# Nucleon-Nucleon Bremsstrahlung: An S-Matrix Approach*

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A covariant formulation of nucleon-nucleon bremsstrahlung is presented which is unitary and gauge invariant, and has a relatively simple form in the soft-photon limit. Applying the soft-photon approximation to experiment, a large discrepancy is found with the noncoplanar data taken in Harvard geometry, while the fit for the coplanar case is good. The inclusion of non-soft-photon dynamics using $S$-matrix methods is discussed.

## I. INTRODUCTION

NUCLEON-NUCLEON bremsstrahlung (the reaction $NN\rightarrow NN\gamma$, henceforth $NN\gamma$) at low and intermediate energies has been the subject of much theoretical and experimental interest as a probe of the nucleon-nucleon ($NN$) interaction. It is felt that due to the weakness of the photon interaction, one can determine the off-energy-shell behavior in this reaction to apply elsewhere, for instance to distinguish between phenomenological $NN$ potentials. A number of theoretical approaches have been taken, including potential models, soft-photon approximations, and one-boson-exchange models.

The potential approach[1–4] depends on the assumption that nuclear and electromagnetic potentials can be separated. Calculations are then made using the two-potential formulation which includes all orders of the nuclear potential but only the first order in the electromagnetic potential. The half-off-energy-shell $NN$ matrix elements required are computed either with phenomenological potentials or by using experimentally determined phase shifts with the off-energy-shell behavior determined in some *ad hoc* manner, as by one-pion exchange.[2] The electromagnetic potential is then approximated by the static nucleon interaction with the electromagnetic field. The velocity-dependent, or nonlocal, part of the nuclear potential requires additional terms for the electromagnetic potential in order for the calculation to be gauge invariant. These are neglected and thought to be small.[3]

Thus the potential approach is partly phenomenological in the sense that a particular off-energy-shell dependence of the $NN$ matrix elements is postulated (usually by choosing a particular potential), and the resulting cross section compared with experiment, with the best agreement determining the best off-energy-shell dependence (or potential). It appears, however, that experiments are insufficiently accurate to distinguish among most such calculations.

An interesting feature of bremsstrahlung amplitudes (and the basis for the second method) is that the soft-photon behavior is known in terms of the static electromagnetic properties of the particles and the nonradiative amplitude. As Low[5] has shown, the first two terms in an expansion of powers of photon energy of the bremsstrahlung amplitude, $O(1/k)$ and $O(1)$, are determined. Nyman[6] has applied Low's method to constructing these terms, and found fair agreement with experiment and potential-model calculations. To the extent that the potential models are gauge invariant and fit elastic $NN$ data, they must produce the correct model-independent soft-photon behavior,[7] and this, Nyman concludes, is the reason for the agreement. Apparently more accurate or more inelastic experiments are required to determine $O(k)$ behavior of the amplitude to contrast with the model calculations.

Felsner[8] has estimated the soft-photon behavior using a simplification of the Low method due to Feshbach and Yennie[9] which is designed to reproduce the $1/k$ term and is expected to be a fair approximation of the $O(1)$ term. His amplitude is not gauge invariant, and his results relatively poor.

A third theoretical approach used single-boson exchange, pion exchange with a phenomenological form factor in the case of Ueda,[10] and $\pi$, $\rho$, $\omega$, $\eta$, $\sigma$, and $\epsilon$ mesons in the work of Baier, Kuhnelt, and Urban.[11] This method has the advantage over potential calculations of being relativistically invariant and gauge invariant, but has the correct soft-photon behavior only to the extent that the model used also describes elastic $NN$ scattering.

In the present work we present another approach,

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formulating the problem in such a way that the known properties of the $NN\gamma$ amplitude are automatically satisfied, that is, gauge invariance, soft-photon behavior, relativistic invariance, and unitarity. In addition, our formulation allows the possibility of calculating dynamics beyond the soft-photon limit. We discuss how this may be done, using the $S$-matrix principles of generalized unitarity and analyticity.

Following the $S$-matrix approach, we find that when we choose variables most convenient for discussing unitarity and analyticity, the soft-photon approximation can be expressed in terms of elastic partial-wave amplitudes in a fairly simple form for calculation, in contrast to the more conventional approach followed by Nyman.[6] By expressing the partial-wave amplitudes in terms of the invariant energies of the initial and final nucleons, we find a form that is also unitary to all orders in the photon energy and does not require derivatives of the elastic partial-wave amplitudes (PWA). We also make a partial-wave expansion for $NN\gamma$, and present the soft-photon approximation for the $NN\gamma$ partial-wave amplitudes.

The interesting aspects of $NN\gamma$ are the dynamics giving rise to non-soft-photon behavior, $O(k)$, and higher terms in the amplitude. For this we propose that $NN\gamma$ be treated with $S$-matrix methods as a production amplitude in a similar manner to the treatments of the reaction $N\pi\rightarrow N\pi\pi$.[12,13] The latter is a strong production reaction, however, and the weakness of the photon causes two important simplifications for $NN\gamma$: (1) one needs only the part of the $NN\gamma\rightarrow NN\gamma$ amplitude for which the photon is disconnected, i.e., the $NN$ amplitude itself; (2) there are no anomalous thresholds. The procedure then is to use generalized unitarity to determine crossed-channel singularities, then to use analyticity in the form of dispersion relations for the partial-wave amplitudes to obtain singular integral equations which can be solved by standard techniques. Since this approach has been successful for $NN$ scattering itself,[14] it seems reasonable that $NN\gamma$ also can be treated this way.

In the hypothetical case of spinless nucleons, the integral equation to be solved is the well-known Omnès equation, and we present an explicit solution. The more interesting physical case of $NN\gamma$ is complicated by the fact that in order to apply dispersion relations, one needs to define kinematic-singularity-free PWA, which we have not done. This point is certainly well understood for four-particle amplitudes, and, we feel, can be resolved with further study. The matrix generalization of the Omnès[15] equation that one obtains in the case of spin apparently does not have a known explicit solution, and we present an iterative method for solving it

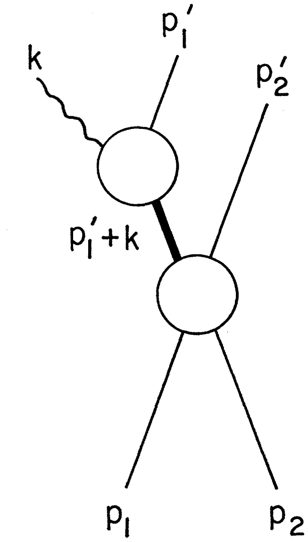


FIG. 1. Kinematics.

which is based on the smallness of the coupling between orbital angular momentum states in $NN$ scattering.

In Sec. II we define our choice of five independent variables, the center-of-mass (c.m.) Lorentz frames for the initial and final nucleons, and the Lorentz transformation necessary to go from one to the other.

In Sec. III we discuss unitarity, finding that the $NN\gamma$ amplitude with the spins of the initial (final) nucleons measured in the c.m. system of the initial (final) nucleon pair has a simple unitarity relation involving only itself and elastic c.m. amplitudes, which are expressed at the invariant energies of initial and final pairs of nucleons. This $NN\gamma$ amplitude turns out to have a simple partial-wave expansion, and we determine the form that unitarity takes for the PWA. Finally we obtain a symmetry condition by crossing the photon from initial to final states and using time-reversal invariance.

We turn, in Sec. IV, to determining the soft-photon approximation for the amplitudes of Sec. III, eventually expressing the results in a form that is unitary and does not involve derivatives of the elastic PWA.

In Sec. V we consider the steps necessary to apply our formulation to experiment, comparing our variables with the usual variables, and present the formulas necessary to obtain cross sections for the two geometries. In addition, we compare the soft-photon approximation with some data not considered by Nyman[6] and find what appears to be a breakdown of the soft-photon approximation.

In Sec. VI, we discuss the possibility of calculating dynamical contributions not included in the soft-photon limit, that is, $O(k)$ and higher terms in the amplitude. We present an explicit solution for spinless nucleons, and, assuming that kinematic-singularity-free amplitudes can be defined, we propose an iterative method for solving the coupled singular integral equations that one obtains.

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## II. KINEMATICS

It is well known that a process involving five particles can be described with five variables. Obviously, there is considerable freedom in the choice of a specific set; our choice is motivated by the form of the unitarity condition and the weakness of the interactions of the photon. We choose two energy and three angular variables; treating the initial and final nucleons in a symmetrical way, we define the energy variables

$$s=\tfrac{1}{2}[(p_1+p_2)^2+(p_1'+p_2')^2],$$
$$\omega=\tfrac{1}{2}[(p_1+p_2)^2-(p_1'+p_2')^2], \qquad (2.1)$$

where $p_1$ and $p_2$ are the momenta of the initial nucleons, $p_1'$ and $p_2'$ the momenta of the final nucleons, and the photon momentum is denoted by $k$ (see Fig. 1). This choice is convenient for considering the limit of zero photon energy where $s$ becomes the elastic energy and $\omega\to 0$; $\omega=k\cdot(p_1+p_2)=k\cdot(p_1'+p_2')$ and is proportional to the photon energy. Note that $s+\omega$ is the energy of the initial nucleons, $s-\omega$, the final, and the physical region is $s\geq 4m_N^2\geq\omega\geq 0$.

To define the angular variables, consider two Lorentz frames defined by the c.m. systems of the initial and final pairs of nucleons, denoting them by $C$ and $C'$, respectively. We choose the $z$ axis in both frames to be in the direction of $-\mathbf{k}$. Then we may choose the $x$ axes to be parallel so that the two frames are related by a boost in the $z$ direction. Finally, we define the angles $\Omega=(\theta,\phi)$ and $\Omega'=(\theta',\phi')$ as the polar angles of $\mathbf{p}_1$ and $\mathbf{p}_1'$ in $C$ and $C'$, respectively. Since we have not specified the orientation of the $x$ and $x'$ axes, we have an additional azimuthal angle beyond that necessary to describe the process; physical quantities can only depend on the difference $\phi'-\phi$ which is invariant under rotations about the direction of the photon in either frame since such rotations commute with the boost along the direction of the photon connecting the two frames.

Defining the momenta $P=p_1+p_2$, $Q=p_1-p_2$, $P'=p_1'+p_2'$, $Q'=p_1'-p_2'$, we have, explicitly,

frame $C$:
$$P=((s+\omega)^{1/2},0,0,0),$$
$$Q=(s+\omega-1)^{1/2}(0,\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta),$$
$$k=[\omega/(s+\omega)^{1/2}](1,0,0,-1),$$
$$\epsilon(k,\lambda)=-2^{-1/2}(0,1,i\lambda,0);$$

frame $C'$: (2.2)
$$P'=((s-\omega)^{1/2},0,0,0),$$
$$Q'=(s-\omega-1)^{1/2}(0,\sin\theta'\cos\varphi',\sin\theta'\sin\varphi',\cos\theta'),$$
$$k=[\omega/(s-\omega)^{1/2}](1,0,0,-1).$$

We have chosen units so that $2m_N=1$ (neglecting the $n$-$p$ mass difference). We have also defined the polarization vector $\epsilon(k,\lambda)$ of the outgoing photon with helicity $\lambda$. Since the photon has no rest frame, the relative phase of the $\lambda=\pm 1$ states is arbitrary. The choice we have made is not the conventional one, but will be convenient later.

We require the parameters of the Lorentz transformation from frame $C'$ to frame $C$. This is easily done noting that since $p'+k=p$, the 3-components of $k$ and $p$ must balance in frame $C'$. If

$$(\text{vector in } C')=\begin{pmatrix}\gamma & 0 & 0 & \gamma_\beta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma_\beta & 0 & 0 & \gamma\end{pmatrix}(\text{vector in } C), \qquad (2.3)$$

then $-\omega/(s-\omega)^{1/2}=\beta\gamma(s+\omega)^{1/2}$ and $\beta=-\omega/s$, $\gamma=(1-\beta^2)^{-1/2}$. We denote this transformation by $L$ in the following sections. Using $L$, we then find

$$Q\cdot Q'=-[(s-1)^2-\omega^2]^{1/2}[(s/(s^2-\omega^2)^{1/2})\cos\theta\cos\theta' +\sin\theta\sin\theta'\cos(\varphi'-\varphi)]. \qquad (2.4)$$

Thus in the zero-photon-energy limit, the angle between the directions $\Omega$ and $\Omega'$, which we denote by $\Delta\theta$,

$$\cos\Delta\theta=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi'-\varphi),$$

becomes identical to the elastic scattering angle.

Finally, using Eqs. (2.2) and (2.3) we present a set of relations useful for obtaining these variables in terms of any other set:

$$\omega=k\cdot(p_1+p_2),$$
$$s=(p_1+p_2)^2-\omega,$$
$$\cos\theta=\omega^{-1}[(s+\omega)/(s+\omega-1)]^{1/2}k\cdot(p_1-p_2), \qquad (2.5)$$
$$\cos\theta'=\omega^{-1}[(s-\omega)/(s-\omega-1)]^{1/2}k\cdot(p_1'-p_2'),$$
$$\cos(\varphi'-\varphi)=-(\sin\theta\sin\theta')^{-1}\left[\frac{(p_1-p_2)\cdot(p_1'-p_2')}{[(s-1)^2-\omega^2]^{1/2}}+\frac{s}{(s^2-\omega^2)^{1/2}}\cos\theta\cos\theta'\right].$$

## III. UNITARITY, PARTIAL-WAVE EXPANSION, AND CROSSING

### A. Unitarity

Since we assume the total energy to be insufficient for pion production, and consider only the lowest order in $e^2$, we include only the channels $N+N$ and $N+N+\gamma$. Now let us denote the amplitude for $NN\to NN$ by $T_{22}$, $NN\to NN\gamma$ by $T_{32}$, $NN\gamma\to NN$ by $T_{23}$, and $NN\gamma\to NN\gamma$ by $T_{33}$. Further, let us allow the subscripts 2 and 3 to represent spin indices in the following way: $2=\lambda_1\lambda_2$, $3=\lambda\lambda_1\lambda_2$, $2'=\lambda_1'\lambda_1'$, etc., where $\lambda$ and $\lambda_1\lambda_2$ are the spin indices of the photon and the two nucleons, respectively.

