Simple Relation between Cross Sections for Neutrino Scattering and Total Muon-Capture Rates by Nuclei

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Using the closure approximation, we derive a simple relation between the total forward cross section for elastic scattering of neutrinos by nuclei and the corresponding total muon-capture rates. This relation is in good agreement with previous estimates of neutrino scattering cross sections, but still in disagreement with recent experimental data.

It has recently been observed[1] experimentally that the *total* cross section for elastic[2] scattering of neutrinos off Fe^{56} does not show any suppression in the forward direction. Such a suppression was expected[3] to occur because of the combined effects of nuclear structure and the Pauli exclusion principle. In this paper we reexamine this problem and present a new estimate of the total forward elastic cross section. Our estimate is based on a simple relation which we derive that expresses the *total* forward cross section for elastic scattering of neutrinos off a nucleus in terms of the *total* muon-capture rate by the same nucleus. The derivation of this relation is based on the observation that the expressions for the total forward cross section for elastic scattering of neutrinos off a nucleus and the corresponding total muon-capture rate by the same nucleus, when analyzed in the context of a current-current weak-interaction Hamiltonian, exhibit striking similarities. There exists at present a considerable amount of experimental and theoretical information concerning total muon-capture rates by medium and heavy nuclei. When used in our relation, these data provide us with the total forward cross section for elastic neutrino scattering by nuclei in a way which is less model dependent than are previous estimates.

Working in the context of a current-current weak-interaction Hamiltonian and using the closure approximation,[4] one can readily show that to order G^2, the *total* muon-capture rate by a nucleus N_i of mass m_i, momentum $\mathbf{k}_i=0$, spin J_i, and third component of spin M_i is simply given by

$$\Gamma(N_i)\simeq D(N_i)\frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\int\frac{d\Omega_\nu}{4\pi}\langle N_i;\mathbf{k}_i=0,M_i|\sum_{l=1}^{3}Q_l^{(+)}(\langle\mathbf{v}\rangle_i)Q_l^{(-)}(\langle\mathbf{v}\rangle_i)$$
$$-Q_4^{(+)}(\langle\mathbf{v}\rangle_i)Q_4^{(-)}(\langle\mathbf{v}\rangle_i)+i\sum_{l=1}^{3}\hat{\nu}_l[Q_l^{(+)}(\langle\mathbf{v}\rangle_i)Q_4^{(-)}(\langle\mathbf{v}\rangle_i)+Q_4^{(+)}(\langle\mathbf{v}\rangle_i)Q_l^{(-)}(\langle\mathbf{v}\rangle_i)]|N_i;\mathbf{k}_i=0,M_i\rangle,\quad(1)$$

where

$$D(N_i)=\frac{G^2\cos^2\theta_C}{2\pi^2}\left(\frac{Z}{a_B}\right)^3C(N_i)m_\mu^2\langle\eta^2\rangle_i,\quad(2)$$

$$\eta^2=\left(\frac{\nu}{m_\mu}\right)^2\left(1-\frac{\nu}{m_\mu+m_i}\right),\quad(3)$$

and

$$Q_\mu^{(\pm)}(\mathbf{k})=\int e^{\pm i\mathbf{k}\cdot\mathbf{x}}J_\mu^{(\pm)}(\mathbf{x},t=0)d\mathbf{x}\quad(\mu=1,2,3,4).\quad(4)$$

In the above, $G=(1.02/m_p^2)\times10^{-5}$ is the Fermi coupling constant (m_p is the proton mass), θ_C is the Cabibbo angle,

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[2] By "elastic" scattering we mean reactions in which the three-momentum transfer to the nucleus (and therefore also the energy transfer, since we neglect terms of the order of the charged lepton mass squared and limit ourselves to forward muon production) is less than 30–40 MeV/c. This automatically excludes reactions in which pions, baryon resonances, and strange particles are produced.

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Z is the proton number of N_i, and a_B is the Bohr radius of the muon. $C(N_i)$ is a correction factor arising from the nonpoint character of the charge distribution of N_i. We have denoted the mass of the muon by m_μ, and $\mathbf{v}$ and ν are the neutrino momentum and energy, respectively. Also, $\langle\ \rangle_i$ denotes an appropriate weighted average over all possible final nuclear states. $J_\mu^{(\pm)}(\mathbf{x},t)$ is the strangeness-conserving weak hadronic current, the superscript $(+)$ and $(-)$ corresponding to the isotopic spin raising and lowering components, respectively. The integration in Eq. (1) is over the neutrino solid angle.

