, but can be considered so for our purposes. The derivations of the results to be presented in this section employ the same techniques used in general relativity, and so we mention again that details are presented only when they differ from the standard approach.

We write the expansions in the form

$$\phi=\phi_0+\epsilon\tilde{\phi}(x^i)+\epsilon^2\hat{\phi}(x^i)+\cdots, \quad (19a)$$

$$g_{ij}=\eta_{ij}+\epsilon\tilde{g}_{ij}(x^i)+\cdots, \quad (19b)$$

$$\lambda(\phi)=\lambda_0+\epsilon\lambda_0'\tilde{\phi}+\epsilon^2(\lambda_0'\hat{\phi}+\tfrac{1}{2}\lambda_0''\tilde{\phi}^2)+\cdots, \quad (19c)$$

$$\psi(\phi)=1+\epsilon\psi_0'\tilde{\phi}+\epsilon^2(\psi_0'\hat{\phi}+\tfrac{1}{2}\psi_0''\tilde{\phi}^2)+\cdots, \quad (19d)$$

$$T^i{}_j=\epsilon\tilde{T}^i{}_j(x^i)+\cdots. \quad (19e)$$

The last equation holds only if the field is also weak near the source. Note that we have normalized $\psi$ so that coordinate intervals equal measured intervals at infinity. The constants $\lambda_0=\lambda(\phi_0)$, $\lambda_0'=(d\lambda/d\phi)(\phi_0)$, etc.

Substituting these expansions into the field equations (12) and (13), the equations corresponding to order $\epsilon^0$ give

$$\lambda_0=\lambda_0'=0. \quad (20)$$

It should be noted that this result remains valid if the sources of the fields are of arbitrary strength, as long as they are localized. Of course, Eq. (20) is a direct consequence of our asymptotic boundary condition on $g_{ij}$ and $\phi$, which in the real world are modified slightly by cosmological effects. However, Eq. (20) is of sufficient accuracy for our purposes.

The field equations corresponding to order $\epsilon$ are then

$$\eta^{km}\tilde{g}_{ij,km}\equiv\Box^2\tilde{g}_{ij}=-16\pi G^*c^{-4}(\tilde{T}_{ij}-\tfrac{1}{2}\eta_{ij}\tilde{T}), \quad (21)$$

$$\Box^2\tilde{\phi}+n\lambda_0''\tilde{\phi}=-8\pi nG^*c^{-4}\psi_0'\tilde{T}, \quad (22)$$

where we have imposed the usual coordinate conditions

$$(\eta^{jm}\tilde{g}_{mi}-\tfrac{1}{2}\delta^j{}_i\eta^{km}\tilde{g}_{km})_{,j}=0. \quad (23)$$

One reason for choosing the representation of the theory given by Eqs. (6) and (7) is that the weak-field equations for $g_{ij}$ and $\phi$ are then uncoupled as indicated in Eqs. (21) and (22). Employing a Fourier decomposition, the well-known solutions to Eqs. (21) and (22) for an infinite domain are

$$\tilde{g}_{ij}=-2\pi^{-1}G^*c^{-4}\int\int\int R^{-1}[\tilde{T}_{ij}(\bar{x}^k)-\tfrac{1}{2}\eta_{ij}\tilde{T}(\bar{x}^k)]\times\exp i[\omega(t-\bar{t})+\omega R/c]d^3\bar{x}^\alpha d\bar{t}d\omega, \quad (24)$$

$$\tilde{\phi}=-n\pi^{-1}G^*c^{-4}\psi_0'\int\int\int R^{-1}\tilde{T}(\bar{x}^k)\exp i[\omega(\bar{t}-t)+(\omega^2/c^2-n\lambda_0'')^{1/2}R]d^3\bar{x}^\alpha d\bar{t}d\omega, \quad (25)$$

where $R=[(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2]^{1/2}$, and the proper integration path in the $\omega$ plane must be taken.[14]

Now in the weak-field slow-motion limit, Eq. (16) assumes the form of Newton's second law if the gravitational potential $\Phi(x^\alpha)$ is identified with

$$\Phi=\epsilon c^2(\tfrac{1}{2}\tilde{g}_{44}+\psi_0'\tilde{\phi}), \quad (26)$$

[14] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 7.

