rather the integral of this quantity multiplied by some weighting function. In such a case the result of this section would not hold.

Thus, in summary of what we have seen, we may say that while the electromagnetic effects may give rise to measurable differences between the decay distributions for $K_L{}^0 \to \pi^- l^+ \nu_l$ and $K_L{}^0 \to \pi^+ l^- \bar{\nu}_l$, their contribution to the difference between the rates is not greater than $\alpha\Gamma(K^0 \to \pi\pi l\nu)$. This latter result removes a possible nagging doubt about the interpretation of $K_L{}^0$ charge asymmetry experiments.

Note added in proof. After submitting this work for publication I became aware of a very similar contribution on this subject by L. B. Okun, Soviet Phys.—JETP Letters **6**, 272 (1967).

ACKNOWLEDGMENTS

My sincere thanks are due Professor D. Speiser and Professor F. Cerulus for their hospitality at the Centre de Physique Nucléaire of the University of Louvain where this work was begun, and to Professor Abdus Salam, Professor P. Budini, and the International Atomic Energy Agency for similar hospitality at the International Centre for Theoretical Physics, Trieste, where the work was completed. I wish also to acknowledge the very considerable assistance I received from Professor J. S. Bell as well as extremely helpful conversations with Dr. L. Picasso and Dr. H. Sugawara. Finally, I am grateful to the School of Theoretical Physics in the Dublin Institute for Advanced Studies for the award of a travel grant.

PHYSICAL REVIEW D VOLUME 1, NUMBER 1 1 JANUARY 1970

Finite-Energy Sum Rules and Their Application to πN Scattering at Fixed u*

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(Received 19 August 1969)

The finite-energy sum rule, and a class of sum rules which can be used to probe the existence of fixed poles, are obtained for amplitudes whose left- and right-hand cuts are not related by crossing symmetry. The finite-energy sum rule is evaluated for each of four independent πN amplitudes with u fixed at $(1236\text{ MeV})^2$, both sides of the resultant four sum rules being obtained from the properties of the low-energy πN resonances. Results are presented for three choices of end point: $(1356\text{ MeV})^2$, $(1808\text{ MeV})^2$, and $(2313\text{ MeV})^2$. For the intermediate end point, all four sum rules work. For the highest one, however, they all fail. These results, while pointing to a failure of the resonance dominance approximation above 1800 MeV, give us a new confirmation of Regge high-energy behavior on the basis of low-energy data alone. In particular, they verify in some detail the relation predicted by Reggeism between the high-energy, fixed-u behavior of the amplitudes and the low-energy u-channel resonances. They also show that for $u=(1236\text{ MeV})^2$, all the πN amplitudes have Regge behavior on the average (duality) above 1800 MeV. The finite-energy sum rules are shown to be violated in a fictitious universe where the lowest particle on each of the leading πN Regge trajectories is accompanied by a degenerate partner of opposite parity.

I. INTRODUCTION

Although they follow very simply from assumptions of analyticity and Regge behavior, finite-energy sum rules (FESR)[1] provide a powerful tool for gaining new information about Regge trajectories and their residues, for obtaining theoretical insight into the nature of physical scattering amplitudes, and for constructing bootstrap models of remarkable computational simplicity.[2] As a test of the assumptions on which the FESR and their practical applications are based, we have investigated whether these sum rules are satisfied in πN scattering with the cross momentum transfer u fixed at $M^{*2} \equiv (1236\text{ MeV})^2$, the mass squared of the 3-3 resonance. Our results provide a new verification of Regge high-energy behavior from low-energy data. They also support the idea that the πN scattering amplitudes, as functions of energy, have Regge behavior on the average even in the intermediate energy region (around 2 BeV) where significant resonance structure is still present. However, the popular resonance dominance approximation appears to fail above 1800 MeV in the particular process we studied, which suggests that this approximation should be used with caution.

In Sec. II we derive the sum rules we have used. These are independent of any fixed poles that may exist at wrong-signature nonsense points in the J plane. We

* Supported in part by the Atomic Energy Commission, under Contract No. AT(30-1)-3668B.

[1] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967); Phys. Rev. **166**, 1768 (1968); A. Logunov, L. Soloviev, and A. Tavkhelidze, Phys. Letters **24B**, 181 (1967); L. Balázs and J. Cornwall, Phys. Rev. **160**, 1313 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967).

[2] For examples of the various applications of FESR, see Ref. 1 and also F. Gilman, H. Harari, and Y. Zarmi, Phys. Rev. Letters **21**, 323 (1968); S. Mandelstam, Phys. Rev. **166**, 1539 (1968); D. Gross, Phys. Rev. Letters **19**, 1303 (1967); C. Schmid, *ibid.* **20**, 628 (1968); V. Barger and R. Phillips, *ibid.* **21**, 865 (1968).

also obtain sum rules which are explicitly sensitive to such fixed poles, and could be used to learn something about them. All of our sum rules apply to amplitudes, such as those for πN scattering at fixed u, whose left- and right-hand cuts are not related by crossing symmetry.

Section III describes our application of FESR to πN scattering in detail. In Sec. IV, we present the results and attempt to interpret them.[3] We also report the result of applying the FESR to a hypothetical universe which differs from the real one by containing a few additional, fictitious particles.

In an Appendix, there is shown to be an interesting theoretical difference between Chew's reciprocal bootstrap of the N and the N_{33}^* [4] and any Reggeized version of this bootstrap, such as might be based on FESR for πN scattering. The difference is that in Chew's bootstrap, direct forces (t-channel singularities) play a very minor role, whereas if one were to bootstrap entire Regge trajectories in the absence of direct forces, these trajectories would necessarily be accompanied by others along which lie ghosts.

II. FINITE-ENERGY AND FIXED-POLE SUM RULES

We begin with the trivial derivation of the FESR which is independent of any fixed poles. Suppose an invariant amplitude $A(s,u)$ [5] for some process has the analyticity properties implied by the Mandelstam representation and the fixed-u asymptotic behavior implied by

$$A(s,u)=R(s,u)+G(s,u)\,, \tag{2.1}$$

where

$$R(s,u)\equiv\sum_i \gamma_i(u)\frac{s^{\alpha_i(u)}-(-s)^{\alpha_i(u)}}{\sin\pi\alpha_i(u)} +\sum_j -\gamma_j(u)\frac{s^{\alpha_j(u)}+(-s)^{\alpha_j(u)}}{\sin\pi\alpha_j(u)} \tag{2.2}$$

and

$$sG(s,u)\to 0 \quad \text{as} \quad |s|\to\infty\,. \tag{2.3}$$

Here i runs over odd-signature terms, j over even-signature terms. At fixed u, A will typically have cuts along much of the real s axis [cf. (2.8) below]; to be general we suppose that A, like R, has a cut along the entire real axis. In view of (2.3), $A-R$ will then obey the superconvergence relation

$$\int_{-\infty}^{\infty}\text{Im}[A(s,u)-R(s,u)]ds=0\,. \tag{2.4}$$

To arrive at a practical sum rule, one must truncate this convergent integration at some finite points, say, N_r and $-N_l$. This leads to the approximate relation

$$\int_{-N_l}^{N_r}\text{Im}[A(s,u)-R(s,u)]ds\cong 0\,, \tag{2.5}$$

which will be well satisfied if $A(s,u)\cong R(s,u)$ for s beyond N_r or $-N_l$. Using (2.2), one may integrate ImR explicitly, obtaining the finite-energy sum rule

$$\int_{-N_l}^{N_r}\text{Im}A(s,u)ds\cong\sum_i \gamma_i(u)\frac{N_r^{\alpha_i(u)+1}+N_l^{\alpha_i(u)+1}}{\alpha_i(u)+1} +\sum_j \gamma_j(u)\frac{N_r^{\alpha_j(u)+1}-N_l^{\alpha_j(u)+1}}{\alpha_j(u)+1}\,. \tag{2.6}$$