In a similar way, one can show that the total forward cross section for elastic[2] scattering of neutrinos of energy $\nu \gg m_\mu$ off a nucleus N_i is[5]

$$\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \simeq \frac{G^2\cos^2\theta_C}{\pi}\nu^2 \frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\langle N_i;\mathbf{k}_i=0,M_i|Q_3^{(-)}(0)Q_3^{(+)}(0) - Q_4^{(-)}(0)Q_{(4)}^{(+)}(0) + i[Q_3^{(-)}(0)Q_4^{(+)}(0)+Q_4^{(-)}(0)Q_3^{(+)}(0)]|N_i;\mathbf{k}_i=0,M_i\rangle. \quad (5)$$

We have chosen the z axis along the incident neutrino beam direction. The angle θ is the angle of the scattered muon relative to the incident neutrino. Note that while in Eqs. (1)–(3) ν refers to the *emitted* neutrino energy, in Eq. (5) it refers to the *incident* neutrino energy.

We use the impulse approximation and assume that

$$J_\alpha^{(\pm)}(\mathbf{x},t=0) \simeq \sum_{a=1}^{A}(\Gamma_\alpha)_a \tau_a^{(\pm)}\delta(\mathbf{x}-\mathbf{r}_a), \quad (6)$$

where A is the nucleon number of N_i, $\mathbf{r}_a$ is the radius vector of the ath nucleon, and $\tau_a^{(+)}$ and $\tau_a^{(-)}$ are the 2×2 isotopic spin raising and lowering matrices, respectively (the subscript a indicates that these matrices act only on the ath nucleon). In the allowed approximation,

$$(\Gamma_\alpha)_a \simeq i\delta_{\alpha,4}g_V - (1-\delta_{\alpha,4})(\sigma_\alpha)_a g_A, \quad (7)$$

where $(\sigma_j)_a$ $(j=1, 2, 3)$ are the Pauli spin matrices that act on the ath nucleon alone, and g_V and g_A are the strangeness-conserving vector and axial-vector renormalization constants, respectively. It is well known that the induced pseudoscalar and weak-magnetism contributions to *total* muon-capture rates are of opposite signs and practically cancel one another.[6] It is therefore quite feasible to use Eq. (7) in calculating the total muon-capture rate as given by Eq. (1). Furthermore, since we are restricting ourselves to forward elastic neutrino scattering, the allowed approximation is also applicable to Eq. (5).

Using Eqs. (6) and (7), one readily sees that the terms in Eqs. (1) and (5) proportional to $Q_l^{(+)}(\langle\mathbf{v}\rangle_i)Q_4^{(-)}(\langle\mathbf{v}\rangle_i)$, $Q_4^{(+)}(\langle\mathbf{v}\rangle_i)Q_l^{(-)}(\langle\mathbf{v}\rangle_i)$, $Q_3^{(-)}(0)Q_4^{(+)}(0)$, and $Q_4^{(-)}(0)Q_3^{(+)}(0)$ vanish in the allowed approximation, so that

$$\frac{1}{D(N_i)}\Gamma(N_i) \simeq \frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\int\frac{d\Omega_\nu}{4\pi}\langle N_i;\mathbf{k}_i=0,M_i|Q_3^{(+)}(\langle\mathbf{v}\rangle_i)Q_3^{(-)}(\langle\mathbf{v}\rangle_i) - Q_4^{(+)}(\langle\mathbf{v}\rangle_i)Q_4^{(-)}(\langle\mathbf{v}\rangle_i)|N_i;\mathbf{k}_i=0,M_i\rangle + B(N_i) \quad (8)$$

and

$$\frac{1}{K}\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \simeq \frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\langle N_i;\mathbf{k}_i=0,M_i|Q_3^{(-)}(0)Q_3^{(+)}(0) - Q_4^{(-)}(0)Q_4^{(+)}(0)|N_i;\mathbf{k}_i=0,M_i\rangle, \quad (9)$$

where

$$K=\frac{G^2\cos^2\theta_C}{\pi}\nu^2 \quad (10)$$

and

$$B(N_i)=\frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\frac{2}{3}\int\frac{d\Omega_\nu}{4\pi}\langle N_i;\mathbf{k}_i=0,M_i|\mathbf{Q}^{(+)}(\langle\mathbf{v}\rangle_i)\cdot\mathbf{Q}^{(-)}(\langle\mathbf{v}\rangle_i)|N_i,\mathbf{k}_i=0,M_i\rangle. \quad (11)$$

[5] Strictly speaking, the quantities $Q_\mu^{(\pm)}$ in Eq. (5) should be functions of the average momentum transfer $\langle q\rangle_i$, which is of the order of 20–30 MeV/c. The inclusion of this correction increases the cross section by about 10%. The same remark applies to the forward cross sections calculated in Ref. 3.

