$O(2,1)$ Decomposition of the Equal-Mass Multipheripheral Equation at $t=0$†

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We extend the results of a group-theoretical analysis of the $t<0$ multiperipheral equation to the case $t=0$ for pairwise equal masses. Using variables discussed in a previous paper, we diagonalize the equation in the Bali-Chew-Pignotti (BCP) model with respect to the $O(2,1)$ group and relate the solutions to the equation so obtained with the solutions obtained after diagonalization with respect to the $O(3,1)$ group. Poles in the $O(3,1)$ partial-wave amplitude give rise to the expected sequence of daughter poles in the $O(2,1)$ partial-wave amplitude. At general momentum transfer, we establish factorization at the $O(1,1)$ poles in the decomposition of the BCP amplitude, and present further simplifications to the diagonalized equations based upon this model.

I. INTRODUCTION

The recent group-theoretical analysis[1–3] of the multiperipheral equation[4–6] with respect to the $O(3,1)$ and $O(2,1)$ groups has provided a natural framework in which to investigate the constraints that unitarity imposes upon the residues and trajectories of the Regge-daughter family near $t=0$. In this paper, we shall examine some preliminary problems in this direction.

Since different sets of variables have been used to write the $t=0$ [6,1] and $t<0$ [2,3] equations, it is important to study first how they match in the limit $t=0$. Moreover, if we take the Bali-Chew-Pignotti[7] (BCP) model for the production amplitudes at $t=0$ as CD did, it is essential to translate this model in the $t<0$ variables by keeping the nonleading powers in the asymptotic expansion.

The BCP variables, used by CD and MM[1] at $t=0$, are essentially the parameters of the $O(2,1)$ groups which preserve the momentum transfers in the multiperipheral chain. The $t<0$ variables,[2,3] which we shall call "three-dimensional BCP variables," are instead the parameters of the little groups of the Lorentz three-vectors $\mathbf{k}$, associated with each upper and lower momentum transfer $Q_{u,l}$ by the formula

$$Q_{u,l}=[\mathbf{k},\, w\pm\tfrac{1}{2}(-t)^{1/2}],$$

valid in a Breit frame of the over-all momentum transfer Q. Since the most important contribution to the phase space comes, for t small, from spacelike $\mathbf{k}$'s,[3] we shall often refer to the three-dimensional BCP variables as "$O(1,1)$ variables" and to the poles in the respective Fourier transforms as "$O(1,1)$ poles."

In this language, the purpose of this paper is (a) to establish the factorization at the $O(1,1)$ poles in the $O(1,1)$ decomposition of the BCP model at general momentum transfer, and (b) to use the three-dimensional BCP variables at $t=0$, giving a relation between the $O(2,1)$ and $O(3,1)$ decompositions of the incomplete absorptive part of the scattering amplitude.

The latter relation, which is model dependent, gives, so to speak, the eigenfunctions of the Regge daughter poles in terms of the ones of the Lorentz poles. It is therefore similar to the off-shell relation found[8] for the Bethe-Salpeter equation. As we mentioned before, that would be the natural starting point for the dynamical study of derivatives and residues of the daughter sequence near $t=0$. However, we have not extended our analysis further in this direction.

The $O(1,1)$ expansion of the BCP model for the production amplitudes has been given in MM[2]. We derive a simplified form of this expression and of the resulting multiperipheral equation in Sec. II, and we show that to each Regge pole in the BCP expansion there corresponds an infinity of integrally spaced $O(1,1)$ poles with factorizable residues.

In Sec. III, we take the $t=0$ limit of this equation for pairwise equal masses and relate the incomplete absorp-

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[1] A. H. Mueller and I. J. Muzinich, Ann. Phys. (N.Y.) (to be published); hereafter referred to as MM[1].

[2] A. H. Mueller and I. J. Muzinich, Ann. Phys. (N.Y.) (to be published); hereafter referred to as MM[2].

[3] M. Ciafaloni, C. DeTar, and M. N. Misheloff, Phys. Rev. **188**, 2522 (1969); hereafter referred to as CDM.

