Infinite-Component Local Field Theory with a Rising Mass Spectrum*

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An infinite-component field theory is proposed to describe the ρ Regge family of particles. Using the generators of the group $O(3,2)$, a class of first-order wave equations is obtained. The simplest of this class of equations is solved to yield a rising mass spectrum with a hydrogenlike accumulation point. Because only finite-dimensional representations of the homogeneous Lorentz group appear in the theory, it is free from many of the difficulties, including noncausality, which have plagued other infinite-component field theories with nontrivial mass spectra.

I. INTRODUCTION

IN recent years, several authors[1–4] have investigated the possibility of using infinite-component fields to describe the spectrum of strongly interacting particles. Unfortunately, a major result of these investigations has been a number of theorems,[5–7] detailing the conditions under which such theories must fall victim to various diseases, e.g., the breakdown of causality,[7] the appearance of spacelike solutions,[5] and mass spectra with totally unrealistic behavior.[5,7]

In this paper, we shall construct a quantum field theory free of these pathologies and describing a Regge family of particles with a tolerably realistic mass spectrum. In particular, we find it useful to consider the ρ trajectory and its daughters as an irreducible infinite-dimensional representation of $SO(3,2)$. This representation decomposes into an infinite sum of finite-dimensional representations of the homogeneous Lorentz group $\mathfrak{L}$; because we do not introduce infinite-dimensional representations of $\mathfrak{L}$, we have no trouble with the causality of the theory.

We remark that we are here using techniques very similar to those developed[2,3] in analogy with the nonrelativistic hydrogen atom[8]; there, the symmetry of the Hamiltonian is $O(4)$ instead of $\mathfrak{L}$, and one finds it enlightening to consider the "spectrum-generating" group $SO(4,1)$ [where we use $SO(3,2)$], an irreducible representation of which contains all the bound states of the system.

In Sec. II we present some of the group-theoretical background; in Sec. III we use the generators of $SO(3,2)$ to write down a class of first-order Lorentz-invariant wave equations. The simplest of these yields a mass spectrum of the form

$$m_k^2=\alpha^2-\beta^2/(k+1)^2,\qquad k=1,2,\ldots$$

where α and β are free parameters. This mass spectrum is monotonically rising, and is hydrogenlike in that it has an accumulation point. This latter feature is not common to the whole class of equations, however; it may well be that another member of the class will yield a spectrum extending to infinity.

In Sec. IV we discuss the consequences and possible generalizations of our equation. Section V is devoted to the construction of local quantized fields satisfying the equation, and Sec. VI presents some conclusions.

II. SOME PROPERTIES OF $O(3,2)$

$O(3,2)$ is the group of all linear homogeneous transformations on the 5-tuplet $x=(x_0, x_1, x_2, x_3, x_5)$, leaving invariant the quadratic form

$$S(x)=x_0^2-x_1^2-x_2^2-x_3^2+x_5^2.$$

Any member T of the proper subgroup of $O(3,2)$ [by proper subgroup (notation: $SO(3,2)$) we mean those transformations continuously derivable from the identity] can be written in the form $T(\alpha)=\exp(\frac{1}{2}\alpha_{\mu\nu}M^{\mu\nu})$, where $M_{\mu\nu}=-M_{\nu\mu}$ ($\mu,\nu=0,1,2,3,5$) are the ten generators of the group. Using standard techniques, we can derive the commutation rules:

$$[M_{\mu\nu},M_{\rho\sigma}]=i(M_{\mu\rho}g_{\nu\sigma}+M_{\nu\sigma}g_{\mu\rho}-M_{\nu\rho}g_{\mu\sigma}-M_{\mu\sigma}g_{\nu\rho}),\qquad(2.1)$$

where we take $g_{00}=g_{55}=+1$, $g_{11}=g_{22}=g_{33}=-1$, and $g_{\mu\nu}=0$ for $\mu\neq\nu$.

It will be convenient to define the operators

$$J_i=-\tfrac{1}{2}\epsilon_{ijk}M^{jk},\qquad i,j,k=1,2,3$$

$$K_i=M_{i0},\qquad i=1,2,3$$

$$L_i=M_{5i},\qquad i=1,2,3$$

$$L_0=M_{50}.$$

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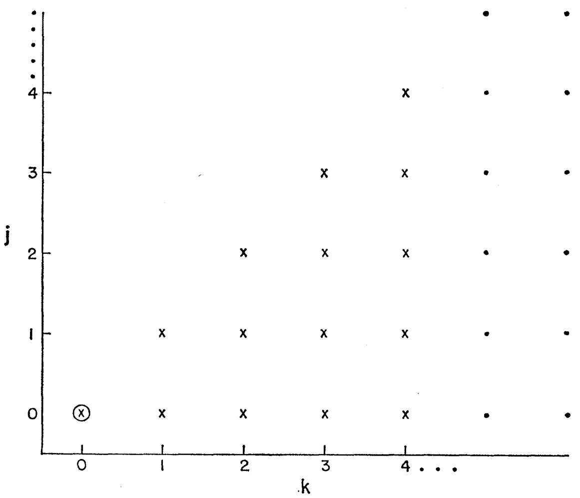

FIG. 1. States to be used in the construction of an infinite-component field theory describing the ρ Regge family. Although the $k=0$, $j=0$ state does not occur in this family, its use as a field component is optional.

Then (2.1) can be written as

$$[J_i, J_j]=i\epsilon_{ijk}J_k, \qquad [J_i, K_j]=i\epsilon_{ijk}K_k,$$
$$[J_i, L_j]=i\epsilon_{ijk}L_k, \qquad [J_i, L_0]=0,$$
$$[K_i, K_j]=-i\epsilon_{ijk}J_k, \qquad [L_i, L_j]=-i\epsilon_{ijk}J_k, \qquad (2.2)$$
$$[K_i, L_j]=ig_{ij}L_0, \qquad [K_i, L_0]=-iL_i,$$
$$[L_i, L_0]=iK_i.$$

We see that the two sets $\{\mathbf{J}, \mathbf{K}\}$ and $\{\mathbf{J}, \mathbf{L}\}$ *separately* generate the subgroup $\mathfrak{L}$. Furthermore, we notice that the operators L_μ ($\mu=0, 1, 2, 3$) transform as a 4-vector under the action of the group generated by $\{\mathbf{J}, \mathbf{K}\}$.[9]

From (2.1) we can also verify that the two quantities $Q=-M_{\mu\nu}M^{\mu\nu}=\mathbf{K}^2-\mathbf{J}^2-L_\mu L^\mu$ and $\bar{W}=S_\mu S^\mu$, $S_\mu=\epsilon_{\mu\nu\lambda\rho\sigma}M^{\nu\lambda}M^{\rho\sigma}$, commute with all ten generators.