Note from (2.5) that the sum rule may still hold even if N_r is at some point N_{r_1} in the low-energy region where A still has resonance wiggles and is not yet well represented by R. This will happen if A has Regge behavior on the average beyond N_{r_1}, in the sense that

$$\int_{N_{r_1}}^{N_{r_2}}\text{Im}A(s,u)ds\cong\int_{N_{r_1}}^{N_{r_2}}\text{Im}R(s,u)ds\,, \tag{2.7}$$

where N_{r_2} is a high energy beyond which $A(s,u)\cong R(s,u)$ does hold locally.

The FESR (2.6) makes no reference to the fixed poles which may occur at wrong-signature nonsense points in the u-channel J plane.[6] Moreover, such poles cannot lead to errors in (2.6), because they do not affect the asymptopic behavior of the physical scattering amplitude,[6] and that behavior is all that this sum rule depends on.

Sum rules for higher moments of the amplitude may be obtained exactly as the zeroth-moment sum rule was.[7]

To obtain sum rules which explicitly contain a term related to the leading wrong-signature nonsense fixed pole, one has to work with some quantity whose asymptotic behavior is affected by this pole. A suitable quantity is the dispersion integral over only the t-channel, or only the s-channel, absorptive part. Imagine that we are dealing with πN scattering, but with spin neglected for simplicity. Suppose that for some range of u above u-channel threshold, the amplitude $A(s,u)$ whose asymptotic behavior is given by (2.1)–(2.3) obeys

[3] A brief statement of our procedure and main results was presented in B. Kayser, Phys. Rev. Letters **21**, 1292 (1968).

[4] G. Chew, Phys. Rev. Letters **9**, 233 (1962).

[5] Here s is the usual Mandelstam energy variable, and u may be thought of as either the momentum transfer or the cross momentum transfer.

[6] S. Mandelstam and L. Wang, Phys. Rev. **160**, 1490 (1967).

[7] The zeroth-moment sum rule for the special case where A is odd under crossing with "u" fixed (Dolen, Horn, and Schmid, Ref. 1) follows immediately from (2.6) when "s" is replaced by the variable in which A is odd. For an amplitude which is crossing-even, there is no nontrivial zeroth-moment sum rule which is independent of fixed poles.

an N-subtracted dispersion relation:

$$A(s,u)=\sum_{n=0}^{N-1}c_n(u)s^n+\frac{1}{\pi}\int_{(2\mu_\pi)^2}^{\infty}\frac{dt'}{t'-t}\rho_t(t',u)\left(\frac{t-\bar{t}}{t'-\bar{t}}\right)^N$$
$$+\frac{1}{\pi}\int_{(M_N+\mu_\pi)^2}^{\infty}\frac{ds'}{s'-s}\rho_s(s',u)\left(\frac{s-\bar{s}}{s'-\bar{s}}\right)^N. \quad (2.8)$$

Denoting the t and s integrals in this relation by $r_t(s,u)$ and $r_s(s,u)$, respectively, we see from (2.1)–(2.3) that

$$r_s(s,u)=-\sum_i\gamma_i\frac{(-s)^{\alpha_i}}{\sin\pi\alpha_i}-\sum_j\gamma_j\frac{(-s)^{\alpha_j}}{\sin\pi\alpha_j}+Q_s(s,u), \quad (2.9)$$

$$r_t(s,u)=\sum_i\gamma_i\frac{s^{\alpha_i}}{\sin\pi\alpha_i}-\sum_i\gamma_j\frac{s^{\alpha_j}}{\sin\pi\alpha_j}+Q_t(s,u), \quad (2.10)$$

where $Q_{s(t)}$ has a right (left) s cut, and

$$s\,\mathrm{Im}Q_{s(t)}\xrightarrow[s\to\infty(-\infty)]{}0.$$

Then Q_s, for example, can be written as

$$Q_s(s,u)=P_Q(s,u)+\frac{1}{\pi}\int_0^\infty ds'\frac{\mathrm{Im}Q_s(s',u)}{s'-s}, \quad (2.11)$$

with P_Q a polynomial in s. Thus, setting[8]

$$\int_0^\infty ds'\,\mathrm{Im}Q_s(s',u)\equiv f_s(u),$$

we have

$$Q_s(s,u)\xrightarrow[|s|\to\infty]{}P_Q(s,u)-\frac{1}{\pi}\frac{f_s(u)}{s}$$
$$+\text{terms which vanish faster than } 1/s. \quad (2.12)$$

Integration of the function

$$r_s(s,u)+\sum_i\gamma_i\frac{(-s)^{\alpha_i}}{\sin\pi\alpha_i}+\sum_j\gamma_j\frac{(-s)^{\alpha_j}}{\sin\pi\alpha_j}-P_Q(s,u)+\frac{1}{\pi}\frac{f_s(u)}{s}$$

along a contour surrounding the positive real s axis and the pole at $s=0$ then leads immediately to

$$0=\int_0^\infty[\rho_s(s,u)-\sum_i\gamma_is^{\alpha_i}-\sum_j\gamma_js^{\alpha_j}]ds-f_s(u). \quad (2.13)$$

Truncation of the convergent integral in (2.13) at N_r yields the sum rule

$$\int_{(M_N+\mu_\pi^2)}^{N_r}\rho_s(s,u)ds\cong\sum_i\gamma_i(u)\frac{N_r^{\alpha_i(u)+1}}{\alpha_i(u)+1}$$
$$+\sum_j\gamma_j(u)\frac{N_r^{\alpha_j(u)+1}}{\alpha_j(u)+1}+f_s(u). \quad (2.14)$$

(Here the integral on the left-hand side is understood to include any Born contribution.)