[4] G. F. Chew, M. L. Goldberger, and F. Low, Phys. Rev. Letters **22**, 208 (1969).

[5] I. G. Halliday and L. M. Saunders, Nuovo Cimento **60A**, 177 (1969).

[6] G. F. Chew and C. DeTar, Phys. Rev. **180**, 1577 (1969); hereafter referred to as CD.

[7] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967); hereafter referred to as BCP.

[8] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

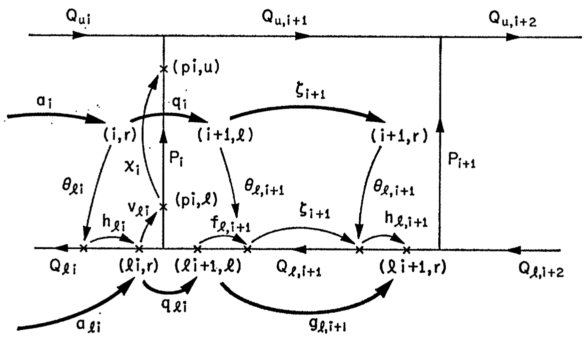

FIG. 1. Connection between $t<0$ and $t=0$ frames in the middle of the chain. Only the lower $t=0$ frames are shown. The $t<0$ frames are shown halfway between upper and lower momenta. The notation is defined in the text.

tive part in this limit with that of the $t=0$ equation of CD. This relationship then implies a connection between the $O(3,1)$ and $O(2,1)$ decompositions of the respective incomplete absorptive parts, from which we can derive the eigenfunctions of the Regge poles in the daughter sequence from that of a given Lorentz pole.

In Appendix B we also simplify the diagonalized $t=0$ equation of MM[1], using a technique similar to that developed by CDM for the $t<0$ equation. In Appendix D a model of the Amati-Stanghellini-Fubini (AFS) type is treated as an example.

II. $t<0$ EQUATION FOR BCP MODEL

We begin with a review of the three-dimensional and four-dimensional BCP variables, which we have indicated schematically for an internal segment of the multiperipheral ladder in Fig. 1 and for the end of the ladder in Fig. 2. The three-dimensional BCP variables (cf. CDM and MM[2]), consisting of the x boosts q_i and y boosts ζ_i, build up the $O(2,1)$ transformation a_i, defined recursively,[9]

$$a_{i+1}=a_i q_i \zeta_{i+1}, \tag{2.1}$$

while the four-dimensional BCP variables, for the lower amplitude, consisting of the z boosts q_{li} and $O(2,1)$ transformation $g_{li}=r_z(\mu_{li})b_x(\zeta_{li})r_z(\nu_{li})$, build up the $O(3,1)$ transformations a_{li}, defined recursively,

$$a_{l,i+1}=a_{li}q_{li}g_{l,i+1}. \tag{2.2}$$

An analogous set of four-dimensional BCP variables is defined for the upper part of the ladder, which we distinguish with the label u: q_{ui}, g_{ui}, etc. The initial transformations a_0 and a_{l0} are defined, respectively, in terms of the initial z rotation ϕ_a and initial rotation[10] $r_{la}=r_z(\phi_a)r_y(\beta_{la})$:

$$a_0=\phi_a, \qquad a_{l0}=r_{la}. \tag{2.3}$$

[9] For the sake of economy, we use the same label for a one-parameter transformation as for the parameter itself.

[10] We have set equal to unity the arbitrary initial Lorentz transformations, mentioned in previous approaches.

A similar set of variables defines transformations at the other end of the ladder, and we obtain the transformations b_b and b_{lb} defined in CD, CDM, and MM[1] and MM[2]:

$$b_b=a_{n+1}q_{n+1}\phi_b, \qquad b_{lb}=a_{l,n+1}q_{l,n+1}r_{lb}, \tag{2.4}$$

where $r_{lb}=r_y(\beta_{lb})r_z(\phi_b)$.