Then, if the $T$ matrix is defined in terms of the $S$ matrix as

$$S_{fi}=\delta_{fi}-(2\pi)^4 i\delta^4(P_f-P_i)T_{fi}, \qquad (3.1)$$

unitarity can be written in the form $i(T_{fi}-T_{if}^*)=\sum_n T_{fn}T_{in}^*$ and we have

$$i[T_{2'2}(p_1'p_2';p_1p_2)-T_{22'}^*(p_1p_2;p_1'p_2')]$$
$$=\sum_{2''}T_{2'2''}(p_1'p_2';p_1''p_2'')T_{22''}^*(p_1p_2;p_1''p_2'')$$
$$+\sum_{3''}T_{2'3''}(p_1'p_2';k''p_1''p_2'')T_{23''}^*(p_1p_2;k''p_1''p_2''), \qquad (3.2)$$

$$i[T_{3'2}(k'p_1'p_2';p_1p_2)-T_{23'}^*(p_1p_2;k'p_1'p_2')]$$
$$=\sum_{2''}T_{3'2''}(k'p_1'p_2';p_1''p_2'')T_{22''}^*(p_1p_2;p_1''p_2'')$$
$$+\sum_{3''}T_{3'3''}(k'p_1'p_2';k''p_1''p_2'')T_{33''}(p_1p_2;k''p_1''p_2''), \qquad (3.3)$$

where $\sum_{2''}$ and $\sum_{3''}$ represent integrals and summations over intermediate momenta and spin indices for the two- and three-particle states. Since $T_{23}$ is $O(e)$ compared with $T_{22}$, the second term in Eq. (3.2) is $O(e^2)$ and may be neglected, so in this approximation $T_{22}$ satisfies elastic unitarity. Similarly, we keep only the portion of $T_{33}$ in which the photon is noninteracting, that is,

$$T_{3'3''}(k'p_1'p_2';k''p_1''p_2'')=2k^{0'}(2\pi)^3\delta^3(\mathbf{k}'-\mathbf{k}'')$$
$$\times T_{2'2''}(p_1'p_2';p_1''p_2'')+O(e^2). \qquad (3.4)$$

Substituting Eq. (3.4) into (3.3), we obtain

$$i[T_{\lambda 2';2}(kp_1'p_2';p_1p_2)-T_{2;\lambda 2'}^*(p_1p_2;kp_1'p_2')]$$
$$=\sum_{2''}T_{\lambda 2';2''}(kp_1'p_2';p_1''p_2'')T_{22''}^*(p_1p_2;p_1''p_2'')$$
$$+\sum_{2''}{}'\,T_{2'2''}(p_1'p_2';p_1''p_2'')T_{2;\lambda 2''}^*(p_1p_2;kp_1''p_2''), \qquad (3.5)$$

where we now display the photon helicity index and have dropped the primes from photon variables. The prime on $\sum_{2''}{}'$ is to indicate that $(p_1''+p_2'')=P'$ instead of $P$.

We now specialize to the c.m. system of the incoming nucleons, $C$, and consider all spin indices to be helicities. Thus, we define the elastic c.m. helicity amplitude

$$H_{2'2}(s,\Omega',\Omega)=T_{2'2}(q_1'q_2';q_1q_2), \qquad (3.6)$$

where $s$ is the invariant energy and $\Omega$ ($\Omega'$) the polar angles of $\mathbf{q}_1$ ($\mathbf{q}_1'$) with respect to a fixed $z$ axis. Physical quantities of course depend only on the difference between initial and final angles, but it will be convenient to retain this redundancy.

At this point we must consider the fact that if we express the amplitudes in Eq. (3.5) in the c.m. system of the *initial* nucleons, the elastic amplitude appearing in the first term is a c.m. amplitude, while the elastic amplitude in the second term is not. In order to relate it to a c.m. amplitude, we must apply the Lorentz transformation from $C$ to $C'$. This requires a unitary transformation corresponding to the well-known Wigner rotation. If $L$ is the boost transformation from $C$ to $C'$ defined in Sec. II, then[16]

$$T_{2'2''}(p_1'p_2';p_1''p_2'')=[Q^{-1}(L;p_1'p_2')$$
$$\times T(Lp_1',Lp_2';Lp_1'',Lp_2'')Q^{-1\dagger}(L;p_1''p_2'')]_{2'2''}$$
$$=[Q^{-1}(L;p_1'p_2')H(s-\omega,\Omega',\Omega'')$$
$$\times Q^{-1\dagger}(L;p_1''p_2'')]_{2'2''}, \qquad (3.7)$$

where $Q(L;p_1p_2)$ is a direct product of unitary matrices corresponding to Wigner rotations for the two nucleons. That is, $Q_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(L,p_1'p_2')=Q_{\lambda_1'\lambda_1}(L,p_1')Q_{\lambda_2'\lambda_2}(L_1'p_2')$. We will specify these functions later.

We wish to define an $NN\gamma$ amplitude such that the initial nucleon spins are helicities in frame $C$, while final spins are helicities in $C'$. This puts the final spin indices on a different footing from the initial, changing the transformation properties. It is accomplished by a unitary transformation, in fact the same one introduced in Eq. (3.7). We thus define the $NN\to NN\gamma$ and $NN\gamma\to NN$ c.m. helicity amplitudes by

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=Q_{2'2''}(L,p_1'p_2')T_{\lambda 2'';2}(kp_1'p_2';p_1p_2),$$
$$G_{2;\lambda 2'}(s,\omega,\Omega,\Omega')=T_{2;\lambda 2''}(p_1p_2;kp_1'p_2')Q_{2''2}{}^{\dagger}(L,p_1'p_2'). \qquad (3.8)$$

With our normalization conventions, we have

$$\sum_{2''}=[1/(2\pi)^6]\int(d^3p_1''/2p_1^{0''})(d^3p_2''/2p_2^{0''})$$
$$\times(2\pi)^4\delta^4(p_1''+p_2''-p_1-p_2)$$
$$=-2\rho(s+\omega)\int d\Omega'', \qquad (3.9)$$

where $\rho(s)=-[(s-1)/s]^{1/2}/N$ and $N=64\pi^2$ for neutron-proton bremsstrahlung and $128\pi^2$ for proton-proton bremsstrahlung. Similarly,

$$\sum_{2''}{}'=-2\rho(s-\omega)\int d\Omega''.$$

Now we use Eqs. (3.6)–(3.9) to reexpress Eq. (3.5) in the form

$$i[Q_{2'2''}{}^{-1}(L;p_1'p_2')G_{\lambda 2'';2}(s,\omega,\Omega',\Omega)$$
$$-G_{2;\lambda 2''}{}^*(s,\omega,\Omega,\Omega')Q_{2''2}{}^*(L;p_1'p_2')]$$
$$=-2\rho(s+\omega)\int d\Omega''\,Q^{-1}{}_{2''2'''}(L;p_1'p_2')G_{\lambda 2''';2}$$
$$\times(s,\omega,\Omega',\Omega'')H_{22''}(s+\omega,\Omega,\Omega'')-2\rho(s-\omega)\int d\Omega''$$
$$\times Q^{-1}(L;p_1'p_2')H(s-\omega,\Omega',\Omega'')Q(L;p_1''p_2'')$$
$$\times G_{2;\lambda 2'''}{}^*(s,\omega,\Omega,\Omega'')Q_{2'''2'}{}^*(L;p_1''p_2''). \qquad (3.10)$$

Taking out a common factor $Q^{-1}(L,\ p_1'p_2')$, we

[16] We will use the conventions of A. O. Barut, *The Theory of the Scattering Matrix* (MacMillan, New York, 1967), whereby outgoing particles with spin $s$ transform according to the representation $(0, s)$ of the Lorentz group, and incoming particles according to the conjugate representation.

finally have the simple form

$$(1/2i)[G_{\lambda 2';2}(s,\omega,\Omega',\Omega)-G_{2;\lambda 2'}{}^{*}(s,\omega,\Omega,\Omega')]$$
$$=\rho(s+\omega)\int d\Omega''\, G_{\lambda 2';2''}(s,\omega,\Omega',\Omega'')H_{22''}{}^{*}(s+\omega,\Omega,\Omega'')$$
$$+\rho(s-\omega)\int d\Omega''\, H_{2'2''}(s-\omega,\Omega',\Omega'')G_{2;\lambda 2''}{}^{*}(s,\omega,\Omega,\Omega''). \quad (3.11)$$

### B. Partial-Wave Expansions

Using the methods of Jacob and Wick,[17] we form the two-particle helicity state of two nucleons in their c.m. system:

$$|\,p_1p_2;2\rangle_{\text{c.m.}}=\sum_{Jm}N_J\,|\,Jm;2\rangle d_{m\mu}{}^{J}(\theta)e^{-im\varphi}, \quad (3.12)$$

where $N_J{}^2=(2J+1)/4\pi$, $\theta$ and $\phi$ are the polar angles of $\mathbf{p}_1$, we continue to use the notation $2=\lambda_1\lambda_2$, and $\mu=\lambda_1-\lambda_2$. The partial-wave expansion of the elastic amplitude is[18]

$$H_{2'2}(s,\Omega',\Omega)=\sum_{Jm}N_J{}^2h_{2'2}{}^{J}(s)d_{m\mu'}{}^{J}(\theta')d_{m\mu}{}^{J}(\theta)\times\exp[im(\varphi'-\varphi)]. \quad (3.13)$$

In order to form the partial-wave expansion for the $NN\gamma$ amplitude, we must combine the final $NN$ state with the photon. Thus we must boost the state with an expansion as in (3.12) from frame $C'$ to frame $C$, which introduces the same Wigner rotation matrix discussed above. Thus the final $NN$ state in frame $C$ has the expansion

$$\langle p_1'p_2';2'\,|=Q^{-1}{}_{2'2''}(L;p_1'p_2')\langle Lp_1',Lp_2';2''\,|_{\text{c.m.}}$$
$$=Q^{-1}{}_{2'2''}(L;p_1'p_2')\sum_{J'm'}N_{J'}$$
$$\times\langle J'm';2''\,|\,d_{m'\mu''}{}^{J'}(\theta')\exp(im'\varphi'). \quad (3.14)$$

We then combine this state, which has total momentum in the $z$ direction, with the photon in the negative $z$ direction, to obtain a sum over states of total angular momentum $J$ and $z$ component $m$:

$$\langle kp_1'p_2';\lambda 2'\,|=Q^{-1}{}_{2'2''}(L,p_1'p_2')\sum_{JmJ'm'}N_JN_{J'}$$
$$\times\langle Jm,J'm';\lambda 2''\,|\exp(i'm'\varphi)\delta_{m,m'-\lambda}. \quad (3.15)$$

Thus we obtain

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=\sum_{JJ'm'}2\pi N_J{}^2N_{J'}{}^2g_{\lambda 2';2}{}^{JJ'm}(s,\omega)$$
$$\times d_{m'\mu'}{}^{J'}(\theta')d_{m'-\lambda,\mu}{}^{J}(\theta)\exp(im'\varphi')\exp[-i(m'-\lambda)\varphi]. \quad (3.16)$$

The particular normalization factors are chosen to simplify the partial-wave projection formula. The PWA $g^{JJ'm'}$ represents the transition amplitude from a pair of nucleons with angular momentum $J$ to a photon with helicity $\lambda$ plus a pair of nucleons with angular momentum $J'$ and helicity $m'$. In a similar manner we have, dropping the prime on the $m$ in the following,

$$G_{2;\lambda 2'}(s,\omega,\Omega,\Omega')=\sum_{JJ'm}2\pi N_J{}^2N_{J'}{}^2g_{2;\lambda 2'}{}^{JJ'm}(s,\omega)$$
$$\times d_{m\mu'}{}^{J'}(\theta')d_{m-\lambda,\mu}{}^{J}(\theta)\exp(-im\varphi')\exp[i(m-\lambda)\varphi]. \quad (3.17)$$

Substituting this result and the elastic expansion in (3.11), we have

$$(1/2i)[g_{\lambda 2';2}{}^{JJ'm}(s,\omega)-g_{2;\lambda 2'}{}^{JJ'm*}(s,\omega)]$$
$$=\rho(s+\omega)g_{\lambda 2';2''}{}^{JJ'm}(s,\omega)h_{22''}{}^{J*}(s+\omega)$$
$$+\rho(s-\omega)h_{2'2''}{}^{J'}(s-\omega)g_{2;\lambda 2''}{}^{JJ'm*}(s,\omega). \quad (3.18)$$

Finally, we assume time-reversal invariance. For the elastic amplitude, we have $h_{2'2}{}^{J}(s)=h_{22'}{}^{J}(s)$, and we specify the relative phase of the processes $NN\gamma\to NN$ and $NN\to NN\gamma$ by $g_{2;\lambda 2'}{}^{JJ'm}(s,\omega)=g_{\lambda 2';3}{}^{JJ'm}(s,\omega)$ to obtain

$$\text{Im}g_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}{}^{JJ'm}(s,\omega)=\rho(s+\omega)g_{\lambda\lambda_1'\lambda_2';\lambda_1''\lambda_2''}{}^{JJ'm}(s,\omega)$$
$$\times h_{\lambda_1''\lambda_2'';\lambda_1\lambda_2}{}^{J*}(s+\omega)+\rho(s-\omega)h_{\lambda_1'\lambda_2';\lambda_1''\lambda_2''}{}^{J'}(s-\omega)$$
$$\times g_{\lambda\lambda_1''\lambda_2'';\lambda_1\lambda_2}{}^{JJ'm*}(s,\omega). \quad (3.19)$$

For completeness, unitarity for the elastic PWA is

$$\text{Im}h_{\lambda_1'\lambda_2';\lambda_1\lambda_2}{}^{J}(s)=\rho(s)h_{\lambda_1'\lambda_2';\lambda_1''\lambda_2''}{}^{J}(s)h_{\lambda_1''\lambda_2'';\lambda_1\lambda_2}{}^{J*}(s). \quad (3.20)$$

We will require the partial-wave projection formula to invert Eq. (3.16). In terms of $G$, it is given by

$$g_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}{}^{JJ'm}(s,\omega)=\int_{-1}^{1}d\cos\theta\, d_{m-\lambda,\mu}{}^{J}(\theta)$$
$$\times\int_{-1}^{1}d\cos\theta' d_{m\mu'}{}^{J'}(\theta')\int_0^{2\pi}d\varphi'\exp(-im\varphi')$$
$$\times G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}(s,\omega,\Omega',\Omega)\Big|_{\varphi=0}. \quad (3.21)$$

### C. A Symmetry Condition

If we cross the photon from the final state to the initial state, then consider the relation between the $NN\gamma$ amplitude and the time-reversed process, we obtain a symmetry condition for the $NN\gamma$ amplitude.

Crossing relates the $NN\gamma\to NN$ and $NN\to NN\gamma$ amplitudes in the following way:

$$T_{\lambda 2';2}(kp_1'p_2';p_1p_2)=T_{2';-\lambda 2}(p_1'p_2';-kp_1p_2). \quad (3.22)$$

Multiplying by the Wigner rotation to obtain $G$, we have

$$Q_{2'2''}(L;p_1'p_2')T_{\lambda 2'';2}(kp_1'p_2';p_1p_2)$$
$$=Q_{2'2''}(L,p_1'p_2')T_{2'';-\lambda 2}(p_1'p_2';-kp_1p_2)$$
$$=T_{2';-\lambda 2''}(Lp_1',Lp_2';-Lk,Lp_1,Lp_2)Q_{2''2}(L;p_1p_2), \quad (3.23)$$

[17] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959); G. C. Wick, *ibid.* **18**, 65 (1962). Note that our helicity amplitudes differ by a phase and some kinematic factors from these authors'.