The analogous treatment of r_t and Q_t gives the sum rule $[\Sigma\equiv2(M_N^2+\mu_\pi^2)]$

$$-\int_{-N_l}^{\Sigma-u-4\mu_\pi^2}\rho_t(\Sigma-u-s,\,u)ds\cong\sum_i\gamma_i(u)\frac{N_l^{\alpha_i(u)+1}}{\alpha_i(u)+1}$$
$$-\sum_j\gamma_j(u)\frac{N_l^{\alpha_j(u)+1}}{\alpha_j(u)+1}+f_t(u), \quad (2.15)$$

where

$$f_t(u)\equiv\int_{-\infty}^{\Sigma-n-4\mu_\pi^2}ds'\,\mathrm{Im}Q_t(s',u).$$

The coefficients f_s and f_t of the $1/s$ terms in the asymptotic forms of r_s and r_t are simply related to the residue of the leading fixed pole. To establish the connection, one requires a Sommerfeld-Watson representation for r_s or r_t. Since u is above its threshold, one can make a u-channel partial-wave expansion of A and of $\bar{A}\equiv r_t-r_s$. Obviously, for $l\geq N$, the partial-wave projections $f_l(u)$ of A and $\bar{f}_l(u)$ of $\bar{A}$ will differ only by the sign of the contribution from r_s. Thus the usual partial-wave amplitudes of definite signature, $f^\pm(l,u)$, and the corresponding analytic continuations from even-l and odd-l ($l\geq N$) values of $\bar{f}_l(u)$, $\bar{f}^\pm(l,u)$, are related by

$$\bar{f}^\pm(l,u)=f^\mp(l,u).$$

Then

$$r_t(s,u)=\tfrac{1}{2}(A+\bar{A})-\sum_{n=0}^{N-1}\tfrac{1}{2}c_ns^n$$
$$=\sum_{l=0}^{N-1}(2l+1)\tfrac{1}{2}[f_l(u)+\bar{f}_l(u)]P_l(z_u)$$
$$+\sum_{l=N}^{\infty}(2l+1)\tfrac{1}{2}[f^+(l,u)+f^-(l,u)]P_l(z_u)-\sum_{n=0}^{N-1}\tfrac{1}{2}c_ns^n$$
$$=\sum_{l=N}^{\infty}(2l+1)\tfrac{1}{2}[f^+(l,u)+f^-(l,u)]$$
$$\times P_l(z_u)+P(s,u), \quad (2.16)$$

where z_u is the cosine of the u-channel scattering angle and $P(s,u)$ is a polynomial in s. All factors in the Legendre series of (2.16) are analytic in l, so that the Sommerfeld-Watson transformation may be applied. If $f^+(l,u)$ has a fixed pole of residue $\lambda(u)(q_u^2/s_0)^{-1}$ at $l=-1$, the leading wrong-signature nonsense point,[9] one easily finds that $r_t(s,u)$ will have a corresponding term $\lambda(u)(s_0/s)$ in its asymptotic behavior. It follows that

$$f_t(u)=-\pi s_0\lambda(u). \quad (2.17)$$

For $f_s(u)$, we simply note that since A has no $1/s$ term, $f_s(u)=-f_t(u)$.

[8] It is assumed that $\mathrm{Im}Q_s$ is dominated at large s by a Regge pole with $\alpha<-1$, so that $s^{1+\epsilon}\,\mathrm{Im}Q_s\to0$.

[9] We assume f^- cannot have a pole at this point, which would be right-signature for it.

From analyticity, it is apparent that the sum rules (2.14) and (2.15) are valid both for u above and below u-channel threshold. When A is superconvergent and there is no fixed pole, these fixed-pole sum rules (FPSR) reduce to the Schwarz superconvergence relations.[10] Given what one knows and what one wants to find out, one may take linear combinations of (2.14) and (2.15) to obtain the most suitable FPSR. Note that if one adds (2.14) and (2.15), the fixed-pole terms f_s and f_t cancel, and one recovers the FESR (2.6), with ImA now expressed in terms of the absorptive parts.

III. APPLICATION TO πN SCATTERING AT $u=(1236\text{ MeV})^2$

The sum rule (2.6) (with α replaced by $\alpha-\frac{1}{2}$, as appropriate for πN scattering) has been evaluated for both of the πN invariant amplitudes[11] $A(s,u)$ and $B(s,u)$ in each of the isospin states $I_u=\frac{3}{2}$ and $I_t=1$. Each of these four evaluations was made at $u=M^{*2}$. The choice of this value of u, and of the isospin states, is governed by the desire to have sum rules which provide a meaningful test of their underlying assumptions, and which, therefore, should involve a minimum of poorly known quantities.

Choosing u in the physical region of the u channel makes it possible to obtain the Regge parameters required for the right-hand side of (2.6) directly from the πN resonance spectrum.[12] It is then unnecessary to make any use of the Regge fits, whose details are somewhat uncertain, to high-energy backward data. In descending order, the leading πN trajectories (along with a familiar particle on each) are Δ_δ (3-3 resonance), N_α (nucleon), and $N_\gamma[\frac{1}{2}(\frac{3}{2}^-)1518]$.[13] With the specific choice $u=M^{*2}$, the Δ_δ contribution to the right-hand side of (2.6) depends only on the properties of the 3-3 resonance and on the slope of the trajectory at $u=M^{*2}$. Thus the $I_u=\frac{3}{2}$ sum rules have an accurately known right-hand side. However, the t-channel contributions to the left-hand sides of these sum rules [cf. (2.8)] involve both $I_t=0$ and $I_t=1$. Now the only t-channel effect which we know well enough to be able to include is the ρ-meson pole. To avoid the unknown contributions of higher states, particularly the f^0 and g, we choose $-N_l=-0.8$ BeV^2. At $u=M^{*2}$, this corresponds to a t-channel cutoff midway between the ρ and the f^0.[14] With this N_l, the omission of a term from the low-energy ($I_t=0$) $\pi\pi$ s-wave resonance remains as an error in the sum rule for $A^{I_u=3/2}$. (The $\pi\pi$ s wave does not couple to the spin-flip amplitude B.)

In computing the contribution of the ρ, we use the narrow-resonance approximation, and take the $\rho\pi\pi$ coupling to correspond to a ρ width of 130 MeV.[15] The γ_μ-type ρNN coupling is inferred from $g_{\rho\pi\pi}$, assuming universal coupling of ρ to the isospin current.[16] The $\sigma_{\mu\nu}q_\nu$ ρNN coupling is then inferred from the γ_μ coupling, assuming ρ dominance of the electric and magnetic isovector nucleon form factors. (Discrepancies between various determinations of the ρNN couplings and the uncertainty over the ρ width suggest that our estimates of the ρ contributions could be off by $\sim 50\%$. It will later be clear, however, that 50% changes in these terms would have no qualitative effect on our results.)

In contrast to the $I_u=\frac{3}{2}$ sum rules, those for $I_t=1$ are free of unknown $I_t=0$ contributions to the left-hand side. These sum rules, however, involve the N_α and N_γ trajectories, in addition to the Δ_δ, on the right-hand side. Since $u=(1236\text{ MeV})^2$ is not the position of a physical particle for either N_α or N_γ, it is necessary to parametrize these trajectories and their residues in some detail to obtain the terms appearing in the FESR. We have assumed that the trajectories are linear functions of u[17]:

$$\alpha_{N_\alpha}=\tfrac{1}{2}+1.01(u-M^2)\,, \qquad (3.1\text{a})$$

$$\alpha_{N_\gamma}=\tfrac{3}{2}+0.80(u-M_1^2)\,. \qquad (3.1\text{b})$$

Here u is in BeV^2, $M=0.938$ BeV is the nucleon mass, and $M_1=1.525$ BeV is the mass of the $\frac{1}{2}(\frac{3}{2}^-)1518$. Our α_{N_α} is taken from Barger and Cline.[17] For α_{N_γ}, which shows some curvature, we use a linear form which fits the two lowest N_γ resonances, rather than the Barger-Cline form, which does not fit well below the second resonance [we will be interested in α_{N_γ} at $u=(1236\text{ MeV})^2$].