MM[2] have given the Lorentz transformation, which relates the three-dimensional BCP frames (i,r), in which

$$\begin{aligned} Q&=Q_{ui}-Q_{li}=[0,0,0,(-t)^{1/2}],\\ Q_{li}&=[0,k_i,0,w_i-\tfrac{1}{2}(-t)^{1/2}],\\ Q_{ui}&=[0,k_i,0,w_i+\tfrac{1}{2}(-t)^{1/2}], \end{aligned} \tag{2.5}$$

and $Q_{l,i+1}$ and $Q_{u,i+1}$ lie in the xzt plane, to the four-dimensional BCP frame (li,r) in which

$$Q_{li}=[0,0,0,(-t_{li})^{1/2}] \tag{2.6}$$

and $Q_{l,i+1}$ lies in the tz plane. The transformation consists in a y rotation $\theta_{l,i}$, which brings $Q_{l,i}$ in (2.5) to the form (2.6), followed by an x boost[11] $h_{l,i}$, which removes the x component of $Q_{l,i+1}$.[12] Similarly, we can transform from the frames $(i+1,l)$ to $(li+1,l)$ by a y rotation $\theta_{l,i+1}$ followed by an x boost[11] $f_{l,i+1}^{-1}$. The parameters of these Lorentz transformations may be calculated in terms of k_i, w_i, k_{i+1}, w_{i+1}, M_i, and t, or equivalently in terms of t_{li}, t_{ui}, $t_{l,i+1}$, $t_{u,i+1}$, M_i, and t. The formula for θ_{li} is simply

$$\begin{aligned} \sin\theta_{li}&=k_i/(-t_{li})^{1/2},\\ \cos\theta_{li}&=[w_i-\tfrac{1}{2}(-t)^{1/2}]/(-t_{li})^{1/2}, \end{aligned} \tag{2.7}$$

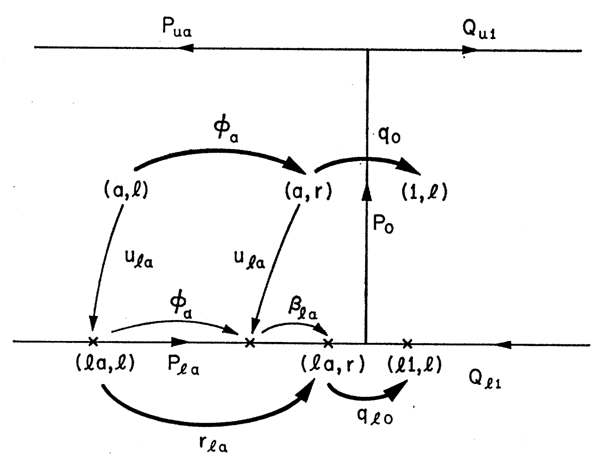

FIG. 2. Connection between $t<0$ and $t=0$ frames at the left end of the chain.

[11] Our notation differs from MM[2]. Our h_{li} is their $u_{i\,i+1}{}^{i\,+}$ and our f_{li} is their $u_{i\,i+1}{}^{i+1\,+}$.

[12] Note that in this way we specify the frame (li,r) completely with no arbitrary z rotation left, as in CD.

while $f_{l,i+1}$ and $h_{l,i}$ depend upon all of these variables.[13] Analogous variables are defined for the upper half of the ladder. The fact that the f's and h's adjacent to one rung of the ladder depend only upon the Lorentz scalars associated with the rung is crucial to the factorization condition.

At the ends of the ladder, the above approach must be modified with θ being replaced by the z boost u_{la} and h by the y rotation β_{la}, as indicated in Fig. 2. From these two figures, one can now read off the important identities relating the three- and four-dimensional BCP variables[14]:

$$g_{li}=f_{li}\zeta_i h_{li}=\mu_{li}\zeta_{li}\nu_{li}, \tag{2.8a}$$

$$\theta_{li}^{-1}q_i\theta_{l,i+1}=h_{li}q_{li}f_{l,i+1}, \tag{2.8b}$$

$$a_{li}=u_{la}^{-1}a_i\theta_{li}h_{li}. \tag{2.8c}$$

The Toller angle $\omega_{li}=\nu_{li}+\mu_{l,i+1}$ is fixed by formula (2.8a) in terms of ζ_i, ζ_{i+1}, and four sets of k_i and w_i; however, to leading order in $\exp|\zeta_i|$ and $\exp|\zeta_{i+1}|$ the dependence is reduced to the variables, $\operatorname{sgn}\zeta_i$, $\operatorname{sgn}\zeta_{i+1}$, k_i, w_i, and k_{i+1}, w_{i+1}.[3] In the same approximation, it is proper to consider the dependence upon the Toller angle as residing in the multi-Regge vertex function, and the reduced kinematical dependence then forms the basis for a simple factorization of the residues as functions of the k's and w's. In general, however, the Toller angles are not convenient kinematical variables for $t<0$. They have, in effect, been replaced by the extra set of momentum-transfer variables.

The procedure for the $O(2,1)$ diagonalization of the $t<0$ equation given by MM[2] and CDM begins with a decomposition of the unitarity integrand with respect to the $O(1,1)$ group parameters ζ_i. We concentrate upon the $O(1,1)$ decomposition of the lower BCP amplitude and later combine lower and upper amplitudes to form the unitarity integrand. We begin with the BCP amplitude for the production of N particles:

$$M_{m_am_0\cdots m_{N+1}m_b}{}^{(N)}\sim\sum_{\gamma_i,l_i,p_i}D_{m_al_0}{}^{s_a}(r_a)G_{l_0m_0p_1}{}^{a\gamma_1}(t_1)\times\tilde{a}_{p_1l_1}{}^{-\alpha_{\gamma_1}(t_1)-1}(g_1)G_{l_1m_1p_2}{}^{\gamma_1\gamma_2}(t_1,t_2)\cdots G_{l_{N+1}m_{N+1}p_{N+2}}{}^{\gamma_{N+1}b}D_{p_{N+2}m_b}{}^{s_b}(r_b), \tag{2.9}$$

where

$$\tilde{a}_{mm'}{}^{-l-1}\equiv U_m{}^l a_{mm'}{}^{-l-1}U_{m'}{}^{-l-1},$$
$$U_m{}^l\equiv\Gamma(l+m+1)/\Gamma(-l+m), \tag{2.10}$$

and a is Toller's[15] $O(2,1)$ representation function of the second kind. For the lower amplitude, m_i is the z component of the spin of particle i in the frame (li,r) for $i=1,\ldots,N+1$, and s_am_a, s_bm_b describe the spins of the initial particles. Conservation of helicity requires that

$$G_{lmp}=\delta_{m,l-p}G_{mp}. \tag{2.11}$$

If we use the formula

$$\tilde{a}_{pl}{}^{-\alpha-1}(g)\sim\sum_{jk}D_{pj}{}^{\alpha}(f)\tilde{a}_{jk}{}^{-\alpha-1}(\zeta)D_{kl}{}^{\alpha}(h) \tag{2.12}$$

for $g=f\zeta h$, which is valid term by term in an asymptotic expansion of both sides in $\exp|\zeta|$, and the $O(1,1)$ decomposition of the a function, given by (A51),[16]

$$\tilde{a}_{jk}{}^{-\alpha-1}(\zeta)\sim\sum_{n=0}^{\infty}\sum_{\tau=\pm}V_{j,n\tau}{}^{\alpha}\exp[\tau\zeta(\alpha-n)]W_{n\tau,k}{}^{\alpha}\theta(\tau\zeta), \tag{2.13}$$

we may write

$$\tilde{a}_{pl}{}^{-\alpha-1}(g)\sim\sum_{n\tau}\tilde{D}_{p,n\tau}{}^{\alpha}(f)\exp[\tau\zeta(\alpha-n)]\theta(\tau\zeta)\tilde{D}_{n\tau,l}{}^{\alpha}(h), \tag{2.14}$$

where

$$\tilde{D}_{p,n\tau}{}^{\alpha}(f)\equiv\sum_j D_{pj}{}^{\alpha}(f)V_{j,n\tau}{}^{\alpha},$$
$$\tilde{D}_{n\tau,l}{}^{\alpha}(h)\equiv\sum_k W_{n\tau,k}{}^{\alpha}D_{kl}{}^{\alpha}(h). \tag{2.15}$$