[18] Our normalizations are such that for an uncoupled amplitude $h^J(s)=\exp[i\delta^J(s)]/\rho(s)$ with $\delta^J(s)$ a real phase.

where we used the transformation properties of $T$ in the second step. If we evaluate this in system $C$, we see that both sides are c.m. amplitudes. From the definitions (3.8), we have

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=G_{2';-\lambda 2}(s,-\omega,\Omega',\Omega). \quad (3.24)$$

Comparing Eqs. (3.17) and (3.18), we see that the $NN\gamma$ amplitude and its time-reversed process are related by

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=G_{2;\lambda 2'}(s,\omega,(\theta,-\varphi),(\theta^1,-\varphi')). \quad (3.25)$$

Thus combining crossing and time-reversal invariance we have the symmetry

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=G_{-\lambda 2;2'}(s,-\omega,(\theta,-\varphi),(\theta',-\varphi')) \quad (3.26)$$

and for $g$, applying the partial-wave expansions,

$$g_{\lambda 2';2}{}^{JJ'm}(s,\omega)=g_{-\lambda 2;2'}{}^{J'Jm-\lambda}(s,-\omega). \quad (3.27)$$

## IV. SOFT-PHOTON APPROXIMATION

### A. Full Amplitude

As we shall see, the $NN\gamma$ amplitude, to lowest order in $e^2$, is analytic in $\omega$ in the physical region, except for a pole at $\omega=0$. This allows a power-series expansion in $\omega$, the first two terms of which can be uniquely determined by the method of Low,[5] using only static electromagnetic properties of the nucleons and the elastic amplitude. One obtains an expression for the $NN\gamma$ amplitude which is linear in the elastic amplitude and its derivatives with respect to physical variables.

Let us define the $NN\gamma$ amplitude

$$\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}{}^{\mu}(kp_1'p_2';p_1p_2)$$

in the following way:

$$T_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}(kp_1'p_2';p_1p_2)=e\epsilon_\mu(k,\lambda)\bar{u}_{\alpha_1'}(p_1',\lambda_1')\bar{u}_{\alpha_2'}(p_2',\lambda_2')\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}{}^{\mu}(kp_1'p_2';p_1p_2)\mu_{\alpha_1}(p_1,\lambda_1)\mu_{\alpha_2}(p_2,\lambda_2). \quad (4.1)$$

Suppose that only particle 1 has charge and anomalous magnetic moment. Then, using a notation due to Burnett and Kroll,[19] we write

$$\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}{}^{\mu}(kp_1'p_2';p_1p_2)=\left[\frac{p_1'^{\mu}}{k\cdot p_1'}-\frac{p_1^{\mu}}{k\cdot p_1}+D_1'^{\mu}(k)+D_1^{\mu}(k)\right]\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}(p_1'p_2';p_1p_2)+\left(\frac{\gamma^\mu k[1+\kappa_1(p_1'+m)]}{2k\cdot p_1'}\right)_{\alpha_1'\beta_1'}\mathfrak{J}_{\beta_1'\alpha_2';\alpha_1\alpha_2}+\mathfrak{J}_{\alpha_1'\alpha_2';\beta_1\alpha_2}\times\left(\frac{[1+\kappa_1(p_1+m)]k\gamma^\mu}{2k\cdot p_1}\right)_{\beta_1\alpha_1}+O(k), \quad (4.2)$$

where the differential operator $D_1^{\mu}(k)$ is given by

$$D_1^{\mu}(k)=(p_1^{\mu}/k\cdot p_1)k\cdot(\partial/\partial p_1)-(\partial/\partial p_{1\mu}), \quad (4.3)$$

and similarly for $D_1'^{\mu}(k)$, and $\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}(p_1'p_2';p_1p_2)$ is given in terms of the elastic amplitude by

$$T_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(p_1'p_2';p_1p_2)=\bar{u}_{\alpha_1'}(p_1',\lambda_1')\bar{u}_{\alpha_2'}(p_2',\lambda_2')\times\mathfrak{J}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}(p_1'p_2';p_1p_2)u_{\alpha_1}(p_1,\lambda_1)u_{\alpha_2}(p_2,\lambda_2) \quad (4.4)$$

except that it must be extended in Eq. (4.2) off the momentum shell for $k\neq 0$, $p_1'+p_2'\neq p_1+p_2$. This is usually accomplished by choosing a representation in terms of $\gamma$ matrices and invariant amplitudes for the elastic $\mathfrak{J}$, then specifying the functional dependence of the elastic invariants in a particular way. For example, $s=(p_1+p_2)^2$ and $s=(p_1'+p_2')^2$ may be identical on the momentum shell, but depend on the momenta in different ways; as emphasized by Burnett and Kroll,[19] the form obtained from Eq. (4.2) depends on this specification, but the result is the same to order $k$.

We choose as variables to describe the elastic process the invariant energy and the c.m. scattering angle, and specify them in the following way:

$$s=\tfrac{1}{2}[(p_1+p_2)^2+(p_1'+p_2')^2],$$
$$\cos\bar{\theta}=-(p_1-p_2)\cdot(p_1'-p_2')/(s-1). \quad (4.5)$$

This is in fact the same $s$ that we defined for $NN\gamma$, and from Eq. (2.5) we see that, in terms of $NN\gamma$ variables,

$$\cos\bar{\theta}=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi'-\varphi)+O(\omega^2). \quad (4.6)$$

Therefore, if the extrapolation off the momentum shell necessary to define the elastic $\mathfrak{J}$ in Eq. (4.2) is defined by changing $\omega$ from zero, with $s$, $\Omega'$, and $\Omega$ fixed, the elastic invariants defined by Eq. (4.5) do not change, to $O(\omega^0)$, and moreover, $\bar{\theta}$ continues to be the difference between initial and final angles, even when these are measured in different frames. Finally, for $NN$ scattering, the $\gamma$-matrix part of $\mathfrak{J}$ can be chosen to be momentum independent,[20] so we may express $\mathfrak{J}$ as a function of only $s$ and $\cos\bar{\theta}$, or $s$, $\Omega'$, and $\Omega$ with the usual redundancy.

Then, since the momentum dependence of $\mathfrak{J}$ is only through $s$, $\Omega'$, and $\Omega$, we use Eq. (4.5) and (4.6) to reexpress the derivatives in terms of the $NN\gamma$ variables. We find, keeping only terms of $O(\omega)$,

$$\epsilon(k,\lambda)\cdot D_1'(k)=E_\lambda(s,\Omega')(\partial/\partial s)+\mathfrak{D}_\lambda(s,\Omega'), \quad (4.7)$$

where we have defined the functions

$$E_\lambda(s,\Omega)=(e^{i\lambda\varphi}/\sqrt{2})\{s^{1/2}\sin\theta'/[\alpha(s)+\cos\theta]\},$$
$$\alpha(s)=[s/(s-1)]^{1/2}, \quad (4.8)$$

and the angular differential operator

$$\mathfrak{D}_\lambda(s,\Omega)=\frac{e^{i\lambda\varphi}}{\sqrt{2}}(s-1)^{-1/2}\left[\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}\frac{\partial}{\partial\theta}+\frac{i\lambda}{\sin\theta}\frac{\partial}{\partial\varphi}\right]. \quad (4.9)$$

[19] T. H. Burnett and N. M. Kroll, Phys. Rev. Letters 20, 86 (1968).

[20] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

In addition, we note that

$$\epsilon\cdot p_1'/k\cdot p_1'=\omega^{-1}E_\lambda(s-\omega,\Omega'),$$
$$\epsilon\cdot p_1/k\cdot p_1=\omega^{-1}E_\lambda(s+\omega,\Omega). \tag{4.10}$$

We wish to compute the Low expression for the amplitude $G$ instead of $T$, so we must multiply the final-state spinors in Eq. (4.1) by the Wigner rotation matrices discussed in Sec. III. These are given by[16]

$$Q(L,p_1')=B^{-1}(Lp_1')A(L)B(p_1'), \tag{4.11}$$

where $A$ is an $SL(2,C)$ representation of the Lorentz transformation $L$, and $B(p)$ is a particular $SL(2,C)$ matrix representing a transformation from the rest frame to $p$. Since this transformation is not unique, we may specify it further to obtain helicity amplitudes. We choose

$$B(p)=(\sigma\cdot p/m)^{1/2}R(\Omega),$$
$$R(\Omega)=\exp(-i\sigma_3\varphi/2)\exp(-i\sigma_2\theta/2), \tag{4.12}$$

where $R(\Omega)$ is a rotation into the direction of motion of the particle, and $(\sigma\cdot p/m)^{1/2}$ is a pure boost along the direction of motion with velocity $|\mathbf{p}|/p^0$, where $\sigma=(1,-\boldsymbol{\sigma})$. Properties of $A$ and $B$ that we will need are

$$A(L)\sigma\cdot pA^\dagger(L)=\sigma\cdot Lp,\qquad B(p)B^\dagger(p)=\sigma\cdot p. \tag{4.13}$$

Now spinors can be represented by

$$\bar{u}(p,\lambda)=\Phi_\lambda{}^\dagger[B^{-1}(p),B^\dagger(p)],$$
$$\Phi_{1/2}=\sqrt{2}^{-1}\begin{pmatrix}1\\0\end{pmatrix},\qquad \Phi_{-1/2}=\sqrt{2}^{-1}\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.14}$$

and in this representation,

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \boldsymbol{\gamma}=\begin{pmatrix}0&-\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix}. \tag{4.15}$$

Now we have

$$\sum_{\lambda_1''}Q(L,p_1')_{\lambda_1'\lambda_1''}\bar{u}(p_1',\lambda_1'')=\sum_{\lambda_1''}\Phi_{\lambda_1'}{}^\dagger B^{-1}(Lp_1')A(L)B(p_1)\Phi_{\lambda_1''}\Phi_{\lambda_1''}\times[B^{-1}(p_1'),B^\dagger(p_1')]$$
$$=\Phi_{\lambda_1'}{}^\dagger[B^{-1}(Lp_1')A(L),B^{-1}(Lp_1')A(L)\times B(p_1')B^\dagger(p_1')]. \tag{4.16}$$

Since

$$A(L)B(p_1')B^\dagger(p_1')=A(L)\sigma\cdot p_1'=\sigma\cdot Lp_1'A^{\dagger-1}(L)=B(Lp_1')B^\dagger(Lp_1')A^{\dagger-1}(L),$$

we obtain

$$\sum_{\lambda_1''}Q(L,p_1')_{\lambda_1'\lambda_1''}\bar{u}(p_1',\lambda_1'')=\bar{u}(Lp_1',\lambda_1')\mathcal{A}(L), \tag{4.17}$$

where we have defined the $4\times4$ matrix

$$\mathcal{A}(L)=\begin{pmatrix}A(L)&0\\0&A^{-1\dagger}(L)\end{pmatrix}. \tag{4.18}$$

We recall that $L$ was defined to be a pure boost from $C$ to $C'$. Then $A(L)$ is Hermitian, and explicitly

$$A(L)=(1/\sqrt{2})[(\gamma+1)^{1/2}-\sigma_3(\gamma-1)^{1/2}],$$
$$A^{-1}(L)=(1/\sqrt{2})[(\gamma+1)^{1/2}+\sigma_3(\gamma-1)^{1/2}], \tag{4.19}$$

where $\gamma=(1-\omega^2/s^2)^{-1/2}$. Expanding to $O(\omega)$, we have

$$A(L)=1-(\omega/2s)\sigma_3,\qquad A^{-1}(L)=1+(\omega/2s)\sigma_3, \tag{4.20}$$

so

$$\mathcal{A}(L)=1-(\omega/2s)\gamma^0\gamma^3+O(\omega^2). \tag{4.21}$$

Combining Eq. (4.17) with Eqs. (4.1) and (4.2), and the definition of $G$, (3.8), we find

$$G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}(s,\omega,\Omega',\Omega)=[\bar{u}(Lp_1',\lambda_1')\mathcal{A}(L)]_{\alpha_1'}[\bar{u}(Lp_2',\lambda_2')\mathcal{A}(L)]_{\alpha_2'}\times\mathcal{G}_{\lambda\alpha_1'\alpha_2';\alpha_1\alpha_2}(s,\omega,\Omega',\Omega)u_{\alpha_1}(p_1,\lambda_1)u_{\alpha_2}(p_2,\lambda_2)+O(\omega), \tag{4.22}$$

$$\mathcal{G}_{\lambda\alpha_1'\alpha_2';\alpha_1\alpha_2}(s,\omega,\Omega',\Omega)=\{(1/\omega)[E_\lambda(s-\omega,\Omega')-E_\lambda(s+\omega,\Omega)]+[E_\lambda(s,\Omega)+E_\lambda(s,\Omega')](\partial/\partial s)+\mathfrak{D}_\lambda(s,\Omega)+\mathfrak{D}_\lambda(s,\Omega')\}\mathfrak{I}_{\alpha_1'\alpha_2';\alpha_1\alpha_2}(s,\Omega',\Omega)+(\boldsymbol{\gamma}\cdot\epsilon_1\boldsymbol{k}[1+\kappa_1(\boldsymbol{p}_1'+m)]/2k\cdot p_1')_{\alpha_1'\beta_1'}\mathfrak{I}_{\beta_1'\alpha_2';\alpha_1\alpha_2}+\mathfrak{I}_{\alpha_1'\alpha_2';\beta_1\alpha_2}([1+\kappa_1(\boldsymbol{p}_1+m)]\boldsymbol{k}\boldsymbol{\gamma}\cdot\epsilon_\lambda/2k\cdot p_1)_{\beta_1\alpha_1}. \tag{4.23}$$

Now we identify powers of $\omega$:

$$G(s,\omega,\Omega',\Omega)=(1/\omega)G^{(0)}(s,\Omega',\Omega)+G^{(1)}(s,\Omega',\Omega)+O(\omega), \tag{4.24}$$

$$G_{\lambda2';2}{}^{(0)}(s,\Omega',\Omega)=[E_\lambda(s,\Omega')-E_\lambda(s,\Omega)]H_{2'2}(s,\Omega',\Omega), \tag{4.25}$$

$$G_{\lambda2';2}{}^{(1)}(s,\Omega',\Omega)=[(\partial/\partial\omega)\omega G_{\lambda2';2}(s,\omega,\Omega',\Omega)]_{\omega=0}$$
$$=\{-[\partial E_\lambda(s,\Omega')/\partial s]-[\partial E_\lambda(s,\Omega)/\partial s]+E_\lambda(s,\Omega')(\partial/\partial s)+E_\lambda(s,\Omega)(\partial/\partial s)+\mathfrak{D}_\lambda(s,\Omega')+\mathfrak{D}_\lambda(s,\Omega)\}H_{2'2}(s,\Omega',\Omega)+[\bar{u}_1'X_1']\bar{u}_2'\mathfrak{I}u_1u_2+\bar{u}_1'[\bar{u}_2'X_2']\mathfrak{I}u_1u_2+\bar{u}_1'\bar{u}_2'\mathfrak{I}[X_1u_1]u_2+\bar{u}_1'\bar{u}_2'\mathfrak{I}u_1[X_2u_2], \tag{4.26}$$

where

$$X_2' = -2E_\lambda(s,\Omega')(\partial/\partial s) - \mathfrak{D}_\lambda(s,\Omega') - E_\lambda(s,\Omega')(\gamma^0\gamma^3/2s),$$
$$X_1' = X_2' + (\gamma\cdot\epsilon_\lambda k/2k\cdot p_1')[1+\kappa_1(p_1'+m)],$$
$$X_2 = -2E_\lambda(s,\Omega)(\partial/\partial s) - \mathfrak{D}_\lambda(s,\Omega), \tag{4.27}$$
$$X_1 = X_2 - [1+\kappa_1(p_1+m)](\gamma\cdot\epsilon_\lambda k/2k\cdot p_1).$$

We have introduced an abbreviated notation for the spinors, and the derivatives in $X_1'$ and $X_2'$ are understood to operate on $\bar{u}_1'$ and $\bar{u}_2'$, respectively. To derive Eq. (4.26), we added and subtracted the effect of the differential operators on the spinors. Since final-state spinors in Eq. (4.22) are evaluated in frame $C'$, initial in $C$, $\omega$ dependence is restricted to the energy $(s\pm\omega)^{1/2}$, allowing us to convert $\omega$ derivatives to $s$ derivatives. Finally, we have used the relation

$$\mathfrak{A}_{\alpha_1'\alpha_1''}\mathfrak{A}_{\alpha_2'\alpha_2''}\mathfrak{I}_{\alpha_1''\alpha_2'';\alpha_1\alpha_2} = \mathfrak{I}_{\alpha_1'\alpha_2';\alpha_1''\alpha_2''}\mathfrak{A}_{\alpha_1''\alpha_1}\mathfrak{A}_{\alpha_2''\alpha_2}, \tag{4.28}$$

obtained from the transformation properties of $\mathfrak{I}$ and the fact that it depends only on scalar invariants.