assuming that the fields also change slowly with time. Employing Eqs. (24) and (25) in the static limit, and noting that Eq. (17) gives $\epsilon \tilde{T}_{ij}=\rho_0 c^2\delta^4{}_i\delta^4{}_j$, where $\rho_0$ is the proper rest mass density, we obtain

$$\Phi=-G^*\int \rho_0 R^{-1}\{1+2n(\psi_0')^2 \times\exp[-(n\lambda_0'')^{1/2}R]\}d^3\bar{x}^\alpha. \quad (27)$$

The Newtonian result $\Phi=-G\int\rho_0 R^{-1}d^3\bar{x}^\alpha$ has been well verified over distances $1\lesssim R\lesssim 10^{22}$ cm. Thus this constraint upon theory yields two possible restrictions on $\lambda_0''$ and $G^*$, either

$$0<(n\lambda_0'')^{-1/2}\lesssim 1\text{ cm}, \quad (28a)$$

$$G^*=G \quad (28b)$$

or

$$|\lambda_0''|^{-1/2}\gtrsim 10^{22}\text{ cm}, \quad (29a)$$

$$G^*=G[1+2n(\psi_0')^2]^{-1}. \quad (29b)$$

The first choice corresponds to a short-range scalar field of mass $m=\hbar|\lambda_0''|^{1/2}/c\gtrsim 3\times10^{-11}m_e$. Although the existence of such a component of the "gravitational" field is certainly possible as long as $\psi_0'$ is not too large, it would not be detectable except at very small distances from its source, and so is not of interest for this investigation. We therefore consider the second set of constraints, Eqs. (29a) and (29b), to hold, where $G=6.67\times10^{-8}$ dyn cm$^2$ g$^{-2}$ is the usual gravitational constant. Note that we have imposed no condition on the sign of $n\lambda_0''$ in this case, so that oscillatory solutions are also allowed.

The equation for the redshift $Z$ of a photon emitted in a time-independent gravitational field and received at infinity is found from Eq. (4) to be $1+Z=\psi^{-1}g_{44}{}^{-1/2}$, where $\psi$ and $g_{44}$ are evaluated at the (stationary) point of emission. In a weak field this reduces to the result $1+Z=1-c^{-2}\Phi$, or $\Delta Z=-c^{-2}\Delta\Phi$, using Eq. (26). This result agrees with experiment, as it must for any theory in which inertial and passive gravitational masses are equivalent, energy content is proportional to inertial mass, and energy is conserved.

The weak-field solutions can also be used to compute the bending of light by the sun. A straightforward calculation yields the result

$$\Delta\theta=(G^*/G)\times(\text{Einstein value}) =[1+2n(\psi_0')^2]^{-1}\times(\text{Einstein value}) \quad (30)$$

for the angular deflection. The radar timing experiments[15] are sensitive at present to the same weak-field components as the light-bending experiments, and so their predicted results may also be obtained by merely replacing $G$ by $G^*$ in the general relativistic prediction.

[15] I. I. Shapiro, Phys. Rev. Letters **13**, 789 (1964); D. K. Ross and L. I. Schiff, Phys. Rev. **141**, 1215 (1966); I. I. Shapiro, *ibid.* **141**, 1219 (1966); **145**, 1005 (1966).

At present, the most precise observational result[16] yields the limits $+0.3\geq 2n(\psi_0')^2\geq -0.1$.

In order to compute the advance of the perihelion of Mercury, one must include in addition terms of order $\epsilon^2$ in the calculation. Again proceeding in the same manner as in general relativity, one obtains the result

$$\Delta\alpha=\frac{1+\frac{2}{3}(\psi_0')^2[2n-\psi_0''-(\psi_0')^2]}{[1+2n(\psi_0')^2]^2}\times(\text{Einstein value}) \quad (31)$$

for the perihelion shift per revolution. Thus in principle this test also places limits upon $\psi_0''$. Using the fact that $(\psi_0')^2\lesssim 0.15$ from the radar measurements, we may approximate Eq. (31) by

$$\Delta\alpha=[1-(8n/3)(\psi_0')^2+6(\psi_0')^4 -\tfrac{2}{3}(\psi_0')^2\psi_0''+\cdots]\times(\text{Einstein value}),$$

assuming $|\psi_0''|\lesssim(\psi_0')^2$.