We are specifically interested in the representations of $SO(3,2)$ whose basis vectors occupy the lattice sites of Fig. 1. There is an ambiguity as to whether the $k=0$, $j=0$ site should be included. This state is not present in the ρ Regge family, but it may be excluded either at this point by choosing a representation without a $k=j=0$ state, or at a later stage by simply not including the $k=j=0$ state among the particles from which we shall build our fields. The latter course (which corresponds roughly to the familiar case of describing the ρ meson by a vector field ρ_μ and then excluding the $j=0$ part by imposing the condition $\partial_\mu\rho^\mu=0$) will turn out to be mathematically more tractable; we shall compare the two procedures in Sec. III. We choose $\{\mathbf{J}, \mathbf{K}\}$ to be the generators of the homogeneous Lorentz group under which the states of each tower transform among themselves, as described in the Introduction. Since each tower is a representation of the form $(\frac{1}{2}k, \frac{1}{2}k)$, we must have $\mathbf{J}\cdot\mathbf{K}=0$. But from (2.2) we find

$$[\mathbf{J}\cdot\mathbf{K}, L_0]=-i\mathbf{J}\cdot\mathbf{L},$$
$$[\mathbf{J}\cdot\mathbf{K}, L_i]=-i(J_iL_0+\epsilon_{ijk}L_kK_j).$$

So in order to have $\mathbf{J}\cdot\mathbf{K}=0$, we must have also $\mathbf{J}\cdot\mathbf{L}=0$ and $J_iL_0+\epsilon_{ijk}L_kK_j=0$. But this amounts to putting $S_\mu=0$, and so the representation we want must have $\bar{W}=0$.

We label our basis states $|kj\sigma\rangle$, where k tells us which tower we are in, j tells us the spin of the state, $\mathbf{J}^2|kj\sigma\rangle=j(j+1)|kj\sigma\rangle$, and $J_3|kj\sigma\rangle=\sigma|kj\sigma\rangle$. The matrix elements of $\mathbf{J}$ and $\mathbf{K}$ are well known; clearly

$$\langle kj\sigma|\mathbf{J}, \mathbf{K}|k'j'\sigma'\rangle\propto\delta_{kk'}.$$

To evaluate the matrix elements of L_μ, we use the techniques of Thomas,[10] who analyzed the unitary representations of $O(4,1)$. If we define $\mathbf{N}=i\mathbf{K}$ and $N_0=iL_0$, then the set $\{\mathbf{J}, \mathbf{L}, N_\mu\}$ will generate a unitary representation of $O(4,1)$. So we can apply Thomas's results directly to our case, provided we choose $\mathbf{L}$ Hermitian and L_0 anti-Hermitian. Some of the details are provided in Appendix A.

We record the matrix elements of L_0, which will play an important role in what follows:

$$\langle kj\sigma|L_0|k'j'\sigma'\rangle=[a_{kj}\delta_{k,k'+1}-\delta_{k+1,k'}a_{k'j}]\delta_{jj'}\delta_{\sigma\sigma'}, \qquad (2.3)$$

where for the representation with the $k=j=0$ state,

$$a_{kj}=\tfrac{1}{2}[(k-j)(k+j+1)]^{1/2}, \qquad (2.4a)$$

and for the representation without the $k=j=0$ state,

$$a_{kj}=\tfrac{1}{2}[(k-1)(k+2)/k(k+1)]^{1/2}\times[(k-j)(k+j+1)]^{1/2}. \qquad (2.4b)$$

The value of Q for the representation without the $k=j=0$ state is determined, in fact, by the condition that this state be absent. This necessitates $Q=0$. In the case where we include the $k=0$ state, $Q=2$ (see Appendix A).

III. WAVE EQUATION

We have used the states in the ρ Regge family to construct irreducible representations of $O(3,2)$, but we have not yet made an assumption that will allow us to calculate any of the physical parameters, such as the masses or coupling constants of these particles. To do this, we shall make use of the existence of the 4-vector set of matrices L_μ to write the first-order wave equation

$$(\partial_\mu L^\mu - M)\phi(x)=0. \qquad (3.1)$$

More explicitly, this is an infinite-dimensional matrix

[9] It is generally true that in going from an orthogonal group in n dimensions to one in $n+1$ dimensions, one adds a set of n generators which transform as a vector under the action of the n-dimensional group.

[10] L. H. Thomas, Ann. Math. **42**, 113 (1941); see also T. D. Newton, *ibid.* **51**, 730 (1950).

equation, with $\phi(x)$ a column vector:

$$\sum_{k'=1}^{\infty}\sum_{j'=0}^{k'}\sum_{\sigma'=-j'}^{j'}[\partial_\mu(L^\mu)_{kj\sigma;k'j'\sigma'}-M_{kj\sigma;k'j'\sigma'}]\phi_{k'j'\sigma'}(x)=0.$$

This equation will be Lorentz invariant if M satisfies

$$D(\Lambda^{-1})MD(\Lambda)=M \tag{3.2}$$

for every Lorentz transformation Λ. Since in our representation

$$D(\Lambda)_{kj\sigma;k'j'\sigma'}=D_{j\sigma,j'\sigma'}{}^{(k/2,k/2)}(\Lambda)\delta_{kk'}, \tag{3.3}$$

we will satisfy (3.2) if

$$M_{kj\sigma;k'j'\sigma'}=m_k\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}. \tag{3.4}$$

Here m_k is an arbitrary function of k.

To explore the mass spectrum allowed by (3.1), we consider the Fourier transform

$$\tilde{\phi}(p)=\int d^4x\,\exp(ip\cdot x)\phi(x), \tag{3.5}$$

which must satisfy

$$(ip_\mu L^\mu-M)\tilde{\phi}(p)=0. \tag{3.6}$$

In the rest frame, $p_\mu=(p_0,0,0,0)$ and

$$(ip_0L^0-M)\tilde{\phi}(p_0)=0. \tag{3.7}$$

To examine under what conditions p_0 will be real, we assume $m_k\neq 0$, in which case we can rewrite (3.7) as $[M^{-1}(iL^0)-p_0{}^{-1}I]\tilde{\phi}=0$; thus p_0 will be real if the matrix $M^{-1}(iL_0)=SHS^{-1}$, where H is Hermitian and S is an arbitrary invertible matrix. But $M^{-1}(iL_0)=M^{-1/2}[M^{-1/2}(iL_0)M^{-1/2}]M^{1/2}$. Identifying $H=M^{-1/2}\times(iL_0)M^{-1/2}$, and recalling that iL_0 is Hermitian, we see that p_0 will be real if $M^{-1/2}$ is either Hermitian or anti-Hermitian, that is, if m_k is real and of definite sign.