We now express all of the terms in Eq. (4.26) in terms of elastic helicity amplitudes. The details can be found in the Appendix. Recombining $G^{(0)}$ and $G^{(1)}$, the results are

$$G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2} = [(e^{i\lambda\varphi'}/\sqrt{2})M_0' + (e^{i\lambda\varphi}/\sqrt{2})M_0]H_{\lambda_1'\lambda_2';\lambda_1\lambda_2} + (e^{i\lambda\varphi'}/\sqrt{2})(M_1'H_{-\lambda_1'\lambda_2';\lambda_1\lambda_2} + M_2'H_{\lambda_1',-\lambda_2';\lambda_1\lambda_2})$$
$$+ (e^{i\lambda\varphi}/\sqrt{2})(M_1H_{\lambda_1'\lambda_2';-\lambda_1\lambda_2} + M_2H_{\lambda_1'\lambda_2';\lambda_1,-\lambda_2}), \tag{4.29}$$

where

$$M_0' = \frac{s^{1/2}\sin\theta'}{\alpha(s)+\cos\theta'}\left(\omega^{-1}+\frac{\partial}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{s^{1/2}\sin\theta'}{\alpha(s)+\cos\theta}\right) + (s-1)^{-1/2}\left[\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'}\frac{\partial}{\partial\theta'} + \frac{i\lambda}{\sin\theta'}\frac{\partial}{\partial\varphi'}\right]$$
$$+ \frac{\lambda}{(s-1)^{1/2}}\left[\frac{\mu'\cos\theta'}{\sin\theta'} + (1+\kappa_1)\frac{2\lambda_1'\sin\theta'}{\alpha(s)+\cos\theta'}\right],$$

$$M_0 = \frac{s^{1/2}\sin\theta}{\alpha(s)+\cos\theta}\left(-\omega^{-1}+\frac{\partial}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{s^{1/2}\sin\theta}{\alpha(s)+\cos\theta}\right) + (s-1)^{-1/2}\left[\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}\frac{\partial}{\partial\theta} + \frac{i\lambda}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$
$$- \frac{\lambda}{(s-1)^{1/2}}\left[\frac{\mu\cos\theta}{\sin\theta} + (1+\kappa_1)\frac{2\lambda_1\sin\theta}{\alpha(s)+\cos\theta}\right],$$

$$M_1' = \left[2\lambda_1'\lambda\kappa_1 - \frac{[\alpha(s)-2\lambda_1'\lambda]^2}{2\alpha(s)}\right]\left[2\lambda_1'\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta} + \lambda\right],$$

$$M_2' = \frac{\alpha(s)^2-1}{2\alpha(s)}\left[2\lambda_2'\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'} - \lambda\right],$$

$$M_1 = \left[-2\lambda_1\lambda\kappa_1 - \frac{[\alpha(s)+2\lambda_1\lambda]^2}{2\alpha(s)}\right]\left[2\lambda_1\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta} - \lambda\right],$$

$$M_2 = \frac{\alpha(s)^2-1}{2\alpha(s)}\left[2\lambda_2\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta} + \lambda\right].$$

Recall that we have considered so far only terms proportional to the change and magnetic moment of particle 1. If particle 2 has the same charge and anomalous magnetic moment $\kappa_2$, we obtain an expression for which the $M$'s, as defined above, differ in the following respects: over-all sign change; $\cos\theta$ and $\cos\theta'$ have the opposite sign; $\lambda_1$ and $\lambda_2$, and $\lambda_1'$ and $\lambda_2'$ interchanged; $M_1$ and $M_2$, and $M_1'$ and $M_2'$ interchanged; $\kappa_1$ replaced by $\kappa_2$. If particle 2 is a neutron, we have of course only the terms proportional to $\kappa_2$.

### B. Partial-Wave Amplitude

Our examination of unitarity in Sec. III indicates that the $NN\gamma$ PWA are closely related to the elastic PWA; if the nucleons were spinless, Eq. (3.19) can be shown to require that the phase of $g_\lambda^{JJ'm}(s,\omega)$ be the sum of the phase shifts of $h^J(s+\omega)$ and $h^{J'}(s-\omega)$. More generally, if we write

$$g_{\lambda 2';2}^{JJ'm}(s,\omega) = [e/\sqrt{2}\rho(s)][\rho(s+\omega)f_{\lambda 2';2''}^{JJ'm}(s,\omega)h_{2''2}^{J}(s+\omega) - \rho(s-\omega)h_{2'2''}^{J'}(s-\omega)f_{\lambda 2'';2}^{JJ'm}(s,\omega)], \tag{4.30}$$

unitarity is seen to be satisfied if the functions $f^{JJ'm}(s,\omega)$ introduced above are real. Since Eq. (4.29) is already linear in the elastic amplitude one is led to believe that one can use the soft-photon limit to determine the functions $f^{JJ'm}(s,\omega)$, to $0(1)$, in terms of the static electromagnetic properties of the nucleons, independently of the elastic amplitude. This turns out to be the case. The computation of the PWA from Eq. (4.29), and the determination of the functions $f^{JJ'm}$ follows.

First let us examined the partial-wave projection of the term proportional to $M_0'$ in Eq. (4.29). Using the projection formula, Eq. (3.22), and the partial-wave expansion of $H_{2'2}$, Eq. (3.13), we have the following contribution to $g_\lambda^{JJ'm}$:

$$\sqrt{2}^{-1}\int d\cos\theta' d_{m\mu'}^{J'}(\theta')\bigg[\frac{s^{1/2}\sin\theta'}{\alpha(s)+\cos\theta'}\left(\omega^{-1}+\frac{\partial}{\partial s}\right)-\frac{\partial}{\partial s}\left(\frac{s^{1/2}\sin\theta'}{\alpha(s)+\cos\theta'}\right)$$
$$+(s-1)^{-1/2}\left(\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'}\frac{\partial}{\partial\theta'}-\frac{\lambda(m-\lambda-\mu'\cos\theta')}{\sin\theta'}+(1+\kappa_1)\frac{2\lambda_1\lambda\sin\theta'}{\alpha(s)+\cos\theta'}\right)\bigg]d_{m-\lambda\mu}^{J}(\theta')h_{2'2}^{J}(s). \quad (4.31)$$

Then if we integrate half of the term with the $\partial/\partial\theta$ by parts, using the identity

$$-\frac{\partial}{\partial s}\left[\frac{s^{1/2}\sin\theta}{\alpha(s)+\cos\theta}\right]-\frac{(s-1)^{-1/2}}{2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}\right)-1\right]=\frac{s^{1/2}\sin\theta}{\alpha(s)+\cos\theta}[\rho(s)]^{-1}\frac{d\rho(s)}{ds},$$

Eq. (4.31) becomes

$$\sqrt{2}^{-1}\int d\cos\theta'\bigg\{\bigg[\omega^{-1}\frac{s^{1/2}\sin\theta'}{\alpha(s)+\cos\theta'}+\lambda(s-1)^{-1/2}\left(\frac{m-\frac{1}{2}\lambda-\mu'\cos\theta'}{\sin\theta'}-(1+\kappa_1)\frac{2\lambda_1'\sin\theta'}{\alpha(s)+\cos\theta'}\right)\bigg]d_{m-\lambda,\mu}^{J}(\theta)d_{m\mu}^{J'}(\theta)$$
$$-\tfrac{1}{2}(s-1)^{-1/2}\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'}W[d_{m-\lambda,\mu}^{J}(\theta'),d_{m\mu}^{J'}(\theta')]\bigg\}\frac{\rho(s+\omega)h_{2'2}(s+\omega)}{\rho(s)}+O(\omega), \quad (4.32)$$

where $W(a,b)=ab'-a'b$. Comparing with Eq. (4.30), we see that the integral is then the soft-photon limit of $-f_{\lambda\lambda_1\lambda_2;\lambda_1\lambda_2}^{JJ'm}(s,\omega)$. The nondiagonal terms of $f$ are obtained by partial-wave projections of the terms in Eq. (4.29) proportional to $M_1$ and $M_2$ (we would get the same results using $M_1'$ and $M_2'$). The results are, for particle 1 still,

$$f_{\lambda\lambda_1\lambda_2;\lambda_1\lambda_2}^{JJ'm}(s,\omega)=\int d\cos\theta\bigg\{\bigg[\frac{\sin\theta}{\alpha(s)+\cos\theta}\left(\frac{s^{1/2}}{\omega}+\frac{2\lambda_1\lambda(1+\kappa_1)}{(s-1)^{1/2}}\right)-\lambda(s-1)^{-1/2}\frac{m-\frac{1}{2}\lambda-\mu\cos\theta}{\sin\theta}\bigg]d_{m-\lambda,\mu}^{J}(\theta)d_{m\mu}^{J'}(\theta)$$
$$-\tfrac{1}{2}(s-1)^{-1/2}\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}W[d_{m-\lambda,\mu}^{J}(\theta),d_{m\mu}^{J'}(\theta)]\bigg\}+O(\omega),$$

$$f_{\lambda\lambda_1\lambda_2;-\lambda_1\lambda_2}^{JJ'm}(s,\omega)=-\left[2\lambda_1\lambda\kappa_1-\frac{[\alpha(s)-2\lambda_1\lambda]^2}{2\alpha(s)}\right]\int d\cos\theta\, d_{m-\lambda,-\lambda_1-\lambda_2}^{J}(\theta)d_{m\mu}^{J'}(\theta)\left[2\lambda_1\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}+\lambda\right]+O(\omega), \quad (4.33)$$

$$f_{\lambda\lambda_1\lambda_2;\lambda_1,-\lambda_2}^{JJ'm}(s,\omega)=\tfrac{1}{2}[s(s-1)]^{-1/2}\int d\cos\theta\, d_{m-\lambda,\lambda_1+\lambda_2}^{J}(\theta)d_{m\mu}^{J'}(\theta)\left[2\lambda_2\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}-\lambda\right]+O(\omega),$$

$$f_{\lambda\lambda_1\lambda_2;-\lambda_1,-\lambda_2}^{JJ'm}(s,\omega)=O(\omega).$$

Using the partial-wave expansion, Eq. (3.16), the expression for the PWA, Eq. (4.30), and the soft-photon approximation to the functions $f^{JJ'm}(s,\omega)$, Eq. (4.33), one can now construct an amplitude that not only has the correct soft-photon behavior, but is unitary. Now we shall reexpress our results in a form more suitable for calculation. First let us combine Eq. (3.16) and (4.30):

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=[e/\sqrt{2}\rho(s)]\sum_{JJ'm}N_{JJ'}d_{m\mu'}^{J'}(\theta')e^{im\varphi'}d_{m-\lambda,\mu}^{J}(\theta)\exp[-i(m-\lambda)\varphi]$$
$$\times[\rho(s+\omega)f_{\lambda 2';2''}^{JJ'm}(s,\omega)h_{2''2}^{J}(s+\omega)-\rho(s-\omega)h_{2'2''}^{J'}(s-\omega)f_{\lambda 2'';2}^{JJ'm}(s,\omega)]$$
$$=[e/\sqrt{2}\rho(s)]\{e^{i\lambda\varphi'}\rho(s+\omega)\sum_{Jm}N_J^2F_{\lambda 2';2''}^{Jm}(s,\omega,\theta')h_{2''2}^{J}(s+\omega)d_{m\mu}^{J}(\theta)\exp[im(\varphi'-\varphi)]$$
$$-e^{i\lambda\varphi}\rho(s-\omega)\sum_{J'm}N_{J'}^2d_{m\mu'}^{J'}(\theta')h_{2'2''}^{J'}(s-\omega)F'_{\lambda 2'';2}^{J'm}(s,\omega,\theta)\exp[im(\varphi'-\varphi)]\}. \quad (4.34)$$

In the second step we have defined the functions

$$F_{\lambda 2';2}{}^{Jm}(s,\omega,\theta')=\sum_{J'}\tfrac{1}{2}(2J'+1)d_{m+\lambda,\mu'}{}^{J'}(\theta')f_{\lambda 2';2}{}^{JJ'm+\lambda}(s,\omega), \tag{4.35}$$

$$F'_{\lambda 2';2}{}^{J'm}(s,\omega,\theta)=\sum_{J}\tfrac{1}{2}(2J+1)f_{\lambda 2';2}{}^{JJ'm}(s,\omega)d_{m-\lambda,\mu}{}^{J}(\theta). \tag{4.36}$$

Now the symmetry for $G$, Eq. (3.27), implies that

$$f_{\lambda 2';2}{}^{JJ'm}(s,\omega)=-f_{-\lambda 2;2'}{}^{J'Jm-\lambda}(s,-\omega) \tag{4.37}$$

[note that the soft-photon approximation to $f$, Eq. (4.33), is consistent with this, as it must be], so

$$F'_{\lambda 2';2}{}^{J'm}(s,\omega,\theta)=-F_{-\lambda 2;2'}{}^{J'm}(s,-\omega,\theta). \tag{4.38}$$

Thus we obtain

$$G_{\lambda 2';2}(s,\omega,\Omega',\Omega)=[e/\sqrt{2}\rho(s)]\sum_{Jm}N_J{}^2[\exp(i\lambda\varphi')\rho(s+\omega)F_{\lambda 2';2''}{}^{Jm}(s,\omega,\theta')h_{2''2}{}^{J}(s+\omega)d_{m\mu}{}^{J}(\theta)$$
$$+e^{i\lambda\varphi}\rho(s-\omega)d_{m\mu'}{}^{J}(\theta')h_{2'2''}{}^{J}(s-\omega)F_{-\lambda 2;2''}{}^{Jm}(s,-\omega,\theta)]\exp[im(\varphi'-\varphi)]. \tag{4.39}$$