Now if the measured solar oblateness[17] represents an equivalent quadrupole moment, the measured perihelion shift[18] leads to $\Delta\alpha=(0.914\pm0.025)\times$(Einstein value), giving $n(\psi_0')^2=+0.032\pm0.009$. On the other hand, if the quadrupole moment of the Sun is negligible, $n(\psi_0')^2=+0.003\pm0.008$. In any case, we can say that

$$(\psi_0')^2\leq 0.04 \quad \text{for} \quad n=+1$$
$$\leq 0.005 \quad \text{for} \quad n=-1, \quad (32)$$

assuming that $|\psi_0''|\ll 1$. The observational evidence gives no information about $\psi_0''$ if it is as small as $(\psi_0')^2$. On the other hand, this general theory does predict a definite relationship between the light-bending and perihelion-shift measurements, since they are both effectively only functions of $n(\psi_0')^2$. The evidence also indicates that $n=+1$ if the Sun does have a large quadrupole moment.

## IV. GRAVITATIONAL WAVES

In this section we are concerned with the radiative components of the gravitational field at very large distances from their (arbitrarily strong) source, where the weak-field equations are applicable. In this case, the locally valid plane-wave solutions to Eqs. (21) and (22) are

$$\tilde{g}_{ij}=\int\tilde{g}_{ij}(\omega)e^{i(k_t z-\omega t)}d\omega, \quad (33)$$

$$\tilde{\phi}=\int\tilde{\phi}(\omega)e^{i(k_s z-\omega t)}d\omega, \quad (34)$$

[16] I. I. Shapiro, G. H. Pettengill, M. E. Ash, M. L. Stone, W. B. Smith, R. P. Ingalls, and R. A. Brockelman, Phys. Rev. Letters **20**, 1265 (1968).

[17] R. H. Dicke and H. M. Goldenberg, Phys. Rev. Letters **18**, 313 (1967).

[18] P. A. Wayman, Quart. J. Roy. Astron. Soc. **7**, 138 (1966).

where the wave numbers are

$$k_t=\omega/c, \quad k_s=(\omega^2/c^2-n\lambda_0'')^{1/2}, \tag{35}$$

and the spatial coordinates have been chosen so that locally the waves are traveling in the $z$ direction. Thus the scalar waves propagate as though they traveled through a medium with an index of refraction

$$N(\omega)\cong 1-\tfrac{1}{2}nc^2\lambda_0''\omega^{-2} \tag{36}$$

for wavelengths $c/\omega \ll |n\lambda_0''|^{-1/2}$. This leads to a group velocity $v_g\cong cN(\omega)$, implying $v_g>c$ (and a phase velocity $<c$) if $n\lambda_0''<0$. Thus there exists the possibility of scalar Cherenkov radiation from a moving particle.

This possible difference in the velocity of propagation of the scalar and tensor components of the gravitational field could lead to other observable consequences. It is seen from Eq. (36) that the scalar component of a gravitational wave which has traveled a distance $D$ takes a time

$$\Delta t\cong(1-N)(D/c)=\tfrac{1}{2}ncD\lambda_0''\omega^{-2} \tag{37}$$

longer to travel that distance than the tensor component. However, the observational limitation $|\Delta t| \gtrsim \omega^{-1}$ means that the effect would be easily detectable only for frequencies $\omega \lesssim cD|\lambda_0''| \lesssim 3\times 10^{-33}D$ sec$^{-1}$, using inequality (29a). Even for sources at cosmological distances $D\sim 10^{28}$ cm, such frequencies are lower than those which seem to offer the best hope for detection.[5,7]

It is beyond the scope of this paper to investigate in detail such consequences of nonzero $\lambda_0''$, especially since its small upper limit makes it most likely that $\lambda_0''=0$.