Incidentally, since $(i\mathbf{L})$ is anti-Hermitian, this same argument shows that if we choose p^μ spacelike, the components of p^μ will turn out to be pure imaginary—that is, there are no spacelike solutions.

We can reduce Eq. (3.7) to its bare essentials by noting, from (2.3) and (3.4), that both L_0 and M are diagonal in j and σ. So for fixed j, σ, we have

$$(ip_0\bar{L}_0{}^{(j)}-\bar{M})_{kk'}\tilde{\phi}_{k'}{}^{(j\sigma)}(p_0)=0, \tag{3.8}$$

where $(\bar{L}_0{}^{(j)})_{kk'}=(a_{kj}\delta_{k,k'+1}-a_{k'j}\delta_{k+1,k'})$ and $\bar{M}_{kk'}=m_k\delta_{kk'}$.

We are now prepared to use the explicit form of the a_{kj} [Eq. (2.4)] and so we must treat the two representations ($Q=2$ and $Q=0$) separately.

We begin with the case $Q=2$; the a_{kj} are given by Eq. (2.4a). We write

$$i\bar{L}_0{}^{(j)}=-H_y{}^{(j)} \tag{3.9}$$

and we define two additional matrices

$$(H_x{}^{(j)})_{kk'}=a_{kj}\delta_{k,k'+1}+a_{k'j}\delta_{k+1,k'}, \tag{3.10}$$

$$(H_0)_{kk'}=(k+1)\delta_{kk'}. \tag{3.11}$$

These three Hermitian matrices satisfy the commutation rules

$$\begin{aligned}[][H_x{}^{(j)},H_y{}^{(j)}]&=-iH_0,\\ [H_x{}^{(j)},H_0]&=-iH_y{}^{(j)},\\ [H_y{}^{(j)},H_0]&=+iH_x{}^{(j)}.\end{aligned} \tag{3.12}$$

That is, they generate a unitary representation of $SO(2,1)$, the homogeneous Lorentz group in three dimensions.

Using (3.9), (3.8) takes the form

$$(p_0H_y{}^{(j)}+\bar{M})\tilde{\phi}=0. \tag{3.13}$$

Now $\bar{M}$ is an arbitrary diagonal matrix, so we can write

$$[p_0H_y{}^{(j)}+f(H_0)]\tilde{\phi}=0, \tag{3.14}$$

where f is an arbitrary function.

We can solve (3.14) in one simple case: when f is a linear function, $f=\alpha H_0+\beta$. Thus

$$[p_0H_y{}^{(j)}+\alpha H_0+\beta]\tilde{\phi}=0. \tag{3.15}$$

(H_0,H_y,H_x) transform as a vector in three-dimensional Minkowski space, so there exists a transformation $U(\alpha,p_0)$ with the property

$$U(\alpha,p_0)[p_0H_y+\alpha H_0]U^{-1}(\alpha,p_0)=\gamma H_0,$$

with $\gamma=(\alpha^2-p_0{}^2)^{1/2}$. We assume here that $p_0<\alpha$. Defining $h^{(j)}(p_0)=U(\alpha,p_0)\tilde{\phi}^{(j)}(p_0)$, we have

$$(\gamma H_0+\beta)h=0. \tag{3.16}$$

Since H_0 is diagonal, the solutions to (3.16) are trivially found. The eigenvectors h are

$$h_{k'}{}^{(j)}(k)=\delta_{kk'},\qquad k=j,j+1,\ldots$$

and the associated value of γ is $\gamma(k)=-\beta/(k+1)$. That is,

$$p_0{}^2(k)=\alpha^2-\beta^2/(k+1)^2. \tag{3.17}$$

Making the identification of k with the trajectory function $\alpha(p_0{}^2)$, we can plot the Regge trajectory given by (3.17) in Fig. 2.

Having obtained a solution in the case that a_{kj} is given by (2.4a), we now discuss briefly the modifications which are necessary if $Q=0$, i.e., when a_{kj} is given by (2.4b).

We can use the arbitrary nature of $\bar{M}$ in (3.8) to factor out the extra k dependence in (2.4b). We let $M=A^{1/2}BA^{1/2}$, where A is the matrix

$$A_{kk'}=[k(k+2)/(k+1)^2]\delta_{kk'}, \tag{3.18}$$

and define

$$g_k{}^{(j\sigma)}=A_{kk'}{}^{1/2}\tilde{\phi}_{k'}{}^{(j\sigma)}(p_0). \tag{3.19}$$

A has been chosen so that

$$i\bar{L}_0{}^{(j)}=-A^{1/2}H_y{}^{(j)}A^{1/2}, \tag{3.20}$$

where $H_y{}^{(j)}$ is exactly the same matrix defined in (3.9) [that is, $\bar{L}_0{}^{(j)}$ in (3.9) is still given by the a_{kj} as defined

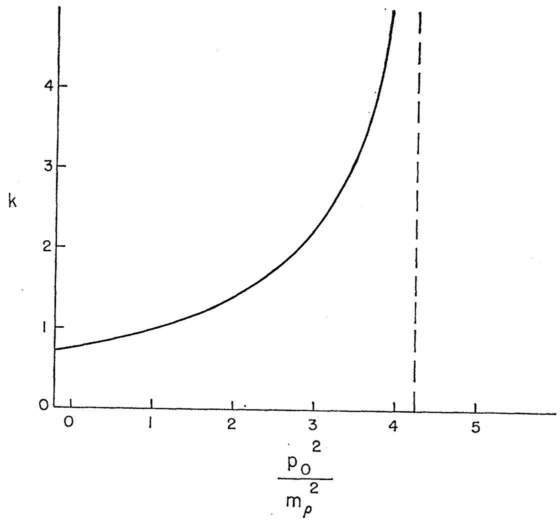

FIG. 2. Mass spectrum given by Eq. (3.17). Values of α and β are the same as in Table I: $\alpha=(8/\sqrt{15})m_\rho$; $\beta=(14/\sqrt{15})m_\rho$.

in (2.4a)]. Then (3.8) becomes

$$(p_0 H_y{}^{(j)}+B)g^{(j)}=0, \qquad (3.21)$$

and by choosing $B=\alpha H_0+\beta$ we can employ exactly the same method of solution as in the previous case, and we obtain the same spectrum (3.17).