This form explicitly satisfies crossing symmetry for $G$, and is convenient for calculation. The soft-photon approximation to $F$ is, if particle $i$ has charge $n_ie$ and anomalous magnetic moment $\kappa_i$,

$$F_{\lambda\lambda_1\lambda_2;\lambda_1\lambda_2}{}^{Jm}(s,\omega,\theta)=n_1\Bigg[\frac{\sin\theta}{\alpha(s)+\cos\theta}\left(\frac{s^{1/2}}{\omega}+\frac{2\lambda_1\lambda}{(s-1)^{1/2}}-\frac{\alpha(s)}{2(s-1)^{1/2}}-\frac{1}{2(s-1)^{3/2}[\alpha(s)+\cos\theta]}\right)$$
$$-\frac{\lambda}{(s-1)^{1/2}}\frac{m-\mu\cos\theta}{\sin\theta}+\frac{1}{2(s-1)^{1/2}}\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}\frac{d}{d\theta}\Bigg]d_{m\mu}{}^{J}(\theta)$$
$$-n_2\Bigg[\frac{\sin\theta}{\alpha(s)-\cos\theta}\left(\frac{s^{1/2}}{\omega}+\frac{2\lambda_2\lambda}{(s-1)^{1/2}}-\frac{\alpha(s)}{2(s-1)^{1/2}}-\frac{1}{2(s-1)^{3/2}[\alpha(s)-\cos\theta]}\right)$$
$$-\frac{\lambda}{(s-1)^{1/2}}\frac{m-\mu\cos\theta}{\sin\theta}-\frac{1}{2(s-1)^{1/2}}\frac{1-\alpha(s)\cos\theta}{\alpha(s)-\cos\theta}\frac{d}{d\theta}\Bigg]d_{m\mu}{}^{J}(\theta)$$
$$+\frac{\lambda\sin\theta}{(s-1)^{1/2}}\left[\frac{2\lambda_1\kappa_1}{\alpha(s)+\cos\theta}-\frac{2\lambda_2\kappa_2}{\alpha(s)-\cos\theta}\right]d_{m\mu}{}^{J}(\theta), \tag{4.40}$$

$$F_{\lambda\lambda_1\lambda_2;-\lambda_1\lambda_2}{}^{Jm}(s,\omega,\theta)=\Bigg[\left(2\lambda_1\lambda\kappa_1-n_1\frac{[\alpha(s)-2\lambda_1\lambda]^2}{2\alpha(s)}\right)\left(2\lambda_1\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}+\lambda\right)$$
$$-\frac{n_2}{2[s(s-1)]^{1/2}}\left(2\lambda_1\frac{1-\alpha(s)\cos\theta}{\alpha(s)-\cos\theta}-\lambda\right)\Bigg]d_{m,-\lambda_1-\lambda_2}{}^{J}(\theta),$$

$$F_{\lambda\lambda_1\lambda_2;\lambda_1,-\lambda_2}{}^{Jm}(s,\omega,\theta)=\Bigg[\left(-2\lambda_2\lambda\kappa_2+n_2\frac{[\alpha(s)-2\lambda_2\lambda]^2}{2\alpha(s)}\right)\left(2\lambda_2\frac{1-\alpha(s)\cos\theta}{\alpha(s)-\cos\theta}+\lambda\right)$$
$$+\frac{n_1}{2[s(s-1)]^{1/2}}\left(2\lambda_2\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}-\lambda\right)\Bigg]d_{m,\lambda_1+\lambda_2}{}^{J}(\theta),$$

$$F_{\lambda\lambda_1\lambda_2;-\lambda_1-\lambda_2}{}^{Jm}(s,\omega,\theta)=0.$$

In summary, we have expressed the PWA in terms of real functions such that unitarity is satisfied, found the soft-photon, or small-$\omega$, behavior of these functions, and, using crossing symmetry, expressed the result in a compact manner in Eqs. (4.39) and (4.40).

## V. APPLICATION TO EXPERIMENT

In this section, we discuss the application of our formalism to the two experimental geometries, presenting the relations necessary to compute cross sections. We compare the soft-photon approximation with the unintegrated proton-proton bremsstrahlung data of Gottschalk, Shlaer, and Wang (GSW)[21] taken at 158 MeV in the Harvard geometry, finding good agreement with the coplanar data, but very poor agreement with the noncoplanar data.

[21] B. Gottschalk, W. J. Shlaer, and K. H. Wang, Nucl. Phys. **A94**, 491 (1967).

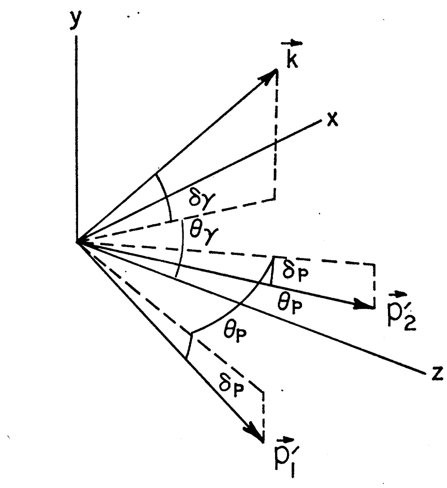


FIG. 2. Definitions of laboratory angles for final particles in the Harvard geometry.

### A. Harvard Geometry

For experiments performed using the Harvard geometry,[21,22] two counters are placed symmetrically on either side of the beam to detect the outgoing protons and measure their energy. If the angle the two protons make with the beam is less than 45°, that is, their opening angle is less than 90°, the event must be inelastic. The parameters of the missing photon, which must be in the plane containing the beam and the counters, can then be determined. Finite experimental resolution will allow some noncoplanarity, and moreover, one experiment[21] of this type was intentionally noncoplanar. Therefore, we shall analyze the more general noncoplanar case.

The directions of the outgoing particles in the lab frame are shown in Fig. 2. The beam is along the $z$ axis, and the $x$-$z$ plane is chosen so that outgoing protons dip down out of the plane by equal angles $\delta_p$ (called $\bar{\phi}$ by GSW). The counters are arranged so that the projected angles $\theta_p$ are approximately equal. The photon is then tipped up out of the plane by an angle $\delta_\gamma$, with a projected angle $\theta_\gamma$. The coplanar case is of course $\delta_p=\delta_\gamma=0$.

With three final-state particles, there are nine parameters to specify the final state. Energy-momentum conservation reduces this to five, and one remains if the final proton angles are fixed. In the coplanar case, this is taken to be the photon angle $\theta_\gamma$ which has a range $0$–$2\pi$. When the configuration is not coplanar, however, this angle is not very convenient, since it becomes restricted, and a cross section in this variable is singular at the kinematic endpoints. GSW define a different angle $\psi$ which reduces to $\theta_\gamma$ in the coplanar case and does not have this difficulty. It is defined in terms of the maximum kinematically allowed photon dip angle, which we call $\alpha$, in the following way:

$$\tan\psi=\sin\theta_\gamma/(\cos\theta_\gamma-\tan\delta_\gamma\cot\alpha). \tag{5.1}$$

The cross section in the general noncoplanar case is

$$d\sigma/d\Omega_1'd\Omega_2'd\psi=[m^3\mathfrak{F}/p_1(2\pi)^5]\,\tfrac{1}{4}\sum_{\text{spins}}|G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}(s,\omega,\Omega',\Omega)|^2, \tag{5.2}$$

where the phase-space factor $\mathfrak{F}$ has been evaluated by Baier, Kuhnelt, and Urban[11] and turns out to be

$$\mathfrak{F}=\frac{(p_1'p_2')^2}{2E_1'E_2'}\frac{(\cos\theta_\gamma-\tan\delta_\gamma\cot\alpha)^2+\sin^2\theta_\gamma}{\cos\delta_p}$$
$$\times\Bigg\{\bigg[-\sin2\theta_p\cos\delta_p\cos\delta_\gamma+\frac{p_1'}{E_1'}\sin(\theta_p+\theta_\gamma)$$
$$+\frac{p_2'}{E_2'}\sin(\theta_p-\theta_\gamma)+2\sin\theta_p\cos\theta_\gamma\sin\delta_p\sin\delta_\gamma\bigg]$$
$$\times\left(1-\frac{\tan\delta_\gamma\cot\alpha}{\cos\theta_\gamma}\right)+(\tan\delta_\gamma\cos\theta_p+\tan\delta_p\cos\theta_\gamma)$$
$$\times\tan\theta_\gamma\cot\alpha\left(\frac{p_1'}{E_1'}-\frac{p_2'}{E_2'}-\sin\theta_\gamma\cos\delta_\gamma\right)\Bigg\}^{-1}. \tag{5.3}$$

The usual coplanar limit is evident. Applying (2.5) we obtain equations for the variables defined in Sec. II:

$$\omega=k(W-p_1\cos\theta_\gamma\cos\delta_\gamma),$$
$$s=W-\omega,$$
$$\cos\theta=(k/\omega)[(s+\omega)/(s+\omega-1)]^{1/2}(T-p_1\cos\theta_\gamma\cos\delta_\gamma), \tag{5.4}$$
$$\cos\theta'=(k/\omega)[(s-\omega)/(s-\omega-1)]^{1/2}$$
$$\times[E_1'-E_2'-(p_1'+p_2')\sin\theta_p\cos\delta_p\cos\delta_\gamma\sin\delta_\gamma$$
$$-(p_1'-p_2')(\cos\theta_p\cos\delta_p\cos\theta_\gamma\cos\delta_\gamma-\sin\delta_p\sin\delta_\gamma)],$$
$$\cos(\varphi'-\varphi)=-[\sin\theta\sin\theta']^{-1}$$
$$\times\left[\frac{T(E_1'-E_2')-p_1(p_1'-p_2')\cos\theta_p\cos\delta_p}{[(s-1)^2-\omega^2]^{1/2}}+\frac{s}{(s^2-\omega^2)^{1/2}}\cos\theta\cos\theta'\right],$$

where $T$ is the incident energy and $W=T+1$.

As we have discussed previously, $\omega$ is less than $s-1$ in the physical region. Thus $x=\omega/(s-1)$ is a convenient invariant measure of the inelasticity, ranging from zero to 1. In the Harvard geometry at 158 MeV for in-

[22] I. Slaus, J. W. Verba, J. R. Richardson, R. F. Carlson, W. T. H. van Oers, and L. S. August, Phys. Rev. Letters **17**, 536 (1966); R. E. Warner, Can. J. Phys. **44**, 1225 (1966); R. E. Warner, J. C. Young, and S. I. H. Naqui, Phys. Rev. Letters **18**, 933 (1967); M. L. Halbert, D. L. Mason, and L. C. Northcliffe, Phys. Rev. **168**, 1130 (1968); **176**, 1159 (1969); F. P. Brady, J. C. Young, and C. Badrinathan, Phys. Rev. Letters **20**, 750 (1968); E. A. Silverstein and K. G. Kibler, *ibid.* **21**, 922 (1968); A. Bahnsen and R. L. Burman, Phys. Letters **26B**, 585 (1968); G. M. Crawley, D. L. Powell, and B. V. Narasimha Rao, *ibid.* **26B**, 576 (1968); F. Sannes, J. Trischuk, and D. G. Staris, Phys. Rev. Letters **21**, 1474 (1968).

stance, it has a range 0.42–0.57 at $\theta_p=30°$, and 0.27–0.40 at $\theta_p=35°$, while the laboratory photon momentum varies by a factor of 2. In Figs. 3 and 4 we illustrate the relation between the $NN\gamma$ variables and the laboratory variables $\delta_p$ and $\Psi$ for the case $T=158$ MeV and $\theta_p=30°$.

We now compare the amplitude presented in Sec. IV with the experimental results of GSW, which allows a detailed comparison of the unintegrated cross section over a large range of our angular parameters. The results are presented in Fig. 5 using both the unitary soft-photon amplitude Eq. (4.39) and (4.40), and the amplitude obtained by keeping only the first two orders in $\omega$, which is of course not unitary. The difference, which is never large, gives an idea of the magnitude of $O(\omega)$ and higher terms in the unitary amplitude. The agreement with the data, which is quite good for small $\delta_p$, becomes unaccountably poor for the more noncoplanar data. This is especially evident for the cross sections integrated over $\Psi$, shown in Fig. 6.

The marked decrease in the experimental results with $\delta_p$ is evidently not due to the phase space which, we find, remains fairly constant even to the kinematic endpoint (see Fig. 7). The $1/\omega$ terms of the soft-photon amplitude does become small, however, vanishing for $\theta'=\theta=\frac{1}{2}\pi$, which occurs near the endpoint, and accounts for the decrease that we do find. This term is not large enough to cause such a dramatic change, as is evidenced by the fact that its characteristic quadrupole behavior is not the dominant feature of the coplanar data. While this discrepancy may be evidence of a breakdown of the soft-photon approximation, it is difficult to understand why the soft-photon approximation should be good at $\delta_p=0$, and poor at $\delta_p\neq 0$, considering that the inelasticity is essentially the same. It is also puzzling that Dreschsel and Maximon,[4] using the Hamada-Johnson and Ried potentials, find substantial agreement with the integrated cross sections, and that Baier, Kuhnelt, and Urban,[11] using a one-boson-exchange model, agree with the differential data as well. The former authors, in commenting on the decrease, also note that the phase space remains fairly constant. In addition, they remark that the amplitude in the maximum noncoplanar limit is order of $p_1/m$ of the coplanar amplitude. This is not the case for $F$, Eq. (4.40), where electric, $1/\omega$ terms are order $(s-1)/\omega=x^{-1}$ while the magnetic terms are order $(1+\kappa)$ for diagonal terms and order $\kappa$ for nondiagonal. Thus the two types of terms are of the *same* order, which is due to the cancellation in the electric terms caused by the identity of the protons. In the low-proton-energy limit, however, the $1/\omega$ terms dominate since the magnetic terms are multiplied by $P$-wave and higher $NN$ amplitudes while the $S$ wave actually predominates.

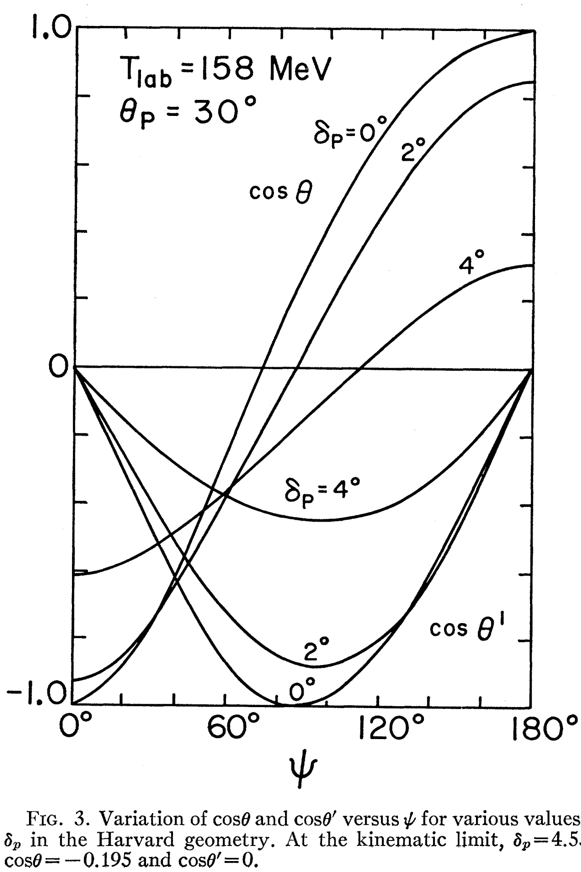


FIG. 3. Variation of $\cos\theta$ and $\cos\theta'$ versus $\psi$ for various values of $\delta_p$ in the Harvard geometry. At the kinematic limit, $\delta_p=4.55°$, $\cos\theta=-0.195$ and $\cos\theta'=0$.