Our residues β are defined in terms of the kinematical-singularity-free u-channel partial-wave amplitudes h:

$$h_{J\mp1/2}{}^{\pm(I_u)}(J,W_u)\equiv\frac{f_{J\mp1/2}{}^{\pm(I_u)}(J,W_u)}{q_u{}^{2(J-1/2)}(E_u\pm M)/W_u}\,. \qquad (3.2)$$

Here J is the total angular momentum, $J\mp\frac{1}{2}$ is the orbital angular momentum, the superscript $\pm$ is the signature, I_u is the isospin, $W_u=\sqrt{u}$ is the c.m. energy, E_u is the nucleon energy, and q_u is the c.m. momentum. f is $(\sin\delta)e^{i\delta}/q_u$. Near their respective Regge poles, we write

$$h_{J+1/2}{}^{+(1/2)}(J,W_u)=\frac{\beta_{N\alpha}(W_u)/s_0{}^J}{J-\alpha_{N\alpha}(u)}\,, \qquad (3.3\text{a})$$

$$h_{J+1/2}{}^{-(1/2)}(J,W_u)=\frac{\beta_{N_\gamma}(W_u)/s_1{}^J}{J-\alpha_{N_\gamma}(u)}\,, \qquad (3.3\text{b})$$

[10] J. Schwarz, Phys. Rev. **159**, 1269 (1967).
[11] The notation is that of G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).
[12] For FESR corresponding to the alternative choice of $u=0$, see C. Chiu and M. DerSarkissian, Nuovo Cimento **55A**, 396 (1968); and V. Barger, C. Michael, and R. Phillips, Phys. Rev. **185**, 1852 (1969).
[13] The notation is $I(J^P)M$, where M is mass in MeV.
[14] For simplicity we use this value of N_l for all four of our sum rules, whose success then depends on the invariant amplitudes having Regge behavior, on the average, to the left of $s=-0.8$ BeV^2.

[15] N. Barash-Schmidt *et al.*, Rev. Mod. Phys. **41**, 109 (1969).
[16] J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).
[17] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).
[18] V. Singh, Phys. Rev. **129**, 1889 (1963), which, however, contains some misprints.

with s_0 and s_1 constants. The Regge-pole contributions to the large-s behavior of the invariant amplitudes are then computed via the Sommerfeld-Watson transformation for the usual amplitudes $f_{1,2}(W_u,s)$.[11,18] For any isospin, we have

$$f_{1,2}(W_u,s)=\frac{1}{4i}\oint_C \frac{dJ}{\cos\pi J}$$
$$\times\{f_{J\mp1/2}^{+}(J,W_u)[P_{J+1/2}'(-z_u)+P_{J+1/2}'(z_u)]$$
$$+f_{J\pm1/2}^{+}(J,W_u)[P_{J-1/2}'(-z_u)-P_{J-1/2}'(z_u)]$$
$$+f_{J\mp1/2}^{-}(J,W_u)[P_{J+1/2}'(-z_u)-P_{J+1/2}'(z_u)]$$
$$+f_{J\pm1/2}^{-}(J,W_u)[P_{J-1/2}'(-z_u)+P_{J-1/2}'(z_u)]\}\,. \quad (3.4)$$

The contour, before distortion, is around the positive real axis in the sense shown, and the upper (lower) signs on the right are for f_1 (f_2).

For large s, we have

$$P_{\alpha+1/2}'(-z_u)\to\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\frac{2}{\sqrt{\pi}}\left(\frac{s}{q_u^2}\right)^{\alpha-1/2}. \quad (3.5)$$

Thus, to avoid poles in the contribution from the trajectory α when $\alpha=-1, -2, \cdots$, we take the quantity $\beta\Gamma(\alpha+1)$ to be a smooth function of W_u. For β_{N_α} we have used two alternative parametrizations:

$$\beta_{N_\alpha}(W_u)\Gamma(\alpha+1)=\gamma_0(1+W_u/M)\,, \quad (3.6a)$$

$$\beta_{N_\alpha}(W_u)\Gamma(\alpha+1)=\gamma_0\left(1+\frac{W_u}{M}\right)\left(\frac{W_u+W_0}{M+W_0}\right)^2, \quad (3.6b)$$

in which γ_0 and W_0 are constants. For (3.6a), γ_0 and s_0 are fitted to $g_{\pi NN}^2/4\pi$ and the width of the first nucleon recurrence at 1688 MeV. The form $(1+W_u/M)$ guarantees that β_{N_α} will have the correct sign for generating resonances of positive width both for $W_u>M+\mu_\pi$ and for $W_u<-(M+\mu_\pi)$.[19] Also, the nonexistent $\frac{1}{2}(\frac{1}{2}^-)938$ expected from the trajectory (3.1a)[20] is extinguished by the property $\beta_{N_\alpha}(-M)=0$.

If one believes that the $\frac{1}{2}(\frac{5}{2}^-)1680$ belongs to the N_α trajectory as the parity-doublet partner of the $\frac{1}{2}(\frac{5}{2}^+)$ 1688,[21] then $\beta_{N_\alpha}(-1680 \text{ MeV})$ ought to correctly predict the $\frac{1}{2}(\frac{5}{2}^-)1680$ width. The form (3.6a) fails to do so by a factor of 4.5. For $u=(1236 \text{ MeV})^2$, the right-hand side of the FESR receives contributions both from $W_u=1236$ MeV and $W_u=-1236$ MeV [cf. (3.4) and the MacDowell rule]. With $\beta_{N_\alpha}(-1680 \text{ MeV})$ apparently very poorly approximated, one suspects that $\beta_{N_\alpha}(-1236 \text{ MeV})$ probably is also, and one fears that this may have serious consequences for the sum rules. To settle this question we have evaluated the sum rules using the alternative parametrization (3.6b), which contains the additional parameter W_0 and can fit simultaneously the widths of the $\frac{1}{2}(\frac{5}{2}^+)1688$ and $\frac{1}{2}(\frac{5}{2}^-)1680$, the πNN coupling constant, and the zero at -938 MeV which extinguishes the $\frac{1}{2}(\frac{1}{2}^-)$. Like (3.6a), (3.6b) has the correct sign both for $W_u>M+\mu_\pi$ and $W_u<-(M+\mu_\pi)$. The harmless double zero of (3.6b) turns out to be at $W_u=-W_0=-606$ MeV. Had we introduced an extra parameter via a linear correction factor of the form (W_u+W_0'), rather than the quadratic correction factor $(W_u+W_0)^2$ of (3.6b), β would have changed sign at $W_u=-W_0'=-2640$ MeV, thus violating unitarity to the left of this point.[19]

The scale factor s_0 is found to be 1.17 BeV^2 when (3.6b) is employed, and 0.79 BeV^2 when (3.6a) is employed.