We now discuss the detection of gravitational waves through their interaction with macroscopic bodies, generalizing the analysis of Weber.[7] We assume that the structure of the detector can be represented by a collection of mass points interacting via nongravitational forces. The equations of motion of any one of the particles in a given gravitational field can then be obtained from the variational principle

$$0=\delta\int[mc\psi(\phi)(g_{ij}U^iU^j)^{1/2}+L_{\rm NG}(x^k)]d\tau, \tag{38}$$

derived from the form of the interaction Lagrangian density (15), with the path-length parameter $\chi$ chosen as proper time $\tau$. Note that the nongravitational interaction Lagrangian $L_{\rm NG}$ is assumed to depend only on the position of the mass point, as would be appropriate for small displacements. The Euler-Lagrange equations of motion corresponding to this variational principle are found to be

$$\frac{d^2x^i}{d\tau^2}+\begin{Bmatrix} i \\ k\ \ l\end{Bmatrix}U^kU^l+\frac{1}{\psi}\frac{d\psi}{d\phi}\left(2U^iU^k-\frac{c^2}{\psi^2}g^{ik}\right)\phi_{,k} =\frac{g^{ik}}{m\psi^2}\frac{dL_{\rm NG}}{dx^k}=\frac{g^{ik}}{m\psi^2}F_k, \tag{39}$$

where $F_k$ is the nongravitational four-force on the particle, and $\begin{Bmatrix} i \\ k\ \ l\end{Bmatrix}$ are the Christoffel symbols formed from $g_{ij}$.

The observational quantity which we wish to obtain is an invariant measure of how the separation of any two particles changes with time as the wave passes by. For this purpose, it is convenient to label a single infinity of particles by a parameter $p$ which is constant along the world line of the particle it identifies. We then define a deviation vector $W^i=\partial x^i/\partial p$, where now $x^i=x^i(p,\tau)$ labels the position of a particle within this restricted set. Using the fact that since $U^i=\partial x^i/\partial\tau$, $\partial U^i/\partial p=\partial W^i/\partial\tau$, plus the equation of motion (39), one obtains as a measure of the relative *coordinate* acceleration of neighboring particles

$$\frac{D^2W^i}{D\tau^2}=R^i{}_{jkl}U^jU^kW^l +\frac{D}{Dp}\left[\frac{1}{\psi}\frac{d\psi}{d\phi}\left(\frac{c^2}{\psi^2}g^{ik}-2U^iU^k\right)\phi_{,k}+\frac{g^{ik}}{m\psi^2}F_k\right], \tag{40}$$

where the total derivative $D/Dq$ of any vector $V^i$ with respect to some scalar $q$ is defined by $DV^i/Dq=V^i{}_{;m}\partial x^m/\partial q$. Equation (40) represents a generalization of Weber's Eq. (8.10),[7] with the opposite sign convention for the Riemann tensor $R^i{}_{jkl}$.

In order to obtain an expression for the *observed* relative acceleration, we return to the weak-field limit, and define an infinitesimal separation vector $\sigma^i$ and relative force vector $f^i$:

$$W^i\Delta p=\sigma^i=\sigma_0{}^i(p)+\epsilon\tilde{\sigma}^i(p,\tau)+\cdots, \tag{41}$$

$$\frac{D}{Dp}\left(\frac{g^{ik}}{\psi^2}F_k\right)\Delta p=\epsilon f^i(p,\tau)+\cdots, \tag{42}$$

noting that the particles are in equilibrium in the absence of the perturbing wave. Equation (40) then gives, to lowest order,

$$\frac{D^2\sigma^i}{D\tau^2}=\epsilon c^2[\tilde{R}^i{}_{44l}+\psi_0'(\eta^{ik}-2\delta^i{}_4\delta^k{}_4)\tilde{\phi}_{,kl}]\sigma_0{}^l+\epsilon m^{-1}f^i, \tag{43}$$

where we have taken $U^i\cong c\delta^i{}_4$. The first-order Riemann tensor $\tilde{R}^i{}_{44l}=\tfrac{1}{2}\eta^{ik}(\tilde{g}_{kl,44}+\tilde{g}_{44,kl}-\tilde{g}_{4l,4k}-\tilde{g}_{4k,4l})$.