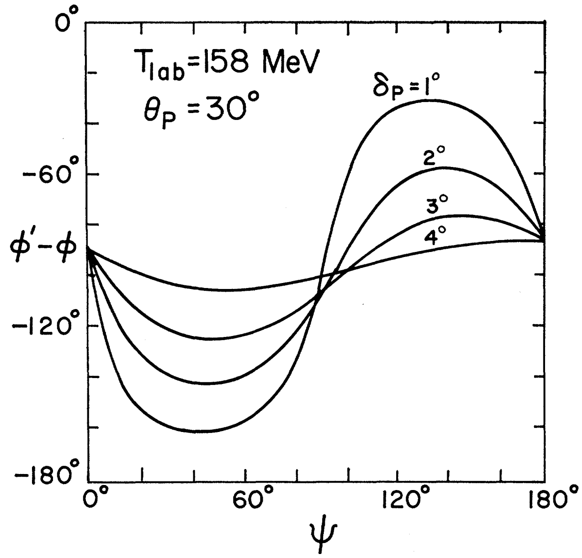


FIG. 4. Variation of $\phi'-\phi$ versus $\psi$ for various values of $\delta_p$. For $\delta_p=0$, $\phi'-\phi=180°$ when $0<\psi<90°$, and $\phi'-\phi=0°$ when $90°<\psi<180°$. At the kinematic limit, $\phi'-\phi=-90°$.

### B. Low-Energy Coplanar Harvard Geometry

At bombarding energies $\leq 10$ MeV, where $T=10/2M_N\simeq 5\times 10^{-3}$ and $p_1\simeq\sqrt{T}\approx 7\times 10^{-2}$, it makes sense to expand in powers of $\sqrt{T}$, keeping only lowest-

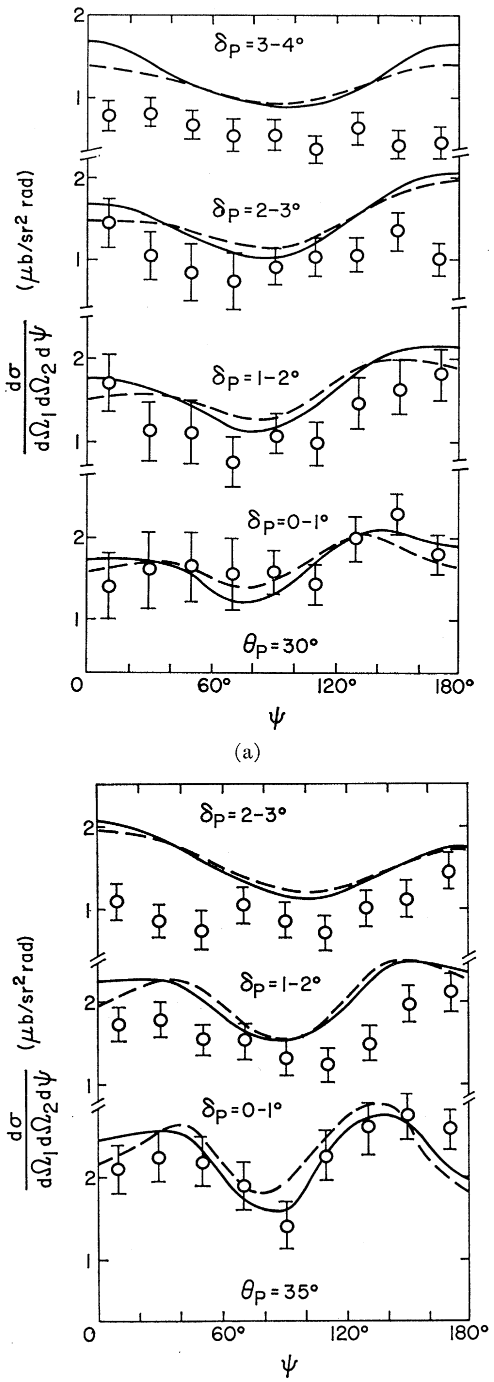


FIG. 5. Comparison of the soft-photon approximation with experimental data of GSW, Ref. 21. The solid line is the unitary formation for the amplitude, the dashed line the first two powers in $\omega$ only. Experimental errors are statistical only.

order terms. From Eqs. (5.13) and energy conservation, we find the relations, valid to $O(\sqrt{T})$,

$$\omega=k=T(\cos 2\theta_p/2\cos^2\theta_p),\qquad s-1=T/2\cos^2\theta_p,$$
$$\cos\theta=-\cos\theta_\gamma,\qquad \cos\theta'=-\sin\theta_\gamma,\qquad (5.5)$$

and thus $x=\omega/(s-1)=\cos 2\theta_p$. The functions $F$ are also much simpler in this limit. Taking $n_1=n_2=1$, $\kappa_1=\kappa_2=\kappa$ in Eqs. (4.40) and keeping lowest-order terms in $\sqrt{T}$, we have

$$F_{\lambda\lambda_1\lambda_2;\lambda_1\lambda_2}{}^{Jm}=[-\sin 2\theta(x^{-1}-\tfrac{3}{2})+\sin^2\theta(d/d\theta)+2\lambda(1+\kappa)\sin\theta(\lambda_1-\lambda_2)]d_{m\mu}{}^J(\theta),$$
$$F_{\lambda-\lambda_1\lambda_2;\lambda_1\lambda_2}{}^{Jm}=[(1+\kappa)(\lambda\cos\theta-2\lambda_1)+2\lambda_1\sin^2\theta]d_{m\mu}{}^J(\theta),\qquad (5.6)$$
$$F_{\lambda\lambda_1-\lambda_2;\lambda_1\lambda_2}{}^{Jm}=[(1+\kappa)(\lambda\cos\theta+2\lambda_2)-2\lambda_2\sin^2\theta]d_{m\mu}{}^J(\theta).$$

If we retain only the singlet $S$-wave $NN$ amplitude in Eq. (4.39), we find

$$G\approx\frac{e\sin 2\theta_\gamma}{\sqrt{2}\rho(s)8\pi}\begin{pmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{pmatrix}\times[(x^{-1}-\tfrac{3}{2})e^{i\delta}\sin\delta+(x^{-1}+\tfrac{3}{2})e^{i\delta'}\sin\delta'],\qquad (5.7)$$

where $\rho(s)=-[(s-1)/s]^{1/2}/128\pi^2$ and $\delta$ ($\delta'$) is the $^1S_0$ phase shift evaluated at the initial (final) energy. Substituting this into (5.4), and using (5.3) with $p_1'p_2'\approx p_1^2/4\cos^2\theta_p$, we obtain

$$\frac{d\sigma}{d\Omega_1'd\Omega_2'}=\frac{\alpha p_1}{8\pi m^3\sin 2\theta_p\cos^2\theta_p}\times|(x^{-1}-\tfrac{3}{2})e^{i\delta}\sin\delta+(x^{-1}+\tfrac{3}{2})e^{i\delta'}\sin\delta'|^2,\qquad (5.8)$$

where $\alpha=e^2/4\pi$. Except for terms within the parentheses of $O(x)$ this can be shown to be the same as the model-independent expression derived by Signell.[23] Since the $^1S_0$ phase shift varies rapidly at low energies, one might expect (5.8) to be better than the Low ex-

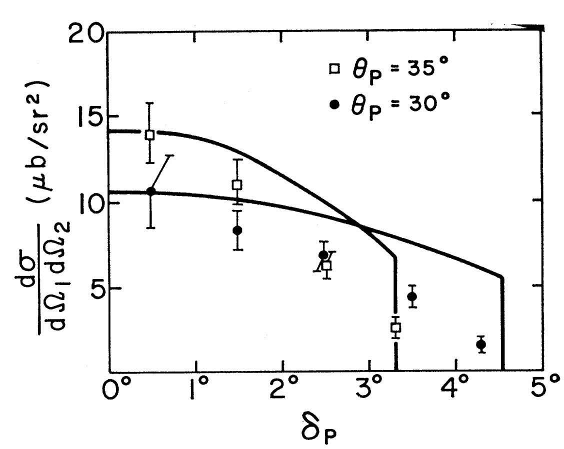


FIG. 6. Comparison of the unitary soft-photon approximation with the GSW data integrated over $\psi$.

[23] P. Signell, in *Proceedings of the International Conference on Light Nuclei, Few Body Problems and Nuclear Forces, Brela, Yugoslavia, 1967* (Gordon and Breach, New York, 1968).

pression obtained from it by expanding $\delta$ and $\delta'$ about a mean energy and keeping terms of $O(x)$, as seems to be the case.[24] The rapid variation of the phase shift is due to the quasi-bound state which could also make a large contribution to the $O(\omega)$ term of $F$ (see Sec. VI); the success of (5.8) indicates that it does not.

### C. Rochester Geometry

All final particles are detected in the geometry employed by experiments at Rochester.[25,26] The experimental parameters are the c.m. polar angles of the photon, $\theta_\gamma$ and $\phi_\gamma$, and the c.m. polar angles of $\mathbf{p}_1'-\mathbf{p}_2'$, $\theta_{\text{c.m.}}$ and $\phi_{\text{c.m.}}$, where the $z$ axis is along the beam direction, in contrast to our convention. Then one uses the c.m. energy of an initial proton, $E_1$, and the c.m. photon energy, $E_\gamma$, to complete the specification. Applying Eq. (2.5), we have

$$\omega=2E_\gamma E_1,\qquad s=4E_1^2-\omega,\qquad \cos\theta=-\cos\theta_\gamma,$$
$$\cos\theta'=[(|\mathbf{p}_1'-\mathbf{p}_2'|)/2E_1][(s-\omega)/(s-\omega-1)]^{1/2}$$
$$\times[\cos\theta_{\text{c.m.}}\cos\theta_\gamma+\sin\theta_{\text{c.m.}}\sin\theta_\gamma\cos(\varphi_{\text{c.m.}}-\varphi_\gamma)],\qquad(5.9)$$
$$\cos(\varphi'-\varphi)=(\sin\theta'\sin\theta)^{-1}$$
$$\times[(s-\omega-1)^{-1/2}|\mathbf{p}_1'-\mathbf{p}_2'|\cos\theta_{\text{c.m.}}-s(s^2-\omega^2)^{-1/2}\cos\theta\cos\theta'].$$

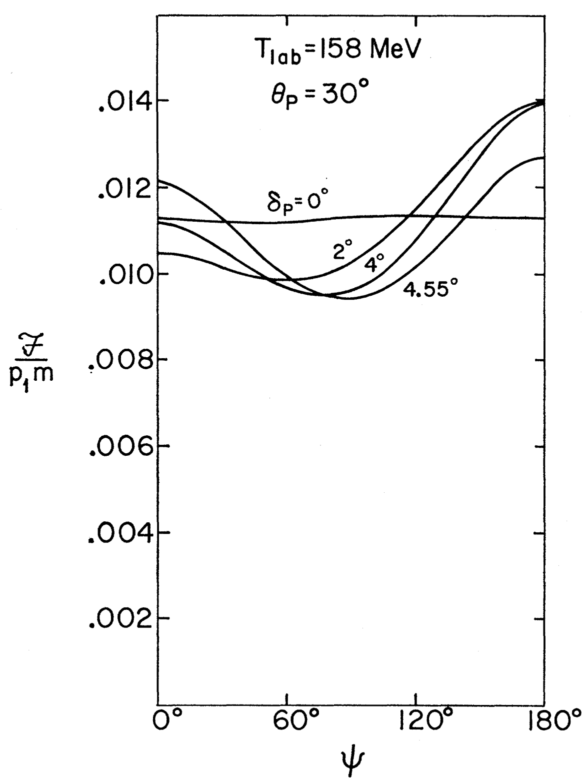


FIG. 7. Dimensionless quantity $\mathfrak{F}/p_1m$ versus $\psi$ for various values of $\delta_p$ in the Harvard geometry.

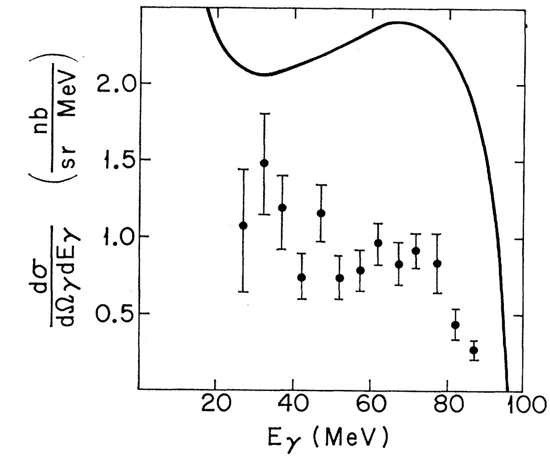


FIG. 8. Comparison of the soft-photon approximation with the Rochester geometry data of Ref. 25, laboratory energy 204 MeV, $\theta_\gamma=108°$.

A simpler expression for $\phi'-\phi$ can be obtained by rotating the coordinate system so that the photon is in the $-z$ direction. In this system

$$\mathbf{p}_1'-\mathbf{p}_2'=|\mathbf{p}_1'-\mathbf{p}_2'|[-\cos\theta_\gamma\sin\theta_{\text{c.m.}}\cos(\varphi_{\text{c.m.}}-\varphi_\gamma)+\sin\theta_\gamma\cos\theta_{\text{c.m.}},\ \sin\theta_{\text{c.m.}}\sin(\varphi_{\text{c.m.}}-\varphi_\gamma),\ -\cos\theta_{\text{c.m.}}\cos\theta_\gamma-\sin\theta_{\text{c.m.}}\sin\theta_\gamma\cos(\varphi_{\text{c.m.}}-\varphi_\gamma)],\qquad(5.10)$$

and we have immediately

$$\tan(\varphi'-\varphi)=(p_1'-p_2')_2/(p_1'-p_2')_1=\frac{\sin(\varphi_{\text{c.m.}}-\varphi_\gamma)}{\sin\theta_\gamma/\tan\theta_{\text{c.m.}}-\cos\theta_\gamma\cos(\varphi_\gamma-\varphi_{\text{c.m.}})}.\qquad(5.11)$$

The c.m. differential cross section in terms of the Rochester variables can be found in Nyman's work.[6] We remark that if one used $\Omega'$ instead of $\Omega_{\text{c.m.}}$, the cross section has the simpler form

$$\frac{d\sigma}{dE_\gamma d\Omega_\gamma d\Omega'}=\frac{m^4E_\gamma}{2(2\pi)^5(s^2-\omega^2)^{1/2}}\left(\frac{s-\omega-1}{s+\omega-1}\right)^{1/2}\times\tfrac{1}{4}\sum_{\text{spins}}|G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}(s,\omega,\Omega',(\pi-\theta_\gamma,\varphi_\gamma))|^2.\qquad(5.12)$$

The basic difference between the Harvard and Rochester geometries is that for the Harvard geometry, the angles of the final protons are fixed while the photon is fixed in the Rochester geometry. In terms of our variables, both $\bar{\theta}$, the difference between $\Omega$ and $\Omega'$, and $x$, the inelasticity, remain fairly constant for the Harvard geometry, while $\Omega$ only is fixed in the Rochester geometry. In the latter case, $x$ is determined by the photon energy and $\Omega'$ by the angles of the final protons. Since the range of $\Omega'$ is restricted by the counter size and efficiency, cross sections presented as integrals over $\Omega'$ depend on assumptions made about behavior of the cross section where it is not observed. In fact, Nyman[6] found that the soft-photon approximation

[24] P. Signell and D. Marker, Phys. Letters **26B**, 559 (1968).
[25] K. W. Rothe, P. F. M. Koehler, and E. H. Thorndike, Phys. Rev. **157**, 1247 (1966).
[26] P. F. M. Koehler, K. W. Rothe, and E. H. Thorndike, Phys. Rev. **168**, 1537 (1968).