The difficulty is that, because of the absence of the $k=0$ state, (3.20) does not hold for $j=0$. However, by inspection of (2.4a) and (2.4b), we notice that

$$iL_0{}^{(0)}=-H_y{}^{(1)}. \qquad (3.22)$$

Thus the $j=0$ equation takes the form

$$[p_0 H_y{}^{(1)}+A(\alpha H_0+\beta)]\tilde{\phi}^{(0)}=0, \qquad (3.23)$$

or, using (3.18),

$$[p_0 H_y{}^{(1)}+[(H_0{}^2-1)/H_0{}^2](\alpha H_0+\beta)]\tilde{\phi}^{(0)}=0. \qquad (3.24)$$

Thus the $j=0$ spectrum will differ from the $j\neq 0$ spectra by virtue of the factor $(H_0{}^2-1)/H_0{}^2$. We have been unable to obtain the solution in this case, and the modification of the $j=0$ spectrum is therefore not known.

IV. DISCUSSION OF WAVE EQUATION

Equation (3.1) is the simplest, but by no means the only, equation we could have written down using the generators (2.2). Furthermore, (3.1) allows us the freedom to choose the constants m_k defined in (3.4), and we have chosen them simply in order to be able to solve explicitly for the mass spectrum. Thus (3.17) is the outcome of both some speculation as to the utility of a group like $SO(3,2)$ in describing Regge trajectories, and a pragmatic approach in selecting an equation that can be easily solved.

It should be pointed out that the properties of Eq. (3.15) were investigated several years ago by Nambu[1] in his search for an infinite-component wave equation to simulate the spectrum of the relativistic hydrogen atom. He rejected this equation on the grounds that the probability density was of alternating sign. Since we intend to use our solutions to construct quantized fields which can be inserted into a Dyson expansion of the S matrix, we foresee no difficulties of this nature.

The fact that $p_0{}^2\to$ const as $k\to\infty$ can be understood on dimensional grounds. The matrix elements of H_y are proportional to k for large k, and so we may expect the large-k behavior of p_0 to be $p_0\to f(H_0)/k$. Since we chose $f(H_0)\propto H_0\to k$, it is reasonable to get $p_0\to$ const.

This suggests, of course, that if we want $p_0{}^2\to k$ for large k, as most Regge theorists would prefer, then the leading power of H_0 should be $H_0{}^{3/2}$.

In Sec. III we did not discuss the case $p_0>\alpha$. By an $SO(2,1)$ rotation similar to $U(p_0,\alpha)$, we can reduce (3.15) in this case to

$$[(p_0{}^2-\alpha^2)^{1/2}H_y+\beta]\tilde{\phi}=0.$$

Since H_y is a noncompact Hermitian generator, it will have a continuous spectrum, and thus

$$p_0{}^2=\alpha^2+\beta^2/\lambda^2, \qquad \lambda^2>0.$$

Therefore, above α (and below $-\alpha$) we have a continuous spectrum of masses. The interpretation of these solutions is uncertain within the present work, but we remark that possibly if we obtain a solution to (3.15) in which the discrete part of the spectrum extends to infinity, the continuous part will cease to exist.

Our trajectory (3.17) has the same degeneracy as the Veneziano model; that is, the mass does not depend on j, but only on k. It seems, however, that this is because of our special choice of $f(H_0)$, and is not an intrinsic property of (3.1). For example, we can test this by adding a term $\lambda H_0{}^2$ to $f(H_0)$, and treating this as a perturbation. Of course, for large k, $\lambda H_0{}^2$ will be the dominant term; but assuming analytic behavior in λ, for small k a perturbation treatment will indicate how the masses are shifted. Writing $p=p_0+\lambda p'$, we find that to first order in λ,

$$p'=-g^*H_0{}^2g/g^*H_yg, \qquad (4.1)$$

where g is the solution to the unperturbed equation. The result is

$$p'(k,j)=p_0(k)[\Delta(k)+j(j+1)/2\beta], \qquad (4.2)$$

TABLE I. Mass spectrum for $\alpha=(8/\sqrt{15})m_\rho$, $\beta=(14/\sqrt{15})m_\rho$.

k	p_0 (MeV)
$\frac{3}{4}$	0
1	780
2	1308
3	1454
4	1504
···	···
∞	1612

where Δ is a function only of k, and β is the same as appears in (3.17). For details see Appendix B.

It is difficult to compare (3.17) directly with experiment. The masses $p_0(k)$ are supposed to represent dipion resonances. We have two parameters α and β at our disposal. One of these is fixed by assigning the ρ to $k=1$. According to current practice,[11,12] we would also like to assign the f_0 to the exchange degenerate leading trajectory. A typical fit is shown in Table I.

We point out that the indicated values of α and β (which cannot be changed too much and still give acceptable values for m_ρ and m_f) yield a largest mass of 1612 MeV, which is much smaller than one would expect for a realistic resonance model. Also, the $m=0$ intercept is at $k=\frac{3}{4}$. Lovelace[12] has shown that the Adler consistency condition requires $\alpha(s)=\frac{1}{2}$ for $s=0$. If we try to fit both $p_0(\frac{1}{2})=0$ and $p_0(1)=m_\rho$ with our formula (3.17), we find that the cutoff mass $p_0(\infty)=1.5m_\rho\approx 1170$ MeV, which is below even the f_0 mass.

We pointed out in Sec. III that if we chose the representation without the $k=0$ state, we could not avoid a more complicated equation, (3.24), for the $j=0$ states. The possibility exists for us to make the "inverse" choice—that is, to choose $\bar{M}$ in (3.8) so that the $j=0$ equation is simple,

$$\bar{M}=\alpha H_0+\beta.$$

Then the $j=0$ spectrum will be (3.17), whereas the $j\neq 0$ equation will be

$$\{p_0H_y^{(j)}+[H_0^2/(H_0^2-1)](\alpha H_0+\beta)\}\tilde{\phi}^{(j)}=0. \quad (4.3)$$

The hope would then be that (4.3) will yield a better spectrum than (3.17), although on dimensional grounds we still would expect the masses to tend to a finite constant at infinity.