[6] The neglect of the weak magnetism and induced pseudoscalar terms as well as terms of the order of $\nu/2m_i$ introduces an error of approximately 8%. In the estimate we give of $F(N_i)$ [see Eqs. (13) and (14)], this error and the one due to the neglect of the $\langle q\rangle_i$ dependence of $Q_\mu^{(\pm)}$ in Eq. (5) (see Ref. 5) approximately cancel one another.

Using Eqs. (6) and (7) and the methods of Ref. 7, one can write[8]

$$B(N_i)\simeq 2g_A{}^2Z[1-(N/2A)\delta] \tag{12}$$

and

$$F(N_i)\simeq\frac{1-(Z/2A)\delta}{1-Z/N}, \tag{13}$$

where

$$F(N_i)\equiv\frac{\int(d\Omega_\nu/4\pi)\langle N_i;\mathbf{k}_i=0,M_i|Q_\alpha{}^{(-)}(\langle\mathbf{v}\rangle_i)Q_\alpha{}^{(+)}(\langle\mathbf{v}\rangle_i)|N_i;\mathbf{k}_i=0,M_i\rangle}{\langle N_i;\mathbf{k}_i=0,M_i|Q_\alpha{}^{(-)}(0)Q_\alpha{}^{(+)}(0)|N_i;\mathbf{k}_i=0,M_i\rangle}\quad\text{(no summation over }\alpha). \tag{14}$$

$N=A-Z$ is the neutron number of N_i. The quantity δ, which depends on nuclear structure effects, has been found to be constant to within a few percent for $A>25$,[4] and is empirically determined by comparing "theoretical" expressions for the total muon-capture rate with the corresponding experimental values. We shall use[9] $\delta=3.11$ in our calculations.

Combining Eqs. (8), (9), and (12)–(14), we find that

$$\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq\frac{K}{F(N_i)}\Bigg(\frac{\Gamma(N_i)}{D(N_i)}+\frac{1}{2J_i+1}\sum_{M_i=-J_i}^{J_i}\int\frac{d\Omega_\nu}{4\pi}\langle N_i;\mathbf{k}_i=0,M_i|$$

$$\times[Q_4{}^{(+)}(\langle\mathbf{v}\rangle_i),Q_4{}^{(-)}(\langle\mathbf{v}\rangle_i)]-[Q_3{}^{(+)}(\langle\mathbf{v}\rangle_i),Q_3{}^{(-)}(\langle\mathbf{v}\rangle_i)]|N_i;\mathbf{k}_i=0,M_i\rangle-B(N_i)\Bigg). \tag{15}$$

We use Eqs. (6) and (7) to estimate the commutators in Eq. (15).[10,11] After substituting Eq. (12), we finally obtain

$$\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq\frac{2}{5}\frac{1}{F(N_i)}\left[\frac{\Gamma(N_i)}{D(N_i)}-2(g_V{}^2+g_A{}^2)I^{(3)}-2g_A{}^2Z\left(1-\frac{N}{2A}\delta\right)\right]\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}, \tag{16}$$

where $I^{(3)}$ is the third component of isotopic spin of N_i and[12]

$$\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq\frac{5}{2}\frac{G^2\cos^2\theta_C}{\pi}\nu^2, \tag{17}$$

is the forward cross section for the reaction $\nu+n\to p+\mu^-$.

In determining $\Gamma(N_i)/D(N_i)$, one can proceed in one of the following two ways.

(a) Use the semiempirical formula[9]

$$[\Gamma(N_i)]_{\text{semiemp.}}=286Z_{\text{eff}}{}^4\langle\eta^2\rangle_i[1-(N/2A)\delta]\ \text{sec}^{-1}, \tag{18}$$

where $Z_{\text{eff}}{}^4=C(N_i)Z^4$. The parameters $\langle\eta^2\rangle_i$ and δ depend upon nuclear structure effects and are usually determined empirically. This formula is in remarkable agreement with experiment when one chooses $\langle\eta^2\rangle_i{}^{1/2}=0.75$ and $\delta=3.11$.

(b) Use the experimental value of $\Gamma(N_i)$.

It is not really obvious which method is better, since, although the $\Gamma(N_i)$ of method (b) is clearly the exact one, the relevant ratio $[\Gamma(N_i)]_{\text{expt}}/D(N_i)$ depends on the empirical parameter $\langle\eta^2\rangle_i$, while the ratio $[\Gamma(N_i)]_{\text{semiemp.}}/D(N_i)$ is independent of $\langle\eta^2\rangle_i$ (it does, however, depend on the empirical parameter δ). In order to demonstrate the usefulness of relation (16) we shall simply use the semiempirical formula (18). Substituting Eqs. (2), (13), and (18) into relation (16), and

[7] H. Primakoff, Ref. 4.