For β_{N_γ} we use the linear form analogous to (3.6a):

$$\beta_{N_\gamma}(W_u)\Gamma(\alpha+1)=\gamma_1(1+W_u/M_1)\,. \quad (3.7)$$

The parameters γ_1 and s_1 are fitted to the widths of the $\frac{1}{2}(\frac{3}{2}^-)1518$ and the $\frac{1}{2}(\frac{7}{2}^-)2190$. The zero at $W_u=-M_1$ extinguishes the nonexistent $\frac{1}{2}(\frac{3}{2}^+)1518$. If one identifies the $\frac{1}{2}(\frac{7}{2}^+)1983$ reported by Donnachie *et al.*[22] as the parity-doublet partner of the $\frac{1}{2}(\frac{7}{2}^-)2190$, one finds that (3.7) does indeed correctly fit the width of the $\frac{7}{2}^+$, so there would be no point in considering a more complicated β_{N_γ} analogous to (3.6b).[23]

For the part of the left-hand side of each sum rule which corresponds to s-channel processes, we use a narrow-resonance approximation, being unable to use πN scattering data directly, since $u=M^{*2}$ is outside the s-channel physical region. For given end point N_r, we include all known πN resonances up to that point, including the new resonances found by Donnachie, Kirsopp, and Lovelace.[22,24] For given isospin, the contributions to A and B of a resonance of mass M_r, orbital angular momentum l, and total angular momentum J, are

$$\text{Im}\frac{A}{4\pi}=(-1)^{J-\frac{1}{2}-l}\pi R\delta(s-M_r^2)\times\left(\frac{M_r+M}{E_r+M}P_{l\pm1}'(z_s)+\frac{M_r-M}{E_r-M}P_l'(z_s)\right), \quad (3.8a)$$

$$\text{Im}\frac{B}{4\pi}=(-1)^{J-\frac{1}{2}-l}\pi R\delta(s-M_r^2)\left(\frac{P_{l\pm1}'(z_s)}{E_r+M}-\frac{P_l'(z_s)}{E_r-M}\right). \quad (3.8b)$$

Here E_r is the nucleon energy at resonance. The cosine of the s-channel scattering angle, z_s, is to be evaluated at

[19] B. Desai, Phys. Rev. Letters **17**, 498 (1966).

[20] C. Chiu and J. Stack, Phys. Rev. **153**, 1575 (1967).

[21] V. Barger, in *Proceedings of the 1967 International Theoretical Physics Conference on Particles and Fields, Rochester, New York, 1967*, edited by C. R. Hagen *et al.* (Wiley-Interscience, Inc., New York, 1968), p. 655.

[22] A. Donnachie, R. Kirsopp, and C. Lovelace, Phys. Letters **26B**, 161 (1968).

[23] Since the existence of the $\frac{1}{2}(\frac{7}{2}^+)1983$ is poorly established (Ref. 15), constraining $\beta_{N\gamma}$ to fit the width of this resonance would be of questionable value anyway.

[24] Properties of long-established (before Ref. 22) resonances are taken from A. Rosenfeld *et al.*, University of California Radiation Laboratory Report No. UCRL-8030-Rev., 1968 (unpublished).

$s=M_r^2$ and $u=(1236\text{ MeV})^2$. R is related to the elastic width of the resonance, Γ_{el}, the resonance mass, and the c.m. momentum at resonance, q_r, by

$$R=\Gamma_{\text{el}}M_r/q_r.$$

Our calculation of ImA or ImB amounts to making a partial-wave expansion in the physical region, approximating the partial waves by the observed resonances, and then analytically continuing the Legendre functions $P_l'(z_s)$ in the partial-wave sum out to $u=(1236\text{ MeV})^2$.[25,26] This value of u corresponds to $z_s \lesssim -2$ for s below 2.2 BeV. Since

$$P_l'(z)\cong\frac{2}{\sqrt{\pi}}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l)}(2z)^{l-1} \qquad (3.9)$$

for large $|z|$, the effects of high-spin resonances are greatly enhanced relative to those of resonances (or background) of lower spin. Lacking this enhancement (and being highly inelastic), the generally low-spin new resonances of Donnachie, Kirsopp, and Lovelace are of no qualitative consequence.[27]

For each amplitude to which it was applied, the FESR (2.6) was evaluated (with $N_l=0.8\text{ BeV}^2$) for each of three values of N_r: $1.84\text{ BeV}^2=(1356\text{ MeV})^2$, $3.27\text{ BeV}^2=(1808\text{ MeV})^2$, and $5.35\text{ BeV}^2=(2313\text{ MeV})^2$. Each of these points is the midpoint (in s) of an $\sim$200-MeV-wide region bounded at each end by a prominent resonance, but free of significant resonances in its interior.[28] Thus our results will not be sensitive to small variations of N_r. Beyond our highest N_r, resonance quantum numbers are not well established.

Since none of our values of N_r is terribly large, we have included on the right-hand side of (2.6) the contributions from the highest *two* terms in the large-s expansions of Δ_δ and N_α exchange. From (3.4) and the fact that our α's depend only on $u=W_u^2$ [cf. (3.1a) and (3.1b)], the second term ($\sim s^{\alpha-3/2}$) from each exchange has two sources. First, for u fixed, z_u has the form $z_u=as+b$. Thus the $z_u^{\alpha-1/2}$ term from $P_{\alpha+1/2}'(z_u)$ goes like $a_1s^{\alpha-1/2}+a_2s^{\alpha-3/2}+\cdots$. Secondly, the leading term in z_u from the $P_{\alpha-1/2}'(z_u)$ term resulting from (3.4) goes like $z_u^{\alpha-3/2}$, hence like $s^{\alpha-3/2}$. These two sources of the $s^{\alpha-3/2}$ contribution have moderately complicated coefficients at any given u, and may tend to reinforce each other or to cancel. Because of this complexity, the $s^{\alpha-3/2}$ term is sometimes a much larger fraction of the leading $s^{\alpha-1/2}$ term than one would naïvely expect.

TABLE I. Resonance contributions to the left-hand sides of the four sum rules. The breaks in the table correspond to the various end points of integration N_r.