V. CONSTRUCTION OF A FIELD THEORY

In this section we show how to make use of the solutions that we found to (3.1) to construct an infinite-component field theory. We shall use the solutions in the case that the $k=0$ state is present, since in the other case the $j=0$ wave functions are not known.

For a given $p_0(k)$, the eigenvector that solves (3.16) is $h_{k'}(k)=\delta_{kk'}$. This in turn leads to the solution

$$\tilde{\phi}_{k'}^{(kj)}=[\exp(i\phi H_x^{(j)})]_{k'k} \quad (5.1)$$

of Eq. (3.8), where

$$\sinh\phi=p_0/(\alpha^2-p_0^2)^{1/2}. \quad (5.2)$$

For a given set $(kj\sigma)$ we can then construct the solution to (3.7),

$$\tilde{\phi}_{k'j'\sigma'}^{(kj\sigma)}=\tilde{\phi}_{k'}^{(kj)}\delta_{jj'}A_{\sigma'\sigma}^{(kj)}, \quad (5.3)$$

where $A^{(kj)}$ is an arbitrary $(2j+1)\times(2j+1)$ matrix. That $\tilde{\phi}^{(kj\sigma)}$ is indeed a solution may be verified by direct substitution into (3.7). The form of $A_{\sigma'\sigma}^{(kj)}$ may be determined from the desired transformation properties of $\tilde{\phi}$. As will be seen below, we want the solution of (3.6) to satisfy

$$\tilde{\phi}_{k_1j_1\sigma_1}^{(kj\sigma)}(\Lambda p)=D_{j_1\sigma_1,j_2\sigma_2}^{(k_1/2,k_1/2)}(\Lambda)\times\phi_{k_1j_2\sigma_2}^{(kj\sigma_3)}(p)D_{\sigma_3\sigma}^{(j)}(R_W^{-1}(k)), \quad (5.4)$$

where $R_W(k,\Lambda,p)$ is the Wigner rotation $L^{-1}(\Lambda p)\Lambda L(\mathbf{p})$. Here $L(\mathbf{p})$ is the boost which takes $(m,0,0,0)$ into $(p_0,\mathbf{p})$. Equation (5.4) implies that

$$\tilde{\phi}^{(kj\sigma)}(p)=D(L(\mathbf{p}))\tilde{\phi}^{(kj\sigma)}. \quad (5.5)$$

It is easy to verify that $\tilde{\phi}(p)$ satsifies (3.6) if $\tilde{\phi}$ satisfies (3.7). In order to satisfy both

$$\tilde{\phi}(\Lambda p)=D(L(\Lambda p))\tilde{\phi}$$

and

$$\tilde{\phi}(\Lambda p)=D(\Lambda)\tilde{\phi}(p)D(R_W^{-1})=D(\Lambda)D(L(p))\tilde{\phi}D(R_W^{-1}),$$

we must have

$$\tilde{\phi}=D(R_W)\tilde{\phi}D(R_W^{-1}); \quad (5.6)$$

when we substitute (5.3) into (5.6), it reduces to

$$A_{\sigma'\sigma}^{(kj)}=D_{\sigma'\sigma_1'}^{(j)}(R_W)A_{\sigma_1'\sigma_1}^{(kj)}D_{\sigma_1\sigma}^{(j)}(R_W^{-1}) \text{ for each } R_W.$$

Hence we must take

$$A_{\sigma'\sigma}^{(kj)}=F(k,j)\delta_{\sigma'\sigma}, \quad (5.7)$$

where the $F(k,j)$ are a set of parameters depending on k,j.

So our solution to (3.6) is

$$\tilde{\phi}_{k'j'\sigma'}^{(kj\sigma)}(p)=D_{j'\sigma',j\sigma}^{(k'/2,k'/2)}(L(\mathbf{p},k))\tilde{\phi}_{k'}^{(kj)}F(k,j), \quad (5.8)$$

with $\tilde{\phi}_{k'}^{(kj)}$ defined in (5.1). $\tilde{\phi}(p)$ also satisfies (5.4). Here we must take $F(0,0)=0$ to ensure the absence of the $k=0$ state, as we promised to do in Sec. II.

Using well-known techniques due to Weinberg,[13] we can now construct the field operator

$$\phi(x)=\phi^{(-)}(x)+\phi^{(+)}(x), \quad (5.9)$$

with

$$\phi_{k'j'\sigma'}^{(-)}(x)=\sum_{kj\sigma}\int[d^3p/2p_0(k)]\tilde{\chi}_{k'j'\sigma'}^{(kj\sigma)}(p)\times a(\mathbf{p},k,j,\sigma)\exp(-ip_k\cdot x) \quad (5.10)$$

and

$$\phi_{k'j'\sigma'}^{(+)}(x)=\sum_{kj\sigma}\int[d^3p/2p_0(k)]\tilde{\phi}_{k'j'\sigma'}^{(kj\sigma)}(p)\times\bar{b}(\mathbf{p},k,j,\sigma)\exp(ip_k\cdot x). \quad (5.11)$$

Here $a(\mathbf{p},k,j,\sigma)$ destroys particles of momentum p, mass $p_0(k)$, spin j, and spin projection σ. $\bar{b}$ is a creation operator for antiparticles, related to the adjoint of the

[11] J. Shapiro, Phys. Rev. **179**, 1345 (1969).
[12] C. Lovelace, Phys. Letters **28B**, 264 (1968).
[13] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **181**, 1893 (1969); see also Ref. 7.

destruction operator by

$$\bar{b}(j,\sigma)=[C^{(j)-1}]_{\sigma\sigma'}b^\dagger(j,\sigma'),\qquad(5.12)$$

where $C^{(j)}$ is a charge-conjugation matrix satisfying

$$D^{(j)*}(R)=C^{(j)}D^{(j)}(R)C^{(j)-1}.$$

We have distinguished between $\tilde{\chi}$ and $\tilde{\phi}$, because in order to satisfy (3.1), we see that

$$(ip_\mu L^\mu-M)\tilde{\phi}=0,\qquad(-ip_\mu L^\mu-M)\tilde{\chi}=0.$$

Thus $\tilde{\phi}$ is the function defined in (5.8). $\tilde{\chi}$ differs from $\tilde{\phi}$ only in Eq. (5.1), where we must take

$$\tilde{\phi}_{k'}{}^{(kj)}=[\exp(-i\phi H_x{}^{(j)})]_{k'k},\qquad \sinh\phi=p_0/(\alpha^2-p_0{}^2)^{1/2}.$$

Using the well-known Lorentz-transformation properties of a and $\bar{b}$, and Eq. (5.4), we verify that $\phi(x)$ satisfies the desired transformation law

$$U(\Lambda)\phi(x)U^{-1}(\Lambda)=D(\Lambda^{-1})\phi(\Lambda x).\qquad(5.13)$$

To test the causality of our theory, we can compute the commutator

$$[\phi_{k'j'\sigma'}(x),\phi_{k_1'j_1'\sigma_1'}(y)]\equiv C$$

and see whether it vanishes for $(x-y)^2<0$.