Now the measured physical separation $L$ between two particles is given by Eq. (5),

$$\begin{aligned}L&=\psi(\phi)[(g_{4\alpha}g_{4\beta}g_{44}{}^{-1}-g_{\alpha\beta})\sigma^\alpha\sigma^\beta]^{1/2}\\ &=L_0\{1+\epsilon[\psi_0'\tilde{\phi}-L_0{}^{-2}(\eta_{\alpha\beta}\sigma_0{}^\alpha\tilde{\sigma}^\beta +\tfrac{1}{2}\tilde{g}_{\alpha\beta}\sigma_0{}^\alpha\sigma_0{}^\beta)]+\cdots\},\end{aligned} \tag{44}$$

where $L_0(p)=(-\eta_{\alpha\beta}\sigma_0{}^\alpha\sigma_0{}^\beta)^{1/2}$ is the unperturbed separation. The measured relative acceleration is then

$$\frac{\partial^2 L}{\partial\tau^2}=\epsilon\left[L_0\psi_0'\frac{\partial^2\tilde{\phi}}{\partial t^2}-\frac{1}{L_0}\left(\eta_{\alpha\beta}\sigma_0{}^\alpha\frac{\partial^2\tilde{\sigma}^\beta}{\partial\tau^2}+\tfrac{1}{2}\sigma_0{}^\alpha\sigma_0{}^\beta\frac{\partial^2\tilde{g}_{\alpha\beta}}{\partial t^2}\right)\right] \quad (45)$$

from Eq. (44). Recall that $d\tau\cong dt$. Next, using the relations $\sigma_0{}^4=0$ and

$$\frac{D^2\sigma^i}{D\tau^2}=\epsilon\left[\frac{\partial^2\tilde{\sigma}^i}{\partial\tau^2}+\tfrac{1}{2}c^2\sigma_0{}^k\eta^{il}(\tilde{g}_{4l,4k}+\tilde{g}_{lk,44}-\tilde{g}_{4k,4l})\right], \quad (46)$$

as well as Eq. (43), we derive the result

$$\frac{\partial^2 L}{\partial\tau^2}+\epsilon\frac{\sigma_0{}^\alpha}{L_0}\frac{f_\alpha}{m}=\epsilon\frac{c^2}{L_0}[-R_{\alpha44\beta}\sigma_0{}^\alpha\sigma_0{}^\beta+\psi_0'(L_0{}^2\tilde{\phi}_{,44}-\sigma_0{}^\alpha\sigma_0{}^\beta\tilde{\phi}_{,\alpha\beta})]. \quad (47)$$

Employing the unit orientation three-vector $N^\alpha=\sigma_0{}^\alpha/L_0$ plus the field equation (22), we finally obtain

$$\partial^2L/\partial\tau^2+\epsilon m^{-1}N^\alpha f_\alpha=\epsilon c^2L_0[-\tilde{R}_{\alpha44\beta}N^\alpha N^\beta-\psi_0'(\eta^{\alpha\beta}+N^\alpha N^\beta)\tilde{\phi}_{,\alpha\beta}-n\psi_0'\lambda_0''\tilde{\phi}]. \quad (48)$$

This result represents the equation of motion of the particles making up our linear detector, which points in the direction $N^\alpha$. The internal force $f_\alpha$ may be composed of a restoring force, a damping term, etc., while the terms on the right-hand side of Eq. (48) represent the driving force of the oscillator provided by the incident gravitational wave.[7] If the detector is oriented in the direction of propagation of the plane wave, it can easily be shown that $\tilde{R}_{\alpha44\beta}N^\alpha N^\beta=(\eta^{\alpha\beta}+N^\alpha N^\beta)\tilde{\phi}_{,\alpha\beta}=0$. (Recall that $\eta^{\alpha\beta}=-\delta_{\alpha\beta}$.) Thus the wave is purely transverse, except for the contribution proportional to $\lambda_0''$, which will be of order $|\lambda_0''|c^2/\omega^2\ll1$ compared to the transverse components. An observable distinction between the scalar and tensor components lies in their modes of excitation in the transverse plane. Whereas the tensor field leads to the well-known quadrupole deformations, the scalar field induces isotropic expansions and contractions in the transverse directions,[19] as can be seen from the structure of Eq. (48).

Let us next determine the energy carried by the wave. For this purpose it is convenient to rewrite the tensor field equation (12) in the Landau-Lifshitz form[10]

$$[(-g)(g^{ij}g^{lm}-g^{il}g^{jm})]_{,lm}=16\pi G^*c^{-4}(-g)(T^{ij}+S^{ij}+{}^*t^{ij}), \quad (49)$$

where the pseudotensor ${}^*t^{ij}$ is obtained from the usual expression[10] by replacing $G$ by $G^*$. Now in the weak-field limit, it can be shown that ${}^*t^{ij}$ [or $t^{ij}=(G^*/G)\,{}^*t^{ij}$, depending on one's normalization] represents the stress-energy in the tensor components of the gravitational wave.[20] The total stress-energy in the wave is then the sum $S^{ij}+{}^*t^{ij}$ of terms involving each field separately, in this limit.