State	Contribution to FESR involving A^-	B^-	A_Δ	B_Δ
ρ	5.0	12.2	5.0	12.2
$\frac{1}{2}(\frac{1}{2}^+)938$	···	−14.5	···	−29.0
$\frac{3}{2}(\frac{3}{2}^+)1236$	− 5.9	−11.2	5.9	11.2
$\frac{1}{2}(\frac{1}{2}^+)1470$	1.0	− 1.8	2.0	− 3.5
$\frac{1}{2}(\frac{3}{2}^-)1518$	− 4.9	8.1	− 9.7	16.1
$\frac{1}{2}(\frac{1}{2}^-)1550$	− 0.1	0.0	− 0.1	− 0.1
$\frac{3}{2}(\frac{1}{2}^-)1640$	0.1	0.0	− 0.1	0.0
$\frac{1}{2}(\frac{5}{2}^-)1680$	− 3.0	− 5.9	− 5.9	−11.9
$\frac{1}{2}(\frac{5}{2}^+)1688$	24.9	−32.8	49.9	−65.6
$\frac{3}{2}(\frac{3}{2}^+)1688^{a,b}$	− 0.2	− 0.3	0.2	0.3
$\frac{3}{2}(\frac{3}{2}^-)1691^{a}$	1.6	− 2.0	− 1.6	2.0
$\frac{1}{2}(\frac{1}{2}^-)1710$	− 0.3	− 0.1	− 0.7	− 0.2
$\frac{1}{2}(\frac{1}{2}^+)1751^{a}$	0.5	− 0.5	0.9	− 1.1
$\frac{1}{2}(\frac{3}{2}^+)1863^{a,b}$	0.5	0.7	0.9	1.3
$\frac{3}{2}(\frac{5}{2}^+)1913^{a}$	− 7.5	7.8	7.5	− 7.8
$\frac{3}{2}(\frac{1}{2}^+)1934^{a}$	− 0.4	0.4	0.4	− 0.4
$\frac{3}{2}(\frac{7}{2}^+)1950$	− 7.7	−16.6	7.7	16.6
$\frac{3}{2}(\frac{5}{2}^-)1954^{a,b}$	0.9	1.7	− 0.9	− 1.7
$\frac{1}{2}(\frac{7}{2}^+)1983^{a,b}$	2.3	5.0	4.7	9.9
$\frac{1}{2}(\frac{3}{2}^-)2057^{a,b}$	− 1.5	1.3	− 2.9	2.7
$\frac{1}{2}(\frac{7}{2}^-)2200$	−20.4	17.5	−40.9	35.1

[a] New resonance reported by Donnachie *et al.*, Ref. 22.
[b] Existence considered poorly established as of Particle Data Group, Ref. 15.

Our normalizations are as follows: For $I_t=1$, we apply (2.6) to the amplitudes $-A^-/4\pi^2M$ and $-B^-/4\pi^2$.[11] For $I_u=\frac{3}{2}$, we apply it to $-A_\Delta/4\pi^2M$ and $-B_\Delta/4\pi^2$, where A_Δ is an $I_u=\frac{3}{2}$ amplitude defined by

$$A_\Delta(s,u)=\tfrac{2}{3}A^{I_s=1/2}(s,u)+\tfrac{1}{3}A^{I_s=3/2}(s,u), \qquad (3.10)$$

and similarly for B_Δ.

To summarize the main assumptions on which the sum rules and their practical evaluation are based, they are (0) analyticity, (1) Regge asymptotic behavior, (2) Regge behavior, on the average, from our actual end points N_r and N_l in the resonance region out to the asymptotic region, and (3) dominance of resonances over background on the left-hand side of the sum rule.

IV. RESULTS AND INTERPRETATION

Table I gives the resonance contributions to the left-hand sides of the four sum rules. Note that these contributions can have either sign, so that it is possible for a FESR to be so badly violated that the left- and right-hand sides do not even agree in sign. [This contrasts with the situation which one finds, for example, in considering the $\pi\pi$ FESR used by Schmid to bootstrap

[25] That this approach is valid despite the divergence of the real part of the partial-wave series at $u=M^{*2}$, and despite the presence of nonvanishing double spectral functions at this u, has already been commented on by Schmid, Ref. 2.

[26] The resonance dominance technique is not the only way to extrapolate the amplitudes out of the physical region. We are finding out whether other procedures lead to similar results.

[27] Since the existence and parameters of these resonances are not well-established, this circumstance is highly advantageous. [At the time of writing of Ref. 15, the existence of five of these nine resonances was considered poorly established. Now A. Brody *et al.*, Phys. Rev. Letters **22**, 1401 (1969), has called the entire analysis of Ref. 22 into question.]

[28] 1356 MeV is midway (in s) between the 3-3 resonance and the Roper resonance, 1808 MeV is midway between the $\frac{1}{2}(\frac{5}{2}^+)1688$ and the $\frac{3}{2}(\frac{5}{2}^+)1913$, and 2313 MeV is midway between the $\frac{1}{2}(\frac{7}{2}^-)2200$ and the $\frac{3}{2}(?)2420$.

TABLE II. FESR for $I_t=1$. The LHS and RHS columns list, respectively, the left- and right-hand sides of the sum rule. The $\Delta_\delta(1)$ column lists the contribution to the RHS from the leading term of the Δ_δ trajectory; the $\Delta_\delta(2)$ column, the contribution from the second term of this trajectory; etc. The entries in parentheses correspond to the use of (3.6b) for β_{N_α}.

N_r	LHS	RHS	$\Delta_\delta(1)$	$N_\alpha(1)$	$N_\gamma(1)$	$\Delta_\delta(2)$	$N_\alpha(2)$
			A^- sum rule				
1.84	− 0.9	4.3(5.8)	6.58	0.0(−3.16)	2.80	− 2.42	−2.64(1.96)
3.27	18.8	14.7(10.8)	18.5	0.0(−9.79)	5.32	− 5.73	−3.39(2.52)
5.35	−15.0	42.9(27.0)	48.1	0.0(−23.4)	9.72	−10.6	−4.32(3.21)
			B^- sum rule				
1.84	−13.5	− 13.0(−5.7)	− 2.83	−11.6(−11.5)	4.48	− 2.98	−0.09(7.14)
3.27	−48.9	− 42.9(−32.9)	− 7.97	−36.2(−35.5)	8.52	− 7.09	−0.11(9.18)
5.35	−31.2	−104.2(−91.4)	−20.6	−86.0(−85.0)	15.6	−13.1	−0.14(11.7)

the ρ meson.[29] At the value of momentum transfer at which Schmid works, all direct channel resonances contribute to the sum rule with the same sign, regardless of their angular momentum. Their isospin does not matter either; resonances with $I=0$ and $I=1$ contribute with the same sign, and there are no resonances with $I=2$. Thus as the end point N is increased and more resonances are included, the left-hand side of the sum rule necessarily grows, as does the ($\sim N^{\alpha+1}$) right-hand side, and a violent disagreement between the N dependences of the two sides is guaranteed not to occur.]

From Table I, one sees that the ρ term is a small fraction of the left-hand side of any sum rule by the time one gets to an end point of (1808 MeV)^2, so that uncertainties in the ρ contributions will not be important except for the lowest N_r. Also, one sees that the resonances of Donnachie *et al.*[22] make only small contributions, in almost all cases very small contributions. These resonances have no qualitative effect on our results, either individually or collectively.[27]

Table II presents the $I_t=1$ results. The A^- sum rule is not satisfied for $N_r=1.84$ BeV^2, which is a pretty low energy at which to expect Regge behavior, even on the average. At $N_r=3.27$ the sum rule works. Left- and right-hand sides agree in sign and roughly in magnitude. If the numerical discrepancy between them seems large, note that it is a small fraction of the largest single contribution to the left-hand side, which comes from the $\frac{1}{2}(\frac{5}{2}^+)1688$ and is bigger than the entire left-hand side. At $N_r=5.35$, however, the FESR does not work at all, the resonances above 3.27 having caused the left-hand side to change sign rather than to grow.

The results for the B^- sum rule are similar: partial success this time at 1.84, success at 3.27, and failure at 5.35. Note that both the A^- and B^- sum rules are obeyed at 3.27, and both are violated at 5.35, independently of which parametrization we use for the residue β_{N_α}.