[8] $B(N_i)$, which is, essentially, proportional to the total muon-capture rate, has been the subject of considerable discussion (see Ref. 4). The numerator in the definition of $F(N_i)$ is also related to the total muon-capture rate. Since we use the semiempirical total muon-capture rate formula (18) in deriving Eqs. (12) and (13), these expressions are also to be regarded as semiempirical, and therefore to a certain extent model independent.

[9] See, for example, B. Goulard, G. Goulard, and H. Primakoff, Ref. 4.

[10] One might be tempted to use the gauge- or quark-field algebras in estimating these commutators. This is, however, quite unjustified for the following reasons: (a) Simultaneous use of the gauge-field algebra and the closure approximation has been shown to lead to violent disagreement with experiment (see Ref. 11). Furthermore, the gauge-field algebra is quite inconsistent with the nonrelativistic allowed approximation (7) which we have invoked. (b) Although simultaneous use of the closure approximation and the quark-field algebra has not led to disagreement with experiment [on the contrary, one application has led to quite remarkable agreement (see Ref. 11)], the subject is still very controversial [see A. M. Wolsky, Ph.D. dissertation, University of Pennsylvania, 1969 (unpublished)], in view of the relativistic nature of the quark-field algebra and the highly nonrelativistic, and therefore conflicting, features of the closure approximation.

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setting $g_V^2=1$ and $g_A^2=1.51$, we obtain

$$\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq\frac{2}{5}\frac{1-Z/N}{1-(Z/2A)\delta}\times\left[2.65Z\left(1-\frac{N}{2A}\delta\right)-5I^{(3)}\right]\times\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}. \quad (19)$$

Setting $\delta=3.11$ and applying this relation to Al^{27} and Fe^{56}, we find that[13]

$$\left.\frac{d\sigma^{(\nu)}(Al^{27};\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq1.02\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \quad (20)$$

and

$$\left.\frac{d\sigma^{(\nu)}(Fe^{56};\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq4.2\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}. \quad (21)$$

This is in very good agreement with the relation

$$\left.\frac{d\sigma^{(\nu)}(N_i;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\simeq(N-Z)\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}, \quad (22)$$

derived by Goulard and Primakoff,[3] but in violent disagreement with the recent experimental value[1]

$$\left.\frac{d\sigma^{(\nu)}(Fe^{56};\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}^{\text{expt}}\simeq30\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}. \quad (23)$$

Further experiments are clearly needed to clarify the situation.

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[13] Direct use of the experimental value of $\Gamma(N_i)$ with $\langle\eta^2\rangle=(0.75)^2$ does not change the value of the cross section for Al^{27} but increases the value for Fe^{56} by 10%.

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On-Shell Current Algebra and the Radiative Leptonic Decay of K^+†

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Dispersion relations and the hard-meson method of Schnitzer and Weinberg are used to study the radiative leptonic decay of the K^+. It is shown that in the pole-dominance approximation the relevant form factor in the axial-vector amplitude cannot be unsubtracted. Possible alterations of our results arising from relaxing a smoothness approximation are estimated to be small. We discuss and compare various symmetry-breaking schemes for the evaluation of necessary coupling constants. The branching ratio for $K^+\to\gamma e^+\nu$ is calculated to be $\sim2.5\times10^{-5}$ for interesting structure-dependent decays. This is comparable to that for $K\to e\nu$ and two orders of magnitude larger than one would expect from the usual estimates for electromagnetic decays. The feasibility of experimentally observing the decay is discussed, as are the possible effects of electromagnetic violations of time-reversal invariance. From these results, a soft-pion estimate of F_4, the vector form factor in K_{l4}, yields $|F_4|\approx6.9$.

I. INTRODUCTION

THERE have been several discussions, both experimental[1] and theoretical,[2–6] of the strangeness-conserving decay mode $\pi^+\to\gamma l^+\nu$, where l^+ is a positron or a positive muon. In the present paper we study the decay $K^+\to\gamma l^+\nu$ from the point of view of current-algebra and hard-meson methods. The theoretical situation is much less clear for strangeness-changing decays. There has been some discussion of this K^+ decay mode[7–10] in the literature, with widely different results depending upon the particular symmetry-breaking scheme employed. We shall therefore present a comparison of these results together with those of our own method.

The hard-pion method we use, that of Schnitzer and

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