The energy density of the scalar component of the wave is found from Eqs. (18) and (34) to be

$$S^{44}\cong\frac{nc^2}{8\pi G^*}\left(\frac{\partial\phi}{\partial t}\right)^2 \quad (50)$$

in the weak-field limit. Now from Eqs. (29b) and (32), we see that $G^*>0$. Therefore the requirement that $\phi$ be real means that this energy density has the sign of $n$. It is interesting to note that no *observational* evidence has yet ruled out the possibility $n=-1$, although by Eq. (32) it would require that $(\psi_0')^2$ be very small.

## V. RADIATION OF GRAVITATIONAL WAVES

In this section we consider the relationship between the strength of a gravitational wave and its source. Although it is likely that the fields in the vicinity of any object which could produce detectable gravitational waves will be strong ($GM/rc^2\sim1$), we are forced to employ the weak-field expansion even at the source in order to obtain explicit expressions. The expansion parameter $\epsilon$ should now be thought of as related to the strength of the source by $v^2/c^2\sim GM/rc^2\sim u/\rho_0c^2\sim p/\rho_0c^2\sim\epsilon$, where $v$ is a typical velocity of the source, $r$ is its size, and $u$ and $p$ are its internal energy and pressure. It is then seen from Eqs. (24) and (25) that far from the source, both $\tilde{g}_{ij}$ and $\tilde{\phi}$ are independent of time in this approximation. In addition, in what follows we shall neglect the function $\lambda(\phi)$, which we have seen can produce only very small corrections to our results in any case.

Our expansions of $\phi$, $g_{ij}$, and $\psi(\phi)$ are still of the form given by Eqs. (19a), (19b), and (19d). We also have

$$T=\rho_0c^2+\epsilon^2\hat{T}+\cdots, \quad (51)$$

where, for a perfect fluid, we obtain

$$\epsilon^2\hat{T}=u-3p+4\epsilon\tilde{\phi}\psi_0'\rho_0c^2 \quad (52)$$

from Eq. (17). We consider situations in which the observer is at a sufficient distance $R$ from the nonrelativistic source to be in the "radiation zone," so that $r\ll c/\omega\ll R$.

The analysis of the tensor gravitational radiation follows from Eq. (12) in the same way as in general relativity,[10] if one considers the scalar field terms as part of the effective source. To lowest order, the only change is the replacement of $G$ by $G^*$, leading to the

[19] D. C. Robinson and J. Winicour, Phys. Rev. Letters **22**, 198 (1969).

[20] R. A. Isaacson, Phys. Rev. **166**, 1272 (1968).

well-known result

$$-\left(\frac{dE}{dt}\right)_t=\frac{G^*}{45c^5}\eta_{\alpha\gamma}\eta_{\beta\delta}D_{(3)}{}^{\gamma\delta}D_{(3)}{}^{\alpha\beta} \qquad (53)$$

for the rate of energy loss by tensor radiation, where $D_{(3)}{}^{\alpha\beta}$ is the third time derivative of the mass quadrupole moment.

In order to investigate the scalar gravitational radiation, we first note from Eq. (50) that the energy loss rate will be proportional to $(\partial\phi/\partial t)^2$. Thus, since $\tilde{\phi}=\hat{\phi}(x^\alpha)$ at the observer's position, the quantity of interest is $\hat{\phi}(x^k)$. Its field equation is found by expanding Eq. (13), which gives at order $\epsilon^2$,