The $I_u=\frac{3}{2}$ results, shown in Table III, repeat the pattern. Both the A_Δ and B_Δ sum rules are satisfied for $N_r=3.27$, and both are violated for $N_r=5.35$. The numerical discrepancies between left- and right-hand sides at $N_r=3.27$ are small compared with the largest single contribution to the left-hand side. Interestingly enough, it is the A sum rule, which contains an error due to omission of the $\pi\pi$ s-wave resonance, which works best at 3.27. Note from Table III that the second term from Δ_δ exchange, $\Delta_\delta(2)$, is far from negligible compared with $\Delta_\delta(1)$, especially in the B_Δ sum rule when $N_r=3.27$. Indeed, in this instance the $\Delta_\delta(2)$ term is crucial to the obtaining of a reasonably well-obeyed sum rule. The reader may wonder about the importance of the third term from Δ_δ exchange. Because we work at the value of u where $\alpha_{\Delta_\delta}(u)=\frac{3}{2}$, the Legendre functions $P_{\alpha_{\Delta_\delta}\pm 1/2}'(-z_u)$ coming out of (3.4) are only polynomials of the form $as+b$, so that the third and all succeeding contributions vanish identically.[30]

To sum up the results: *All four sum rules work for $N_r=3.27$. All four fail for $N_r=5.35$.*

From the success of all four sum rules at $N_r=3.27$, we conclude that for this value of N_r all of the basic assumptions (0)–(3) are correct. In particular, the relation which Regge theory predicts between the high-s behavior of the amplitudes at fixed u and the resonance spectrum in the u-channel is verified. In the usual Regge-type fits to backward high-energy data, this connection is really not checked in detail except to the extent that it is possible to extrapolate trajectories (fairly easy) and residues (very difficult) from the u-channel resonance region to negative u. To be sure,

TABLE III. FESR for $I_u=\frac{3}{2}$. The column headings $\Delta_\delta(1)$, $\Delta_\delta(2)$ have the same significance as in Table II.

N_r	LHS	RHS	$\Delta_\delta(1)$	$\Delta_\delta(2)$
		A_Δ sum rule		
1.84	10.9	12.4	19.7	− 7.26
3.27	45 8	38.3	55.5	−17.2
5.35	22.3	112.5	144.3	−31.8
		B_Δ sum rule		
1.84	− 5.6	− 17.4	− 8.49	− 8.94
3.27	−69.5	− 45.2	−23.9	−21.3
5.35	−13.8	−101.1	−61.8	−39.3

[29] C. Schmid, Ref. 2.

[30] The author is indebted to F. Hayot for a conversation which led him to realize this.

we have had to continue the low-energy amplitudes in the opposite direction, but it is significant that this totally different procedure leads to results which confirm the relation. Amusingly, by using the sum rules, and by working at $u=M^{*2}$, we have verified the high-energy behavior expected from Reggeism on the basis of low-energy data alone. This was possible because of property (2) (duality), which the success of all four sum rules also implies. Specifically, we find that for $u=M^{*2}$, and for energies between 1808 MeV and the asymptotic region where detailed Regge fits hold, all the πN amplitudes have Regge behavior on the average.

It remains, however, to explain why all the sum rules fail when N_r is increased to 5.35. We suggest that this behavior is due to breakdown of the resonance dominance approximation above 1808 MeV. Evidence for significant background in the 2-BeV region may be found in several features of the backward πN data (the backward region, where $u\approx 0$, is, of course, the part of the physical region closest to $u=M^{*2}$). First, there is a dramatic dip at 2200 MeV in the 180° $\pi^- p$ elastic cross section as a function of energy.[31] Barger and Cline[17] have found that this dip cannot be fitted with a superposition of resonances alone, despite the existence of the $\frac{1}{2}(\frac{7}{2}^-)$2200. However, Dikmen[32] has discovered that if one varies the widths and elasticities of some of the resonances somewhat from their experimentally favored values, a good fit can be obtained. The presence of nonresonant background is less ambiguous when one looks at the $\pi^- p$ angular distributions corresponding to $-1<\cos\theta<-0.7$ and energies between 2.1 and 2.3 BeV. Here Carroll *et al.*[33] find another, equally pronounced, dip (this time in the cross section as a function of angle), which moves forward with increasing energy. In an effort to see whether a pure resonance amplitude would still fit if one departed from 180°, Carroll *et al.*[33] attempted to fit their $\pi^+ p$ and $\pi^- p$ angular distributions over a range of energies around 2 BeV with the known resonances alone, allowing the resonance parameters to vary from their accepted values, within reasonable limits. They were unable to achieve a fit, particularly of the $\pi^- p$ dip. Thus, appreciable nonresonant background is definitely present in the backward region. If this background involves high spin, then it remains important when we go out to $u=M^{*2}$ [cf. (3.9)], and this is why all the $N_r=5.35$ sum rules failed.[34,35]

[31] S. Kormanyos, A. Krisch, J. O'Fallon, K. Ruddick, and L. Ratner, Phys. Rev. Letters **16**, 709 (1966).

[32] F. Ned Dikmen, Phys. Rev. Letters **18**, 798 (1967).

[33] A. Carroll *et al.*, Phys. Rev. Letters **20**, 607 (1968). The author thanks V. Barger for reminding him of these data.

[34] The other, rather unattractive, possibility is that the amplitudes are Reggeistic on the average if one averages from 1808 MeV to the asymptotic region, but not if one averages from 2313 MeV to the asymptotic region.

[35] It would be interesting to have a πN phase-shift analysis extending up to $\sim$2.4 BeV, which could be used as a basis for extrapolation to $u=M^{*2}$ without dependence on resonance dominance.

TABLE IV. FESR for $I_u=\frac{3}{2}$ in doublet universe.

N_r	LHS	RHS	$\Delta_\delta(1)$	$\Delta_\delta(2)$
		A_Δ sum rule		
1.84	−48.2	10.3	17.0	−6.70
3.27	−11.6	31.9	47.8	−15.9
		B_Δ sum rule		
1.84	−31.1	−24.3	−17.0	−7.30
3.27	−93.2	−65.1	−47.8	−17.3

Doublet Universe

To get some further perspective on the fact that with $N_r=(1808 \text{ MeV})^2$ our four sum rules are satisfied in nature, we ask whether they are still satisfied if we change the universe a little by adding a few nonexistent particles, namely, the extinct opposite-parity partners of the lowest states on the leading πN Regge trajectories. There is the question of how big a coupling or width to choose for each of the new particles. For the $\frac{1}{2}(\frac{1}{2}^-)$938, which would form a parity doublet with the nucleon, we take the newly introduced pole in the kinematical-singularity-free $\frac{1}{2}(\frac{1}{2}^-)$ partial-wave amplitude (3.2) to have the same residue as does the nucleon pole in the analogous $\frac{1}{2}(\frac{1}{2}^+)$ partial wave. The same prescription is used for the $\frac{3}{2}(\frac{3}{2}^-)$1236, which would be the parity-doublet partner of the 3-3 resonance. This procedure yields partial waves $f\equiv(\sin\delta)e^{i\delta}/q$ which would have normal threshold properties no matter how close the nucleon or 3-3 poles happened to be to threshold. Since these poles are not too far away, this property seems appropriate. For the new partner to the $\frac{1}{2}(\frac{3}{2}^-)$1518, however, we assume the same elastic width as that of the existing particle. We are guided by the fact that the doublets which actually exist (at higher points on the leading trajectories, or along lower-lying trajectories) tend more to have roughly equal elastic widths than to have equal residues in the kinematical-singularity-free partial waves.[24,22] Note that the existing doublets are found at energies above 1400 MeV, where our imaginary doublet at 1518 occurs.