Equation (2.14) expresses the decomposition of an $O(2,1)$ contribution in terms of a series of factorized $O(1,1)$ contributions, and is valid as an asymptotic relation in $\exp|\zeta|$. If we substitute Eq. (2.14) into (2.9), we obtain the $O(1,1)$ decomposition of the BCP amplitude, a simplification of an expression already given by MM[2]:

$$M^{(N)}\sim\sum_{n_i,\tau_i}\exp(-im_a\phi_a)U_{m_a;n_1,\tau_1}{}^{a,m_0}(E_a,w_a;k_1,w_1)\times\exp[\tau_1\zeta_1(\alpha_1-n_1)]\theta(\tau_1\zeta_1)\times U_{n_1,\tau_1;n_2,\tau_2}{}^{m_1}(k_1,w_1;k_2,w_2)\exp[\tau_2\zeta_2(\alpha_2-n_2)]\theta(\tau_2\zeta_2)\cdots U_{n_{N+1},\tau_{N+1};m_b}{}^{m_{N+1},b}(k_{n+1},w_{n+1};E_b,w_b)\exp(-im_b\phi_b), \tag{2.16}$$

where we have omitted the sum over γ for the sake of clarity. We have defined

$$U_{n,\tau;n'\tau'}{}^{m}=\sum_{l,p'}\tilde{D}_{n\tau,l}{}^{\alpha}(h)G_{lmp'}\tilde{D}_{p',n'\tau'}{}^{\alpha'}(f'). \tag{2.17}$$

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[13] We have, in terms of the lower variables, $\cosh h_{li}=[k_{i+1}\sinh q_i]/[(-t_{l,i+1})^{1/2}\sinh q_{li}]$ and $\cosh f_{l,i+1}=[k_i\sinh q_i]/[(-t_{li})^{1/2}\sinh q_{li}]$.

[14] It would appear that the first two identities, viewed as equations relating the various boost parameters, do not always have a solution. Indeed, when $\mu_{li}\neq0$, $\zeta_{li}=0$, $\nu_{li}=0$, the first equation cannot be solved. There are two reasons for these apparent difficulties. The first has to do with the assumptions about the sign of t_{li} and k_i^2. For $k_i^2<0$ (timelike three-momentum) and $t_{li}<0$ we would replace ζ_i with a z rotation ϕ_i and θ_{li} with a z boost. Equation (2.8a) would then read $f_{li}\phi_ih_{li}=\mu_{li}\zeta_{li}\nu_{li}$, which spans the necessary remaining portion of the $O(2,1)$ group. However, as discussed in CDM, spacelike three-momentum transfers span the most important part of the phase space for small t, and the whole phase space in the limit $t\to0$ if one adheres to the definition (2.5) in this limit. The second reason for the apparent inadequacy of (2.8a) is that our prescription for going from the three-dimensional to the four-dimensional BCP frames does not leave room for an arbitrary z rotation in the four-dimensional BCP frames. This restricts the choice of the BCP $O(2,1)$ transformation.

[15] M. Toller, Nuovo Cimento **37**, 631 (1965).

[16] Note that, although formulas given in the text do not depend formally on the choice of the basis for the representation functions, the actual expression for $\tilde{a}(\zeta)$ of course does. Formulas (A3), (A51), and (A52) are written in Toller's conventions. Since the $\tilde{a}(\zeta)$ is evaluated for a y boost, it differs from the expression given by Toller by a factor $i^{m-m'}$.

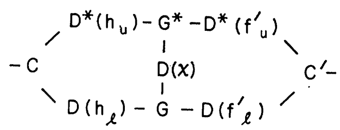

FIG. 3. Index summation scheme for the expression of the residue in Eq. (2.22).