Assuming the usual commutators for creation and destruction operators, and using the fact that

$$[\exp(\pm i\phi H_x)]_{kk'}=(-1)^{k+k'}[\exp(\mp i\phi H_x)]_{kk'},$$

we obtain

$$C=\sum_{kj}d(k,j)\int[d^3p/2p_0(k)]X(p)\{\exp[-ip_k\cdot(x-y)]$$
$$-(-1)^{k'+k_1'}\exp[ip_k\cdot(x-y)]\},\qquad(5.14)$$

where

$$d(k,j)=|F(k,j)|^2\tilde{\phi}_{k_1'}{}^{(kj)}\tilde{\chi}_{k'}{}^{(kj)}$$

and

$$X(p)=\sum_\sigma D_{j'\sigma',j\sigma}{}^{(k'/2,k'/2)}(L(p,k))$$
$$\times D_{j_1'\sigma_1',j\sigma}{}^{(k_1'/2,k_1'/2)*}(L(p,k)).\qquad(5.15)$$

Because the boost matrices in (5.15) belong to finite-dimensional representations of $\mathfrak{L}$, $X(p)$ will be a polynomial in $p_k{}^\mu$.[14] Also, as shown in Appendix C, under the transformation $p^\mu\to-p^\mu$, we have

$$X(-p)=(-1)^{k'+k_1'}X(p),\qquad(5.16)$$

which enables us to write

$$C=\sum_{kj}d(k,j)X(i\partial)\Delta(x-y;p_0(k)).\qquad(5.17)$$

Here $\Delta(x-y;p_0(k))$ is the usual scalar commutator with mass $p_0(k)$.

It can be shown[14] that $X(p)$ is a polynomial of degree $k'+k_1'$, and thus there is a finite upper bound to the number of derivatives acting on Δ in (5.17). Therefore, C vanishes for $(x-y)^2<0$, as required.

VI. CONCLUSIONS

The results of this paper might usefully be extended in two directions: (i) One might attempt to solve (3.14) with a more general choice of $f(H_0)$, thereby obtaining a more general mass spectrum. In particular, the natural modifications to (3.14) suggested by the use of the $Q=0$ rather than the $Q=2$ representation [Eqs. (3.24) and (4.3)] can be further explored. (ii) The construction of causal fields can be viewed as the first step on the road to a relativistically invariant interaction density $H(x)$, which can then be used in a Dyson expansion of the S matrix. The features which one would look for in the S-matrix elements are: (a) resonance structure characteristic of the ρ Regge family and (b) Regge behavior at high energies. The work of Van Hove, Durand, and Blankenbecler and Sugar[15,16] in constructing Feynman-diagram models of Regge amplitudes indicates that one can obtain Regge behavior by summing diagrams in which ever increasing spins are exchanged; it will be interesting to see if our model will reproduce Regge asymptotic behavior in a similar way.

Some dangers which may be inherent in our approach are: (a) Although each individual commutator (5.14) involves only finite-dimensional representations of $\mathfrak{L}$, the construction of $H(x)$ will bring in infinite sums over our Regge fields. One must therefore be wary of divergences and nonlocalities which can be introduced in this way. This same problem is encountered when one sums up an infinite set of diagrams with derivative coupling,[15] because the sum will contain an arbitrarily high number of derivatives acting on the fields. (b) A glance at (5.8) shows that there are a large number of as yet undetermined constants $F(k,j)$ hidden in the fields $\phi(x)$. The constants must be determined by some principle if our theory is eventually to predict values for scattering amplitudes.

In this paper we have pointed out how a rather simple infinite-component field theory may be relevant to the discussion of Regge phenomena. This work is intended as the first step in providing a field-theoretic description of these phenomena, which have hitherto been understood only on the basis of an S-matrix approach. The crucial tests for our theory have yet to be met.

Note added in manuscript. While this manuscript was in preparation, I received a report by H. D. I. Abarbanel, in which an infinite-component field theory is considered for many of the same reasons that motivated the present article. Abarbanel derives conditions under which a theory of this type is strictly localizable [see

[14] For an explicit derivation of $X(p)$, see S. Weinberg, Phys. Rev. **181**, 1893 (1969) [he uses the notation $X(p)=\pi(p/m,j)$].

[15] L. Van Hove, Phys. Letters **24B**, 183 (1967); L. Durand III, Phys. Rev. **154**, 1537 (1967).

[16] R. Blankenbecler and R. L. Suger, Phys. Rev. **168**, 1597 (1968).

Sec. VI, "danger" (a)], and shows that with a suitable choice of constants, an infiite-component field theory can reproduce the Veneziano amplitude. He does not use group-theoretical techniques to discuss either a wave equation or the mass spectrum.

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APPENDIX A

In this appendix we provide a fuller analysis of the representation $Q=0$, $\bar{W}=0$ and $Q=2$, $\bar{W}=0$ of $SO(3,2)$. As noted in the text, the matrix elements of $\mathbf{J}$ and $\mathbf{K}$ are just the usual ones associated with $\mathfrak{L}$:

$$J_3\,|\,kj\sigma\rangle=\sigma\,|\,kj\sigma\rangle,$$

$$J_\pm\,|\,kj\sigma\rangle=[(j\mp\sigma)(j\pm\sigma+1)]^{1/2}\,|\,kj\sigma\pm1\rangle,$$

$$K_3\,|\,kj\sigma\rangle=(j^2-\sigma^2)^{1/2}b_j^{(k)}\,|\,k,j-1,\sigma\rangle+[(j+1)^2-\sigma^2]^{1/2}b_{j+1}^{(k)}\,|\,k,j+1,\sigma\rangle,\quad\text{(A1)}$$

$$K_\pm\,|\,kj\sigma\rangle=[(j\mp\sigma)(j\mp\sigma-1)]^{1/2}b_j^{(k)}\,|\,k,j-1,\sigma\pm1\rangle\mp[(j\pm\sigma+1)(j\pm\sigma+2)]^{1/2}b_{j+1}^{(k)}\,|\,k,j+1,\sigma\pm1\rangle,$$

where

$$b_j^{(k)}=i\{[(k+1)^2-j^2]/(4j^2-1)\}^{1/2}.$$

To evaluate the matrix elements of L_μ, we form the operators

$$\mathfrak{J}^{(\pm)}=\tfrac{1}{2}(\mathbf{J}\pm i\mathbf{K}).\quad\text{(A2)}$$