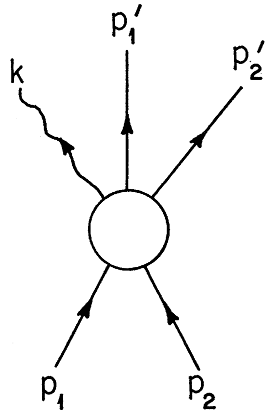


FIG. 9. Determination of singularities in $\omega$. The heavy line represents any physical channel open to $\gamma+p$.

predicts integrated cross sections about a factor of 2 larger than that reported,[25] which agrees with our calculations (see Fig. 8), and he speculates that this may be the cause of the difficulty.

## VI. NON-SOFT-PHOTON DYNAMICS

As long as the experimental data and theoretical calculations are consistent with the soft-photon behavior, study of $NN\gamma$ will not produce anything new. When discrepancies occur, as may be the case for the noncoplanar data considered in Sec. V, it is certainly of interest to learn what underlying dynamics may be the cause. In any case, it would be useful to be able to incorporate reasonable non-soft-photon dynamics into our formulation, which, as we have seen, already has the correct soft-photon behavior, and satisfies unitarity. In this section we propose a method for including such dynamics, basically a means for calculating $O(\omega)$ and higher terms of the function defined in Sec. IV. We make some general comments about the analytic structure of the $NN\gamma$ amplitude, present an explicit solution for the case of spinless nucleons, and finally discuss the problems imposed by spin, proposing an iterative method for solving the equations in the more general case.

### A. Analyticity

First let us consider the analytic properties of the $NN\gamma$ amplitude as a function of $\omega$. From its definition, $\omega=k\cdot(p_1+p_2)=k\cdot(p_1'+p_2')$. This means that dynamical singularities of $\omega$ in the physical region $s>1$, $|\omega|<s-1$ are determined by the possible physical channels which are open to the photon plus a single nucleon as represented by the heavy line in Fig. 10. First, the intermediate state may be a nucleon: $(p_1'+k)^2=m_N^2$. Since this is possible for $\omega=0$, there is a pole in this variable at zero,[27] which in fact the well-known infrared divergence. Secondly, we may have a nucleon plus any number of photons, all producing branch points at $\omega=0$. However, we ignore these since they are $O(e^3)$ and higher. Finally, we dispose of the possibility of a nucleon plus a pion, since the energy is insufficient. This analysis is the same, of course, for any of the nucleons.

We conclude that, for $s$ and $\omega$ physical, the PWA are analytic functions of $\omega$. Furthermore, analyticity in $\omega$ is limited to the physical region since the limiting value, $\omega=s-1$, corresponds to the final nucleons being at threshold.

Analyticity in $\omega$ is an important simplification distinguishing $NN\gamma$ from strong production processes, as $N\pi\to N\pi\pi$. The analysis[12,13] of this latter reaction is complicated by the existence of anomalous thresholds which are a result of the fact that both energy variables have physical cuts.

Considering $\omega$ as a real parameter, let us investigate the analytic properties of the PWA as functions of $s$ and, for the time being, ignore nucleon spin. They contain dynamical singularities which are the physical branch points at $s=1\pm\omega$ and left-hand singularities arising from crossed-channel poles and branch points in the same manner as for elastic PWA, except that there exist two types of crossed channels: the processes $N\gamma\to NN\bar{N}$ and $N\bar{N}\to N\bar{N}\gamma$, characterized by baryon number 1 and zero, respectively. (See Fig. 10.) Thus nearby singularities of the first type would be nucleon poles, and the second, meson poles. The soft-photon approximation discussed in Sec. IV can be understood in this light: To $O(1)$, using gauge invariance, one can show that the nucleon pole is the only singularity in the $\gamma+N$ channel. The residue of this pole is then given by static electromagnetic properties of the nucleon and the elastic scattering amplitude, evaluated in an unphysical region. The $N\bar{N}$ channel-pole singularities are given in terms of the $NN$-meson vertex and meson photoproduction, and, to $O(1)$, the meson photoproduction can be approximated by its Born terms. Thus these singularities are also known, and depend only on the singularities of the elastic amplitude and the static electromagnetic properties of the nucleons. The expression for the PWA, Eq. (4.30), shows that these two types of singularities actually factor: The functions $f_{2'2}{}^{JJ'm}(s,\omega)$ contain the $\gamma N$ channel singularities and the elastic PWA, $h_{2'2}{}^{J}(s\pm\omega)$ determine the $N\bar{N}$ channel singularities.

If the left-hand singularities do not overlap the physical branch cuts there exists a region along the real $s$ axis, $s<1-\omega$, where the PWA are real and may be continued to the lower-half $s$ plane by the relation $g(s,\omega)^*=g(s^*,\omega)$. Then unitarity becomes a relation for the discontinuity of the function.

Then one may write a dispersion relation for $g$:

$$g(s,\omega)=b(s,\omega)+\pi^{-1}\int_{1-\omega}^{\infty}\frac{ds'}{s'-s}\,\mathrm{Im}g(s',\omega),\quad(6.1)$$

[27] We remark that the physical regions for photon emission, for which $\omega\geq 0$, and photon absorption, $\omega\leq 0$, touch at $\omega=0$ allowing analytic continuation from one region to the other, and a power-series expansion about $\omega=0$.

where $b(s,\omega)$ represents left-hand singularities, and we have assumed that no subtractions are necessary. (We will ignore all subtractions in the following.) Then, from the unitarity relation, we write Im$g$ in terms of $g$ itself, obtaining a linear integral equation for $g$. We must also assume that neglecting the effects on Im$g$ due to other channels, e.g., $NN\pi$, is valid. As we have discussed, the first two terms in an expansion in $\omega$ of $b$, and therefore $g$, are known. As emphasized by Nyman,[6] the third term is of a different character entirely. An example of an interesting nearby contribution is the contribution of the $N^*$ to the pion photoproduction amplitude which, as we mentioned above, itself contributes to the residue of the pion pole in the $N\bar{N}\rightarrow N\bar{N}\gamma$ channel. Ueda's model[10] attempts to include this interaction; however, his model unitarizes only through a phenomenological pion form factor obtained from a different process, $NN\rightarrow NN\pi$, and does not have the correct soft-photon behavior.

Since the integral equation for $g$ obtained from Eq. (6.1) is linear in $g$, one may solve for the contribution to $g$ from any particular left-hand singularity independently of any other contribution and the soft-photon part. In the following, we consider how this may be done in the hypothetical case of spinless nucleons.

### B. Solution for Spinless Nucleons

The unitarity relation for spinless nucleons is

$$\mathrm{Im}g_\lambda{}^{JJ'm}(s,\omega)=\rho(s+\omega)g_\lambda{}^{JJ'm}(s,\omega)h^J(s+\omega)^*\theta(s+\omega)+\rho(s-\omega)h^{J'}(s-\omega)g_\lambda{}^{JJ'm}(s,\omega)^*\theta(s-\omega), \quad (6.2)$$

which can be written

$$\mathrm{Im}g_\lambda{}^{JJ'm}(s,\omega)=M^{JJ'}(s,\omega)^*g_\lambda{}^{JJ'm}(s,\omega), \quad (6.3)$$

where

$$M^{JJ'}(s,\omega)=\exp[i\delta^{JJ'}(s,\omega)]\sin\delta^{JJ'}(s,\omega),$$
$$\delta^{JJ'}(s,\omega)=\delta^J(s+\omega)\theta(s+\omega)+\delta^{J'}(s-\omega)\theta(s-\omega), \quad (6.4)$$

and $\delta^J$ and $\delta^{J'}$ are the elastic phase shifts. The dispersion relation, Eq. (6.1), is then

$$g_\lambda{}^{JJ'm}(s,\omega)=b_\lambda{}^{JJ'm}(s,\omega)+\pi^{-1}\int_R\frac{ds'}{s'-s}\times\exp[-i\delta^{JJ'}(s',\omega)]\sin\delta^{JJ'}(s',\omega)g_\lambda{}^{JJ'm}(s',\omega), \quad (6.5)$$

where the integral is over the physical cut, $1-\omega\leq s<\infty$. This is precisely the Omnès[15] equation, and we write the solution in the form[28]

$$g_\lambda{}^{JJ'm}(s,\omega)=[D^{JJ'}(s,\omega)]^{-1}(2\pi i)^{-1}\times\int_L\frac{ds'}{s'-s}D^{JJ'}(s',\omega)\,\mathrm{disc}b_\lambda{}^{JJ'm}(s',\omega), \quad (6.6)$$

where the integral is over the left-hand cuts, which are

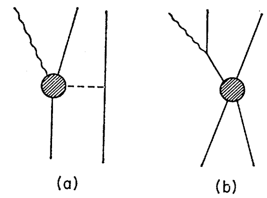

FIG. 10. Illustration of the two types of crossed-channel poles. The blob in (a) is the physical meson photoproduction amplitude, and the blob in (b) is the physical elastic scattering amplitude. Both are evaluated in unphysical regions.

in general complex. The function $D$ is essentially the $D$ of the $N/D$ method. It has no left-hand singularities, and right-hand cuts determined by

$$\mathrm{Im}[D^{JJ'}(s,\omega)]^{-1}=M^{JJ'}(s,\omega)^*[D^{JJ'}(s,\omega)]^{-1}. \quad (6.7)$$

This property, as can be easily checked, guarantees that $g$ is unitary. $D$ has an explicit solution:

$$D^{JJ'}(s,\omega)=\exp\left[-\pi^{-1}\int_R\frac{ds'}{s'-s}\delta^{JJ'}(s',\omega)\right]=\exp[-i\delta^{JJ'}(s,\omega)-\rho^{JJ'}(s,\omega)], \quad (6.8)$$

where we have defined

$$\rho^{JJ'}(s,\omega)=\pi^{-1}P\int_R\frac{ds'}{s'-s}\delta^{JJ'}(s',\omega). \quad (6.9)$$

Then

$$g_\lambda{}^{JJ'm}(s,\omega)=\exp[i\delta^{JJ'}(s,\omega)+\rho^{JJ'}(s,\omega)](2\pi i)^{-1}\times\int_L\frac{ds'}{s'-s}D^{JJ'}(s',\omega)\,\mathrm{disc}b_\lambda{}^{JJ'm}(s',\omega). \quad (6.10)$$

The spinless analog of Eq. (4.30) is

$$g_\lambda{}^{JJ'm}(s,\omega)=[e/\sqrt{2}\rho(s)][\rho(s+\omega)h^J(s+\omega)-\rho(s-\omega)h^{J'}(s-\omega)]f_\lambda{}^{JJ'm}(s,\omega). \quad (6.11)$$

Since both forms are unitary, they must have the same phase, and we find that

$$f_\lambda{}^{JJ'm}(s,\omega)=\frac{\sqrt{2}\rho(s)}{e}\frac{\exp[\rho^{JJ'}(s,\omega)]}{\sin[\delta^J(s+\omega)-\delta^{J'}(s-\omega)]}(2\pi i)^{-1}\times\int_L\frac{ds'}{s'-s}D^{JJ'}(s',\omega)\,\mathrm{disc}b_\lambda{}^{JJ'm}(s',\omega). \quad (6.12)$$

One could in principle take the first two orders in $\omega$ of $b$ and solve for the first two orders in $\omega$ of $f$. This should be the same as that derived using the Low[5] method, and remarkably, would not depend on the elastic amplitude.

### C. Extension to Case of Spin

The realistic case of spin involves some complications which we now discuss. First, in order to apply dispersion theory, one must define kinematic-singularity-free (KSF) partial-wave amplitudes. In the elastic case of four-particle amplitudes, one takes linear combinations of helicity amplitudes and divides out threshold and pseudothreshhold factors which can be determined in various ways. Cook and Lee[12] make a conjecture for the $N\pi\rightarrow N\pi\pi$ amplitude which they prove for a certain

[28] Using the dispersion relation for $D$, Eq. (6.22), partial fractionating, and interchanging orders of integration, Eq. (6.6) can be transformed into the form given by Omnès. See W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1969).

class of diagrams. Perhaps the same procedure will apply here. Let us denote KSF amplitudes, assuming that they can be defined, by $\bar{g}_\lambda{}^{jj'm}(s,\omega)$.

Now we determine the matrix generalizations of the spinless equations. Using single indices to describe $NN$ spin states, unitarity for $\bar{g}$ takes the form

$$\mathrm{Im}\bar{g}_{\lambda ij}{}^{JJ'm}(s,\omega)=\rho(s+\omega)\bar{g}_{\lambda ik}{}^{JJ'm}h_{kj}{}^{J}(s+\omega)^*\theta(s+\omega)$$
$$+\rho(s-\omega)h_{ik}{}^{J'}(s-\omega)\bar{g}_{\lambda kj}{}^{JJ'm}(s,\omega)^*\theta(s-\omega). \quad (6.13)$$

Defining the matrices

$$H_{il;jm}=\rho(s+\omega)h_{ml}{}^{J}(s+\omega)\delta_{ij},$$
$$H'_{il;jm}=\rho(s-\omega)h_{ij}{}^{J'}(s-\omega)\delta_{lm}, \quad (6.14)$$

and suppressing unnecessary indices and variables, we have

$$\mathrm{Im}\bar{g}_{il}=\sum_{jm}[H'_{il;jm}{}^*\bar{g}_{jm}+H_{il;jm}\bar{g}_{jm}{}^*]. \quad (6.15)$$

Let us use a matrix notation where $\bar{g}$ is a $16\times 1$ column vector and $H$ and $H'$ are $16\times 16$ matrices. Then, since Eq. (6.15) is real, we equate it with its complex conjugate to obtain

$$(H-H')\bar{g}^*=(H-H')^*\bar{g}. \quad (6.16)$$

If we let $R=H-H'$ and substitute $\bar{g}^*=R^{-1}R^*\bar{g}$ into Eq. (6.16), we find

$$\mathrm{Im}\bar{g}=(H'^*+HR^{-1}R^*)\bar{g}=(H'^*R+HR^{-1}R^*R)R^{-1}\bar{g}. \quad (6.17)$$

Now $H$ and $H^*$, and $H'$ and $H'^*$ commute using unitarity for the elastic PWA: $HH^*=H^*H=\mathrm{Im}H$. Also, $H$ and $H'$ commute, which follows from their definitions. Thus $R$ and $R^*$ commute. Since

$$\mathrm{Im}R=HH^*-H'H'^*=HR^*+H'^*R, \quad (6.18)$$

we have

$$\mathrm{Im}\bar{g}=(\mathrm{Im}R)R^{-1}\bar{g}=M^*\bar{g}. \quad (6.19)$$

Where the matrix $M$ defined in Eq. (6.19) has the property $\mathrm{Im}M=MM^*=M^*M$ and is the analog to the function $M$ defined by Eq. (6.4). Now the dispersion relation is a set of coupled integral equations:

$$\bar{g}(s)=b(s)+\pi^{-1}\int_R\frac{ds'}{s'-s}M^*(s')\bar{g}(s'). \quad (6.20)$$

The solution is identical in form to Eq. (6.6):

$$\bar{g}(s)=D^{-1}(s)(2\pi i)^{-1}\int_L\frac{ds'}{s'-s}D(s')\,\mathrm{disc}b(s'), \quad (6.21)$$

except that $D$ is now a $16\times 16$ matrix. Its properties are the same, with Eq. (6.7) being understood as a matrix equation. It does not seem to have an explicit solution, however. Using its dispersion relation,

$$D(s)=-\pi^{-1}\int_R\frac{ds'}{s'-s}D(s')M(s'), \quad (6.22)$$

one may construct an iterative series based on the solution for uncoupled amplitudes, Eq. (6.8).