With new particles in the πN spectrum, both sides of the various sum rules are changed. On the left, new resonance contributions are present, while on the right, the γ's are different, because the u-channel partial-wave residues β no longer vanish to extinguish the absent parity partners.

Including the effects of the new particles, we re-evaluate the $I_u=\frac{3}{2}$ sum rules. The right-hand sides of these FESR now depend on the properties of the 3-3 resonance and of its fictitious $\frac{3}{2}^-$ partner.[36] (By not using the $I_t=1$ sum rules we avoid the necessity of making a detailed parametrization of residue functions

[36] For completeness, we include on the left the insignificant contributions from a hypothetical $\frac{3}{2}(\frac{1}{2}^+)$ partner to the $\frac{3}{2}(\frac{1}{2}^-)$1640. The latter particle might be the lowest state on a trajectory with $I=\frac{3}{2}$ and positive signature, for which, however, there is no good evidence.

in a hypothetical universe.) The results are shown in Table IV. For the values of N_r for which it is obeyed in the real world, the A_Δ sum rule is now drastically violated. The main change is the large contribution (−58.0) of the fake $\frac{1}{2}(\frac{1}{2}^-)938$ to the left-hand side. The size and sign of this contribution are responsible for the dramatic violation.

We see that the FESR can distinguish between the real world and at least one rather natural fictitious one. This result underscores the meaningfulness of the success of the sum rules in nature.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge useful conversations with many people. Special thanks are due to Dr. Daniel Freedman, Dr. Benjamin Lee, Dr. Stanley Mandelstam, and Dr. Christoph Schmid for interesting comments and enlightening discussions.

APPENDIX: EXCHANGE ANTIDEGENERACY

In Chew's reciprocal bootstrap of the nucleon and the 3-3 resonance, direct forces (t-channel singularities) are very unimportant. Here we wish to show that if one tries to bootstrap whole Regge trajectories in a model in which there are no direct forces, then exchange-antidegenerate trajectories will result. By this we mean that for every trajectory of given signature and parity, there will exist another of opposite signature and parity, with the same trajectory function and equal but opposite residue function. If normal resonances lie along the one trajectory, resonances of negative width lie along the other.

Let us make a partial-wave analysis in the u channel, denoting the πN amplitudes A and B by A^1 and A^2, respectively. From dispersion relations of the form (2.8), it follows that

$$A_l^i(u)\equiv\int_{-1}^{1}dz_u P_l(z_u)A^i =\frac{1}{\pi q_u^2}\int_{x_0>0}^{\infty}dx'[\rho_t^i(x',u)+(-1)^l\rho_s^i(x'+\beta/u,\,u)]Q_l\left(1+\frac{x'}{2q_u^2}\right) \quad \text{(A1)}$$

for $l\geq N$. Here $\beta\equiv(M_N^2-\mu_\pi^2)^2$, and $i=1,\,2$. Amplitudes $A^{i\pm}(l,u)$ that can be analytically continued in l are defined by replacing the $(-1)^l$ in (A1) by ±1, respectively.

For spinless scattering, there would be just one partial-wave amplitude $A_l(u)$. If there were only exchange forces (s-channel singularities), so that $\rho_t=0$, we would have $A^+(l,u)=-A^-(l,u)$. From this it immediately follows that any trajectory present in $A^+(l,u)$ will be accompanied by a partner with the same α and opposite β in $A^-(l,u)$.

For πN scattering, the signatured partial-wave amplitudes, labeled as in (3.2), are related to the quantities $A^{i\pm}(l,u)$ defined above by relations of the form

$$f_{J+1/2}{}^+(J,W_u)=\sum_{i=1}^{2}[c_i(W_u)A^{i-}(J+\tfrac{1}{2},\,u)-c_i(-W_u)A^{i+}(J-\tfrac{1}{2},\,u)], \quad \text{(A2)}$$

$$f_{J-1/2}{}^+(J,W_u)=-f_{J+1/2}{}^+(J,\,-W_u), \quad \text{(A3)}$$

$$f_{J+1/2}{}^-(J,W_u)=\sum_{i=1}^{2}[c_i(W_u)A^{i+}(J+\tfrac{1}{2},\,u)-c_i(-W_u)A^{i-}(J-\tfrac{1}{2},\,u)], \quad \text{(A4)}$$

$$f_{J-1/2}{}^-(J,W_u)=-f_{J+1/2}{}^-(J,\,-W_u). \quad \text{(A5)}$$

The $c_i(W_u)$ are well-known kinematical factors.[37] If there are no t-channel forces, (A1) gives $A^{i+}(l,u)=-A^{i-}(l,u)$. From (A2)–(A5), it then follows that

$$f_{J+1/2}{}^+(J,W_u)=-f_{J+1/2}{}^-(J,W_u) \quad \text{(A6)}$$

and

$$f_{J-1/2}{}^+(J,W_u)=-f_{J-1/2}{}^-(J,W_u). \quad \text{(A7)}$$

The states for which the two amplitudes related by (A6) are physical have opposite values of both signature and parity, and similarly for the amplitudes of (A7). Thus, for every trajectory of given signature and parity, there will be another with opposite signature and parity, having the same $\alpha(W_u)$ and opposite $\beta(W_u)$.[38]

[37] Chew *et al.*, Ref. 11.

[38] A bootstrap of the N and the 3-3 resonance, using the $I_u=\frac{3}{2}$ FESR at $u=M^{*2}$, and the $I_u=\frac{1}{2}$ superconvergent sum rules which hold at $u=0$ [D. Beder and J. Finkelstein, Phys. Rev. **160**, 1363 (1967)], was considered. In the spirit of a simple bootstrap, it was planned to include only the N, 3-3 resonance, and ρ (thus including at least some t-channel effects) on the left-hand sides of the four sum rules. It is quickly apparent, however, that some of the bootstrap equations coming out of such a model will be very poorly satisfied by the experimental masses and coupling constants, so that such a bootstrap would make no sense. If we include only the ρ, N, and 3-3 on the left, then the $I_u=\frac{3}{2}$ sum rules must be evaluated with N_r not far above the 3-3, or we make a serious error by omitting the contributions of such resonances as the $\frac{1}{2}(\frac{3}{2}^-)1518$ (cf. Table I). But with such a low N_r, the B_Δ sum rule will be badly broken in nature (Table III); duality does not hold to so low an energy. As for the $I_u=\frac{1}{2}$ superconvergence relations, we already know from Beder and Finkelstein that the A sum rule is not satisfied if one omits all t-channel contributions besides that of the ρ. A realistic bootstrap calculation of this type must include more than the ρ, N, and 3-3 resonance.