We shall now apply the above results for the decomposition of the production amplitude to the decomposition of the unitarity integrand. In writing the unitarity integrand with the BCP form (2.9) for the production amplitude, one must use care in summing over the intermediate particle helicities m_i. With the convention adopted above, which gives a simple form (2.10) for the conservation of helicity at the vertex, the helicity of particle i is measured with respect to different axes for the lower and upper amplitudes (see Fig. 1). For the lower amplitude, it is measured along the z axis in the frame (pi, l), a rest frame of particle i, which is related to (li, r) by a z boost $v_{l,i}$. The corresponding frame for the upper amplitude (pi, u) differs from the frame (pi, l) by a y rotation,[17] which we designate by χ_i. (That only a y rotation is required is most easily seen by observing that the sequence of transformations $v_u^{-1}h_u^{-1}\theta_u^{-1}\theta_l h_l v_l$ does not affect the y component.) Naturally, this rotation is zero when $t=0$, since in this limit the frames (li, r) and (ui, r) are equivalent. The rotation χ_i depends upon the variables k_i, w_i, k_{i+1}, w_{i+1}, m_i^2, and t,[18] and therefore introduces no new complications for the factorization condition. To sum over the intermediate helicities, we must therefore insert for each intermediate particle the function $D_{m_l m_u}{}^s(\chi)$, and sum over m_l and m_u, where s is the spin of the intermediate particle.

If we now apply the decomposition (2.16) to the lower and upper amplitudes alike and combine the intermediate particle helicities as prescribed above, we obtain the $O(1,1)$ decomposition of the unitarity integrand. To each pair of Regge trajectories α_{li} and α_{ui}, there corresponds an infinite sequence of $O(1,1)$ contributions, the first of which factorizes directly, the second of which is a sum of two factorizable terms, and so on. The degeneracy comes from the "cross terms" in the product of two series of the form (2.14). The meaning of this degeneracy becomes clear when it is understood that the product of two $\tilde{a}$ functions may be represented asymptotically as a sum of $\tilde{a}$ functions,

$$[\tilde{a}_{j_u k_u}{}^{-\alpha_u-1}(\zeta)]^*[\tilde{a}_{j_l k_l}{}^{-\alpha_l-1}(\zeta)]\sim\sum_\nu C'(\alpha_u,\alpha_l,\nu;j_u,j_l,j)\times\tilde{a}_{jk}{}^{-(\alpha_u+\alpha_l-\nu)-1}(\zeta)C(\alpha_u,\alpha_l,\nu;k_u,k_l,k),\quad(2.18)$$

where α_u, α_l are real, and

$$C(\alpha_u,\alpha_l,\nu;k_u,k_l,k)=\delta_{k,k_l-k_u}C(\alpha_u,\alpha_l,\nu;k_u,k_l),\quad(2.19)$$

and similarly for C'.[19] Each $\tilde{a}$ function in the series contributes in turn a single series of factorizable $O(1,1)$ contributions via Eq. (2.13), beginning with the term $\exp[|\zeta|(\alpha_u+\alpha_l-\nu)]$. Rather than working with Eq. (2.16) directly in the unitarity integrand, we adopt the following strategy, which makes the connection with the $t=0$ formalism more transparent. We substitute Eq. (2.12) in Eq. (2.9), expressing the upper and lower BCP amplitude in terms of the $\tilde{a}(\zeta)$'s. Then we combine the upper and lower amplitudes to form the unitarity integrand. If we then apply formula (2.18) to the product of upper and lower $\tilde{a}$ functions at each link, the result is the following unitarity integrand:

$$M^{l(N)}M^{u(N)*}\sim\sum_{j_a,j_b,\gamma_i,j_i,k_i}\exp(-im_a\phi_a)\times U_{m_a j_1}{}^{j_a,\gamma_1}(E_a,k_a;w_1,k_1)\tilde{a}_{j_1 k_1}{}^{-\alpha_{\gamma 1}-1}(\zeta)\times U_{k_1 j_2}{}^{\gamma_1\gamma_2}(w_1,k_1;w_2,k_2)\tilde{a}_{j_2 k_2}{}^{-\alpha_{\gamma 2}-1}(\zeta_2)\cdots U_{k_{N+1},m_b}{}^{\gamma_{N+1}j_b}(w_{N+1},k_{N+1};E_b,k_b)\exp(-im_b\phi_b),\quad(2.20)$$