$\mathfrak{J}^{(\pm)}$ are two commuting angular momenta, and $\mathbf{J}=\mathfrak{J}^{(+)}+\mathfrak{J}^{(-)}$. We can transform bases via

$$|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle=\sum_{j=0}^{k}|\,k,j,m_1+m_2\rangle\times\langle j,m_1+m_2\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle.\quad\text{(A3)}$$

This new basis has the property

$$(\mathfrak{J}^{(\pm)})^2\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle=\tfrac{1}{4}k(k+2)\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle,$$

$$\mathfrak{J}_3^{(+)}\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle=m_1\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle,\quad\text{(A4)}$$

$$\mathfrak{J}_3^{(-)}\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle=m_2\,|\,\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\rangle.$$

We define the combinations

$$A_{1/2,1/2}=-(L_1+iL_2),\qquad A_{1/2,-1/2}=L_3+L_0,$$
$$A_{-1/2,1/2}=L_3-L_0,\qquad A_{-1/2,-1/2}=L_1-iL_2.\quad\text{(A5)}$$

Then the commutation rules (2.2) tell us that $A_{\kappa;\kappa'}$ transforms as the κ component of a spinor operator with respect to $\mathfrak{J}^{(+)}$, and as the κ' component of a spinor operator with respect to $\mathfrak{J}^{(-)}$. Applying the Wigner-Eckart theorem twice, we can write

$$\langle\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\,|\,A_{\kappa,\kappa'}\,|\,\tfrac{1}{2}k',m_1',\tfrac{1}{2}k',m_2'\rangle=\langle\tfrac{1}{2}k',m_1',\tfrac{1}{2},\kappa\,|\,\tfrac{1}{2}k,m_1\rangle\langle\tfrac{1}{2}k',m_2',\tfrac{1}{2},\kappa'\,|\,\tfrac{1}{2}k,m_2\rangle\times f(\tfrac{1}{2}k,\tfrac{1}{2}k').\quad\text{(A6)}$$

It remains to calculate f. Notice that (A6) tells us that L_μ has nonzero matrix elements only for $k'=k\pm1$.

To calculate f, we use

$$[A_{1/2,-1/2},A_{-1/2,1/2}]=2[\mathfrak{J}_3^{(-)}-\mathfrak{J}_3^{(+)}].\quad\text{(A7)}$$

Taking matrix elements

$$\langle\tfrac{1}{2}k,m_1,\tfrac{1}{2}k,m_2\,|\,(\quad)\,|\,\tfrac{1}{2}k',m_1,\tfrac{1}{2}k',m_2\rangle$$

of this relation, we find that for $k\neq k'$ it is automatically satisfied for any f, while for $k=k'$ we must have

$$f(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2})f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)-f(\tfrac{1}{2}k,\tfrac{1}{2}k+\tfrac{1}{2})f(\tfrac{1}{2}k+\tfrac{1}{2},\tfrac{1}{2}k)=2(k+1).\quad\text{(A8)}$$

This is a simple difference equation, with the solution

$$f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)f(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2})=\alpha_0-k(k+1).\quad\text{(A9)}$$

As noted in Sec. II, we want to choose $\mathbf{L}$ to be Hermitian and L_0 to be anti-Hermitian. [This is consistent with (2.2) and the choices already made, viz., $\mathbf{J}=\mathbf{J}^\dagger$ and $\mathbf{K}=-\mathbf{K}^\dagger$.] We thus require $A_{1/2,-1/2}{}^\dagger=A_{-1/2,1/2}$ which, when inserted into (A6), yields

$$f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)=-[(k+1)/k]f^*(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2});$$

we can choose the phases of our basis states so that f is real, and our condition then becomes

$$f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)=-[(k+1)/k]f(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2}).\quad\text{(A10)}$$

In order completely to determine f, we must know what α_0 is in (A9). If we wish to exclude the $k=0$ state from our representation, we must have $f(0,\tfrac{1}{2})=0$, which implies $\alpha_0=2$. Equations (A9) and (A10) then give us

$$f(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2})=[k(k-1)(k+2)/(k+1)]^{1/2},$$
$$f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)=-[(k+1)(k-1)(k+2)/k]^{1/2}.\quad\text{(A11)}$$

If we want to include the $k=0$ state, we must have $f(0,\tfrac{1}{2})\neq0$; we demand instead that $f(0,-\tfrac{1}{2})=0$, which fixes $\alpha_0=0$. The answer is

$$f(\tfrac{1}{2}k,\tfrac{1}{2}k-\tfrac{1}{2})=k,\qquad f(\tfrac{1}{2}k-\tfrac{1}{2},\tfrac{1}{2}k)=-(k+1).\quad\text{(A12)}$$

Using (A3) we can transform back to the $|\,kj\sigma\rangle$ basis to get (2.3) and (2.4).

To calculate Q, we express $L_\mu L^\mu$ in terms of the $A_{\kappa,\kappa'}$ and use $\mathbf{J}^2-\mathbf{K}^2=k(k+2)$. We find that $L_\mu L^\mu=\alpha_0-2-k(k+2)$ so that $Q=2-\alpha_0$.