Considering low-energy $NN$ scattering, if one uses a representation diagonal in spin and orbital angular momentum, the scattering matrix is almost diagonal so one could use Eq. (6.8) to obtain an approximation to $D$ ignoring the mixing between the coupled triplet amplitudes. Then Eq. (6.22) can be used to iterate, generating a series in the mixing parameters. This is a generalization of the method used by Lebellac, Renard, and Van[29] for solving the same type of equation which arises from treating photodisintegration of the deuteron.

## VII. CONCLUDING REMARKS

We have formulated the problem of $NN\gamma$ from an $S$-matrix point of view, including unitarity, crossing, and the partial-wave expansion. To do this, we defined the $NN\gamma$ amplitude so that spins of the outgoing nucleons are measured in their c.m. system, while the incoming nucleon spins are measured in the over-all c.m. system. This amplitude, we found, has a relatively simple soft-photon expansion, which we derived using the method of Low.[5] Furthermore, we were able to express it in a form unitary to all orders in $\omega$, which involves only the elastic partial-wave amplitudes evaluated at the energies of initial and final nucleons, and a set of functions $F^{jm}(s,\omega,\theta)$ which depend on the charges and magnetic moments of the scattering particles and are independent of the elastic amplitude.

We then presented some comparisons of our soft-photon amplitude with experiment and found an apparent discrepancy, which may indicate a breakdown of the soft-photon approximation.

The interesting dynamics of $NN\gamma$ are in $O(\omega)$ and higher terms. With this in mind we proposed a method for including these effects based on dispersion theory and unitarity, and assuming that it was possible to define kinematic-singularity-free amplitudes. This method, we commented but did not prove, should reproduce the soft-photon results when one includes as interactions only the known $O(1/\omega)$ and $O(1)$ terms.

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## APPENDIX: DETAILS OF SOFT-PHOTON APPROXIMATION

In order to express all of the terms in Eq. (4.26) in terms of elastic helicity amplitudes, we insert wherever

[29] M. LeBellac, F. M. Renard, and J. Tran Thanh Van, Nuovo Cimento **34**, 450 (1964).

necessary the quantity

$$1=\sum_{\lambda}[u(p,\lambda)\bar{u}(p,\lambda)-v(p,\lambda)\bar{v}(\partial,\lambda)],\quad \text{(A1)}$$

where the $v$'s are negative-energy spinors. Since all of the terms in Eq. (4.26) are independent of the representation, we now find it convenient to use the more usual $\gamma_0$-diagonal representation. Then, with $\omega=0$,

$$\begin{aligned}
u(p_1,\lambda_1)&=\begin{pmatrix}(s^{1/2}+1)^{1/2}\\ 2\lambda_1(s^{1/2}-1)^{1/2}\end{pmatrix}R(\Omega)\Phi_{\lambda_1},\\
u(p_2,\lambda_2)&=\begin{pmatrix}(s^{1/2}+1)^{1/2}\\ 2\lambda_2(s^{1/2}-1)^{1/2}\end{pmatrix}R(\Omega)\Phi_{-\lambda_2},\\
\bar{u}(p_1',\lambda_1')&=\Phi_{\lambda_1'}{}^{\dagger}R^{\dagger}(\Omega')\left((s^{1/2}+1)^{1/2},\,-2\lambda_1'(s^{1/2}-1)^{1/2}\right),\\
\bar{u}(p_2',\lambda_2')&=\Phi_{-\lambda_2'}{}^{\dagger}R^{\dagger}(\Omega')\left((s^{1/2}+1)^{1/2},\,-2\lambda_2'(s^{1/2}-1)^{1/2}\right).
\end{aligned}\quad \text{(A2)}$$

Our $v$'s are not quite conventional; they differ from the $u$'s by interchanging the $(s^{1/2}+1)^{1/2}$ and $(s^{1/2}-1)^{1/2}$ factors and from the usual convention by perhaps a sign which does not affect Eq. (A1). With these conventions,

$$\begin{aligned}
\gamma\cdot\epsilon_\lambda \boldsymbol{k}&=\frac{\lambda\omega}{\sqrt{2}s^{1/2}}\begin{pmatrix}1 & -\lambda\\ -\lambda & 1\end{pmatrix}(\sigma_1+i\lambda\sigma_2),\\
\gamma^0\gamma^3&=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sigma_3,
\end{aligned}\quad \text{(A3)}$$

and the matrix elements of the $X$'s defined in Eq. (4.27) are

$$\begin{aligned}
\bar{u}(p_1',\lambda_1')X_1'u(p_1,\lambda_1)&=\frac{e^{i\lambda\varphi'}}{\sqrt{2}}\frac{2\lambda_1'\lambda}{(s-1)^{1/2}}\left[\frac{\cos\theta'}{2\sin\theta'}+(1+\kappa_1)\frac{\sin\theta'}{\alpha(s)+\cos\theta'}\right],\\
\bar{u}(p_1',\lambda_1')X_1'u(p_1,-\lambda_1')&=\frac{e^{i\lambda\varphi'}}{\sqrt{2}}\left\{[2\lambda_1'\lambda(1+\kappa_1)-\tfrac{1}{2}\alpha(s)]\left[2\lambda_1'\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'}+\lambda\right]-\frac{\lambda_1'}{\alpha(s)}\frac{\sin^2\theta'}{\alpha(s)+\cos\theta'}\right\},\\
\bar{u}(p_1',\lambda_1')X_1'v(p_1',\lambda_1')&=-(e^{i\lambda\varphi'}/2\sqrt{2})(\sin\theta'/s^{1/2}),\\
\bar{u}(p_1',\lambda_1')X_1'v(p_1',-\lambda_1')&=(e^{i\lambda\varphi'}/2\sqrt{2})[2\lambda_1'\cos\theta'+\lambda],\\
\bar{u}(p_2',\lambda_2')X_2'u(p_2',\lambda_2')&=-(e^{i\lambda\varphi'}/\sqrt{2})[\lambda\lambda_2'\cos\theta'/(s-1)^{1/2}\sin\theta'],\\
\bar{u}(p_2',\lambda_2')X_2'u(p_2',-\lambda_2')&=\frac{e^{i\lambda\varphi'}}{\sqrt{2}}\left\{\tfrac{1}{2}\alpha(s)\left[2\lambda_2'\frac{1+\alpha(s)\cos\theta'}{\alpha(s)+\cos\theta'}-\lambda\right]-\frac{\lambda_2'}{\alpha(s)}\frac{\sin^2\theta'}{\alpha(s)+\cos\theta}\right\},\\
\bar{u}(p_2',\lambda_2')X_2'v(p_2',\lambda_2')&=(e^{i\lambda\varphi'}/2\sqrt{2})(\sin\theta'/s^{1/2}),\\
\bar{u}(p_2',\lambda_2')X_2'v(p_2',-\lambda_2')&=(e^{i\lambda\varphi'}/2\sqrt{2})[2\lambda_2'\cos\theta'-\lambda],\\
\bar{u}(p_1,\lambda_1)X_1u(p_1,\lambda_1)&=-\frac{e^{i\lambda\varphi}}{\sqrt{2}}\frac{2\lambda_1\lambda}{(s-1)^{1/2}}\left[\frac{\cos\theta}{2\sin\theta}+(1+\kappa_1)\frac{\sin\theta}{\alpha(s)+\cos\theta}\right],\\
\bar{u}(p_1,-\lambda_1)X_1u(p_1,\lambda_1)&=\frac{e^{i\lambda\varphi}}{\sqrt{2}}\left\{[-2\lambda_1\lambda(1+\kappa_1)-\tfrac{1}{2}\alpha(s)]\left[2\lambda_1\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}-\lambda\right]-\frac{\lambda_1}{\alpha(s)}\frac{\sin^2\theta}{\alpha(s)+\cos\theta}\right\},\\
\bar{v}(p_1,\lambda_1)X_1u(p_1,\lambda_1)&=-(e^{i\lambda\varphi}/\sqrt{2})(\sin\theta/s^{1/2}),\\
\bar{v}(p_1,-\lambda_1)X_1u(p_1,\lambda_1)&=(e^{i\lambda\varphi}/2\sqrt{2})[2\lambda_1\cos\theta-\lambda],\\
\bar{u}(p_2,\lambda_2)X_2u(p_2,\lambda_2)&=(e^{i\lambda\varphi}/\sqrt{2})[\lambda\lambda_2\cos\theta/(s-1)^{1/2}\sin\theta],\\
\bar{u}(p_2,-\lambda_2)X_2u(p_2,\lambda_2)&=\frac{e^{i\lambda\varphi}}{\sqrt{2}}\left\{\tfrac{1}{2}\alpha(s)\left[2\lambda_2\frac{1+\alpha(s)\cos\theta}{\alpha(s)+\cos\theta}+\lambda\right]-\frac{\lambda_2}{\alpha(s)}\frac{\sin^2\theta}{\alpha(s)+\cos\theta}\right\},\\
\bar{v}(p_2,\lambda_2)X_2u(p_2,\lambda_2)&=(e^{i\lambda\varphi}/2\sqrt{2})(\sin\theta/s^{1/2}),\\
\bar{v}(p_2,-\lambda_2)X_2u(p_2,\lambda_2)&=(e^{i\lambda\varphi}/2\sqrt{2})[2\lambda_2\cos\theta+\lambda].
\end{aligned}\quad \text{(A4)}$$

Now let us collect all the terms with negative-energy spinors. The sum of these contributions to $G_{\lambda\lambda_1'\lambda_2';\lambda_1\lambda_2}^{(1)}(s,\Omega',\Omega)$ is

$$(e^{i\lambda\varphi'}/2\sqrt{2})[(\sin\theta'/s^{1/2})(T^{(1')}-T^{(2')})-(2\lambda_1'\cos\theta'+\lambda)T_{-\lambda_1'}{}^{(1')}-(2\lambda_2'\cos\theta'-\lambda)T_{-\lambda_2'}{}^{(2')}]$$
$$+(e^{i\lambda\varphi}/2\sqrt{2})[(\sin\theta/s^{1/2})(T^{(1)}-T^{(2)})-(2\lambda_1\cos\theta-\lambda)T_{-\lambda_1}{}^{(1)}-(2\lambda_2\cos\theta+\lambda)T_{-\lambda_2}{}^{(2)}],\quad \text{(A5)}$$

where $T^{(i)}$ means $T$ evaluated with the $u(p_i,\lambda_i)$ replaced by $v(p_i,\lambda_i)$, and we suppressed helicity indices except for the ones that differ. For example,

$$T_{-\lambda_1'}{}^{(1')}=\bar{v}(p_1',-\lambda_1')\bar{u}(p_2',\lambda_2')\mathfrak{I}(s,\Omega',\Omega)u(p_1,\lambda_1)u(p_2,\lambda_2).$$

The $T^{(i)}$'s, after laborious calculation, work out to be

$$T_{++;++}{}^{(i)}=\tfrac{1}{2}[s(s-1)]^{1/2}\left[\frac{f_1}{s}+\frac{f_2}{s-1}+\frac{f_3+\bar{z}f_4}{s(s-1)}+\frac{2\bar{z}f_5}{s-1}\right],$$
$$T_{++;--}{}^{(i)}=\tfrac{1}{2}[s(s-1)]^{1/2}\left[-\frac{f_1}{s}+\frac{f_2}{s-1}-\frac{f_3+\bar{z}f_4}{s(s-1)}\right],\quad \text{(A6)}$$
$$T_{+-;+-}{}^{(i)}=[s/(s-1)]^{1/2}[f_4+f_3+f_5]D_{11}',$$
$$T_{+-;-+}{}^{(i)}=[s/(s-1)]^{1/2}[f_4-f_3+f_5]D_{-11}'$$

for all $i$, where $D_{\mu\mu'}'=\sum_m d_{m\mu}'(\theta)d_{m\mu'}'(\theta')\exp[im(\varphi'-\varphi)]$ are rotation matrices, $\bar{z}=\cos\bar{\theta}$, and $f_i=f_i(S,\bar{Z})$, $i=1,\ldots,5$ are the GGMW[20] kinematic-singularity-free helicity amplitudes. Also,

$$T_{++;+-}{}^{(1)}/D_{10}{}^1=T_{++;-+}{}^{(2)}/D_{-10}{}^1=T_{+-;++}{}^{(1')}/D_{01}{}^1=T_{+-;--}{}^{(2')}/D_{01}{}^1=[2f_4+(s+1)f_5]/[2(s-1)]^{1/2},$$
$$T_{+-;++}{}^{(1)}/D_{01}{}^1=T_{+-;--}{}^{(2)}/D_{01}{}^1=T_{++;+-}{}^{(1')}/D_{10}{}^1=T_{++;-+}{}^{(2')}/D_{-10}{}^1=[f_3-f_4-sf_5]/[2(s-1)]^{1/2},\quad \text{(A7)}$$
$$T_{++;-+}{}^{(1)}/D_{-10}{}^1=T_{++;+-}{}^{(2)}/D_{10}{}^1=T_{+-;--}{}^{(1')}/D_{01}{}^1=T_{+-;++}{}^{(2')}/D_{01}{}^1=-f_5[(s-1)/2]^{1/2},$$
$$T_{+-;--}{}^{(1)}/D_{01}{}^1=T_{+-;++}{}^{(2)}/D_{01}{}^1=T_{++;-+}{}^{(1')}/D_{-10}{}'=T_{++;+-}{}^{(2')}/D_{10}{}^1=[-f_3-f_4-sf_5]/[2(s-1)]^{1/2}.$$

By evaluating Eq. (A5) using Eqs. (A6) and (A7) and explicit formulas for the $D$'s, we find that it is in all cases identical to

$$-(2\sqrt{2})^{-1}[(s-1)/s]^{1/2}\{e^{i\lambda\varphi'}[(2\lambda_1'\cos\theta'+\lambda)T_{-\lambda_1'}+(2\lambda_2'\cos\theta'-\lambda)T_{-\lambda_2'}]$$
$$+e^{i\lambda\varphi}[(2\lambda_1\cos\theta-\lambda)T_{-\lambda_1}+(2\lambda_2\cos\theta+\lambda)T_{-\lambda_2}]\}.\quad \text{(A8)}$$

Combining this with the contributions already calculated, we obtain Eq. (4.29).