$$\epsilon^2\Box^2\hat{\phi}=Q(x^k)+8\pi nG^*c^{-4}s'(x^k), \qquad (54)$$

where $s'=\epsilon\tilde{\phi}\rho_0c^2[(\psi_0')^2-\psi_0'']-\epsilon^2\psi_0'\hat{T}$, and $Q=-\epsilon^2\tilde{g}^{ij}\tilde{\phi}_{,ij}$ when the coordinate condition (23) is employed. Furthermore, using the fact that $n\psi_0'\tilde{g}^{ij}=(\eta^{ij}-2\eta^{4i}\eta^{4j})\tilde{\phi}$ from Eqs. (24) and (25), we obtain $Q=\epsilon^2(n/\psi_0')\tilde{\phi}(-\Box^2\tilde{\phi}+2\tilde{\phi}_{,44})\cong 8\pi G^*c^{-4}\epsilon\tilde{\phi}\rho_0c^2$, using also Eq. (22), and neglecting terms of higher order. Defining a total source

$$s=s'+(nc^4/8\pi G^*)Q=\epsilon\tilde{\phi}\rho_0c^2[n+(\psi_0')^2-\psi_0'']-\epsilon^2\psi_0'\hat{T},$$

the solution to Eq. (54), in analogy with Eq. (22), is then

$$\epsilon^2\hat{\phi}=n\pi^{-1}G^*c^{-4}\int\int\int s(\bar{x}^k)R^{-1}\times\exp[i\omega(\bar{t}-t+R/c)]d^3\bar{x}^\alpha d\bar{t}d\omega. \qquad (55)$$

Taking the time derivative, we obtain

$$\epsilon^2\frac{\partial\hat{\phi}}{\partial t}=\frac{2nG^*}{c^4R_0}\frac{d}{dt}\int s(\bar{x}^\alpha,t_r)d^3\bar{x}^\alpha, \qquad (56)$$

where $R_0\cong R$ is the distance from any point in the source, and $t_r=t-R_0/c$ is the retarded time. All terms of order $\omega r/c$ smaller have been neglected.

Now the energy flux at large distances is purely radial and isotropic, being just equal to $cS^{44}$. Thus we find the scalar energy loss rate to be

$$-\left(\frac{dE}{dt}\right)_s=4\pi R_0^2cS^{44}=\frac{nc^3R_0^2}{2G^*}\left(\epsilon^2\frac{\partial\hat{\phi}}{\partial t}\right)^2=\frac{2nG^*}{c^5}\left(\frac{d}{dt}\int sdV\right)^2, \qquad (57)$$

using Eqs. (50) and (56), with $dV=d^3\bar{x}^\alpha$. For a perfect fluid,

$$s=\psi_0'(3p-u)+\epsilon\tilde{\phi}\rho_0c^2[n-3(\psi_0')^2-\psi_0'']. \qquad (58)$$

Thus in this case the source of the scalar waves involves the integral over the body of the first-order scalar field as well as the thermodynamic properties of the fluid. Note that for a nonrelativistic ideal gas, $3p-u=u$, whereas $3p-u\to 0$ as the gas becomes relativistic. Of course, it is also seen that monopole scalar radiation is permitted.

Let us compare the energy loss rates given by Eqs. (53) and (57). Their ratio is

$$\frac{(dE/dt)_s}{(dE/dt)_t}=90n\left[\left(\int\dot{s}dV\right)^2\Big/D_{(3)}{}^{\alpha\beta}D_{(3)\alpha\beta}\right]. \qquad (59)$$

Thus unless $|\int\dot{s}dV|\ll|D_{(3)}{}^{\alpha\beta}|$, most of the energy will be lost by scalar radiation. Although $s$ is proportional to $\psi_0'$, and by Eq. (32) $|\psi_0'|\leq 0.2$, it would still require sizable deviations from spherical symmetry, for instance, in order that the tensor contribution dominate, assuming $|\psi_0'|\sim 0.1$. Thus the classical tests of gravitation theory do not rule out the possibility of a significant scalar component in gravitational radiation.

## VI. DISCUSSION

The generality of the theory which we have investigated in this paper is contained in the arbitrary functions $\psi(\phi)$ and $\lambda(\phi)$ plus the constant $n=\pm 1$. Some authors[11] have considered such degrees of freedom as strong evidence against the inclusion of a scalar field, since arguments which do not appear to be of a fundamental nature are necessary in order to specify these functions.[12] However, since fundamental arguments which would rule out a scalar field have not been presented, this criticism does not appear to be compelling. In addition, it is certainly possible that new physical principles will be discovered which would lead to a specification of these functions.