APPENDIX B

Given the unperturbed equation

$$(p_0 H_y+\alpha H_0+\beta)g=0, \quad (3.15')$$

we introduce a perturbation λH_0^2, and let $p_0\to p_0+\lambda p'$ and $g\to g+\lambda g'$. Then the equation is

$$[(p_0+\lambda p')H_y+\lambda H_0^2+\alpha H_0+\beta](g+\lambda g')=0. \quad (B1)$$

To first order in λ, we have

$$(p_0 H_y+\alpha H_0+\beta)g+\lambda[(p' H_y+H_0^2)g + (p_0 H_y+\alpha H_0+\beta)g']=0. \quad (B2)$$

The term of order zero vanishes by virtue of (3.15′). Furthermore, we can multiply (B2) on the left by g^*, and note that $g^*(p_0 H_y+\alpha H_0+\beta)=0$ since H_y and H_0 are Hermitian. Thus we demand $g^*(p' H_y+H_0^2)g=0$, or

$$p'=-g^* H_0^2 g/g^* H_y g. \quad (B3)$$

From Sec. III we recall that

$$g_{k'}(k)=U_{k'k}^{-1}, \quad (B4)$$

where U was defined to have the property

$$U(p_0 H_y+\alpha H_0)U^{-1}=\gamma H_0, \quad (B5)$$

with $\gamma=(\alpha^2-p_0^2)^{1/2}$. Equation (B5) implies that

$$U^{-1}H_0 U=\cosh\phi H_0+\sinh\phi\, H_y, \quad (B6)$$

where $\cosh\phi=\alpha/\gamma$, $\sinh\phi=p_0/\gamma$.

Because H_0 generates rotations in the xy plane, the operator $P=\exp(i\pi H_0)$ has the properties

$$PH_0P^{-1}=H_0, \qquad PH_xP^{-1}=-H_x, \qquad PH_yP^{-1}=-H_y. \quad (B7)$$

In particular, since U can be taken to be $\exp(-i\phi H_x)$,

$$PUP^{-1}=U^{-1} \quad \text{and} \quad PU^{-1}P^{-1}=U. \quad (B8)$$

Thus the numerator of (B3) can be written

$$\begin{aligned}(PU^{-1}P^{-1}H_0^2PUP^{-1})_{kk} &=(PU^{-1}H_0^2UP^{-1})_{kk}\\ &=[P(\cosh\phi\, H_0+\sinh\phi\, H_y)^2P^{-1}]_{kk}\\ &=(\cosh\phi\, H_0-\sinh\phi\, H_y)_{kk}^2.\end{aligned}$$

In the denominator, we first use $H_y g=-p_0^{-1}(\alpha H_0+\beta)g$; the denominator is then

$$p_0^{-1}(\alpha\cosh\phi\, H_0-\alpha\sinh\phi\, H_y+\beta)_{kk}.$$

Now terms linear in H_y have no diagonal elements. The diagonal elements of H_0 are $(k+1)\delta_{kk'}$, and those of H_y^2 are $\frac{1}{2}[(k+1)^2-j(j+1)]\delta_{kk'}$. Using this and rearranging, we arrive at

$$p'(k,j)=p_0(k)[\Delta(k)+j(j+1)/2\beta], \quad (B9)$$

where

$$\Delta(k)=-[(k+1)^2/2\beta](2\alpha^2/p_0^2+1).$$

APPENDIX C

Using the fact that

$$\exp(i\boldsymbol{\alpha}\cdot\mathbf{J})\mathbf{K}\exp(-i\boldsymbol{\alpha}\cdot\mathbf{J})=R^{-1}(\alpha)\mathbf{K}, \quad (C1)$$

we can write

$$\langle j'\sigma'|\exp(i\theta_k\hat{p}\cdot\mathbf{K})|j\sigma\rangle=D_{\sigma'\sigma_a'}^{(j')}(R_p)\times\langle j'\sigma_a'|\exp(i\theta_kK_3)|j\sigma_a\rangle D_{\sigma_a\sigma}^{(j)}(R_p^{-1}), \quad (C2)$$

where R_p is a rotation satisfying $R_p\hat{z}=\hat{p}$. Using (C2), $X(p)$ can be written

$$D_{\ '\sigma_a'}^{(j')}(R_p)[\sum_\sigma\langle j'\sigma_a'|\exp(i\theta_kK_3)|j\sigma\rangle^{(k')}\times\langle j\sigma|\exp(i\theta_kK_3)|j_1'\sigma_a\rangle^{(k_1')}]D_{\sigma_a\sigma_1'}^{(j_1')}(R_p^{-1}). \quad (C3)$$

The quantity in square brackets can be expanded as

$$\sum_{\sigma,ab,a'b',cd,c'd'}\{\langle j'\sigma_a'|\tfrac{1}{2}k',a,\tfrac{1}{2}k',b\rangle\times\langle ab|\exp(i\theta_kK_3)|a'b'\rangle\langle\tfrac{1}{2}k',a',\tfrac{1}{2}k',b'|j\sigma\rangle\times\langle j\sigma|\tfrac{1}{2}k_1',c,\tfrac{1}{2}k_1',d\rangle\langle cd|\exp(i\theta_kK_3)|c'd'\rangle\times\langle\tfrac{1}{2}k_1',c',\tfrac{1}{2}k_1',d'|j_1'\sigma_a\rangle\}. \quad (C4)$$

In the "ab" basis, $iK_3|ab\rangle=(a-b)|ab\rangle$. Therefore, (C4) becomes

$$\sum_{\sigma,a,b,c,d}\{\langle j'\sigma_a'|\tfrac{1}{2}k',a,\tfrac{1}{2}k',b\rangle[(p_0+p)/m]^{a-b}\times\langle\tfrac{1}{2}k',a,\tfrac{1}{2}k',b|j\sigma\rangle\langle j\sigma|\tfrac{1}{2}k_1',c,\tfrac{1}{2}k_1',d\rangle\times[(p_0+p)/m]^{c-d}\langle\tfrac{1}{2}k_1',c,\tfrac{1}{2}k_1',d|j_1'\sigma_a\rangle\}, \quad (C5)$$

where we have used $e^{\theta_k}=[p_0(k)+p]/m_k$ and $p=|\mathbf{p}|$. To examine the behavior of $X(p)$ under $p^\lambda\to-p^\lambda$, we must let $p_0\to-p_0$; for the spatial part, we may either let $\hat{p}\to-\hat{p}$ or let $p\to-p$. We choose the latter course. Then the only part of (C3) which changes is the part contained in (C5), and, as it is easy to see, (C5) picks up a factor $(-1)^{a-b+c-d}$. But the Clebsch-Gordan coefficients tell us that $a+b=c+d$ and therefore $a-d=c-b$. So the factor is $(-1)^{2(a-d)}$; but $(-1)^{2a}=(-1)^{k'}$, and $(-1)^{-2d}=(-1)^{k_1'}$. Therefore the factor is $(-1)^{k'+k_1'}$, and we have shown that $X(-p^\lambda)=(-1)^{k'+k_1'}X(+p^\lambda)$.