We have seen that the observational evidence makes it seem likely that $n=+1$ and $\lambda(\phi)$ is negligible, at least for weak fields, so that the theoretical problem reduces to the form of $\psi(\phi)$ in this case. We have also seen that the classical tests of gravitation provide information about $\psi(\phi)$ in the weak-field limit, with the cosmological distribution of matter determining the appropriate asymptotic value of $\phi$ through Eq. (13).

The detection of gravitational waves provides a means of isolating the effects of the scalar field. The use of Eq. (48) allows one to relate the driving force on the detector to the various components of the gravitational field, provided the relative orientation of the propagation direction and detector is known. Note that for scalar and tensor fields of comparable strength $\tilde{\phi}(\omega)\sim\tilde{g}_{ij}(\omega)$, the response of a single detector to the scalar field will be weaker than its response to the tensor field by a factor of the order of $\psi_0'$. However, through the simultaneous monitoring of detectors at various orientations, the transverse monopole modes, which can only be excited by the scalar waves, can be isolated.

Since the ratio of the energy density in the scalar component to that in the tensor components of the same frequency is of the order of $[\tilde{\phi}(\omega)/\tilde{g}_{ij}(\omega)]^2$, it is seen that the ratio of relative detected strength to relative intrinsic strength of the components is of the order of $(\psi_0')^2 \leq 0.04$. Thus a scalar wave which produced the same response in a single detector as a tensor wave would have at least $\sim 25$ times more energy density than the tensor wave. This fact makes it difficult to believe that the signals which Weber is observing[5,6] are due primarily to scalar waves of this type.

As far as the source of the gravitational radiation is concerned, the relative production of the scalar and tensor components is strongly dependent on its configuration, as has been indicated in connection with Eq. (59). Let us investigate in more detail the scalar energy loss rate from a specific type of object, in order to gain a better understanding of how important it might be. (See also Ref. 21.)

Consider a collection of small masses, each of whose internal structure is not changing with time. Then from Eq. (58), we have for any individual mass $A$,

$$\frac{d}{dt}\int_A s\,dV \cong n m_A c^2 \epsilon\left(\frac{d\tilde{\phi}}{dt}\right)_A, \tag{60}$$

where $\epsilon(d\tilde{\phi}/dt)_A$ represents an average over the mass $A$ of the rate of change of the total potential due to all the other masses, and we have assumed that $|\psi_0''| \lesssim (\psi_0')^2$. Then the energy loss rate due to the scalar waves becomes

$$-\left(\frac{dE}{dt}\right)_s \cong \frac{8nG^*}{c^5}\left(\frac{\psi_0' G^*}{G}\right)^2 \left[\sum_A m_A \left(\frac{d\Phi}{dt}\right)_A\right]^2, \tag{61}$$

where we have used the relation $\epsilon\tilde{\phi} = (2n\psi_0' G^*/c^2 G)\Phi$ between the first-order scalar field and the Newtonian potential.

For a system of two masses $m_1$ and $m_2$, Eq. (61) reduces to

$$-\left(\frac{dE}{dt}\right)_s \cong \frac{32nG^*}{c^5}\left(\frac{\psi_0' G^* m_1 m_2}{r^2}\right)^2 \left(\frac{dr}{dt}\right)^2, \tag{62}$$

where $r$ is their separation. It is seen that particles in circular orbits do not radiate scalar waves, although they do radiate tensor waves.[10] For other types of relative motion, however, the scalar radiation is of the order of the tensor radiation if $(\psi_0')^2 = 0.03$, the value needed to produce the observed perihelion advance of Mercury if the solar oblateness represents a quadrupole moment. This fact is illustrated by taking as an example the case in which the two masses are falling directly toward each other. The tensor radiation is found to contribute a loss rate of

$$-\left(\frac{dE}{dt}\right)_t = \frac{8G^*}{15c^5}\left(\frac{Gm_1m_2}{r^2}\right)^2 \left(\frac{dr}{dt}\right)^2 \tag{63}$$

in this case. Thus the energy loss due to scalar radiation is approximately twice that due to tensor radiation if $(\psi_0')^2 = 0.03$.

The simplest situation in which pure scalar radiation is produced remains the radial pulsation of a single, spherically symmetric body.

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[21] R. H. Dicke, in *Gravitation and Relativity*, edited by H. Chiu and W. F. Hoffman (Benjamin, New York, 1964), Chap. 12.