where we have lumped together into γ the sums over γ_u, γ_l, and ν and have written at each link

$$\alpha_\gamma=\alpha_{\gamma u}+\alpha_{\gamma l}-\nu.\quad(2.21)$$

The vertex functions are (see Fig. 3)

$$U_{kj'}{}^{\gamma\gamma'}(w,k;w',k')=\sum_{j_u',j_l',k_u,k_l,m_u,m_l}C(\alpha_u,\alpha_l,\nu;k_u,k_l,k)\times[\sum_{l_u,p_u'}D_{k_u l_u}{}^{\alpha_u}(h_u)G_{l_u m_u p_u'}D_{p_u' j_u'}{}^{\alpha_u'}(f_u')]^*D_{m_u m_l}{}^s(\chi)\times[\sum_{l_l,p_l'}D_{k_l l_l}{}^{\alpha_l}(h_l)G_{l_l m_l p_l'}D_{p_l' j_l'}{}^{\alpha_l'}(f_l')]\times C'(\alpha_u',\alpha_l',\nu';j_u',j_l',j'),\quad(2.22)$$

with similar expressions for U^{j_a,γ_1} and U^{γ_{N+1},j_b}. We have put $m_a=m_{la}-m_{ua}$ and $m_b=m_{lb}-m_{ub}$, and the sum over j_a and j_b includes the usual channel spins at the ends of the ladder.

If we apply the $O(1,1)$ decomposition (2.13) of the $\tilde{a}$ function to Eq. (2.20), we obtain the form of the unitarity integrand required by CDM for the $O(2,1)$ diagonalization of the multiperipheral equation. We define, accordingly, the incomplete absorptive part $B_{m_a,n\tau}{}^\gamma(a)$ and its partial-wave projections $b_{m_a n\tau}{}^{l\gamma}$. For a discussion of the diagonalization of the integral equation, see CDM and MM[2]. After diagonalization the

[17] We are idebted to Michael Misheloff for assistance on this point.

[18] $\cos\chi_i$

$$=\frac{[2M_i^2(t-t_{li}-t_{ui})+(t_{l,i+1}-t_{li}-M_i^2)(t_{u,i+1}-t_{ui}-M_i^2)]}{\lambda^{1/2}(M_{i,}^2\,t_{li},t_{l,i+1})\lambda^{1/2}(M_i^2,t_{ui},t_{u,\,i+1})}.$$

[19] The coefficients C and C' are related to the vector addition coefficients for the representations of $O(2,1)$. [See Kuo-hsiang Wang, UCRL Report No. UCRL-19306, 1969 (unpublished)]. For practical applications involving a few leading terms, they may be obtained directly by comparing asymptotic expressions for $\tilde{a}(\zeta)$.

equation reads

$$\hat{b}_{m_a,n'\tau'}{}^{l\gamma'}(k',w') = {}_{(0)}\hat{b}_{m_a,n'\tau'}{}^{l\gamma'}(k',w') + \sum_{\gamma,n,\tau} \pi \int dk dw\, \hat{b}_{m_a,n\tau}{}^{l\gamma}(k,w)\, U_{n\tau,n'\tau'}{}^{\gamma\gamma'}(k,w;k',w') \times \hat{d}^l{}_{(\alpha\gamma-n),\tau\tau'(\alpha\gamma'-n')}(q^{-1}), \quad (2.23)$$

where the index n refers to the $O(1,1)$ contributions resulting from a single α_γ. The function $\hat{d}$ is described in CDM. The vertex function U is defined through Eqs. (2.13) and (2.22):

$$U_{n\tau,n'\tau'}{}^{\gamma\gamma'} = \sum_{k,j'} W_{n\tau,k}{}^{\alpha_\gamma} U_{kj'}{}^{\gamma\gamma'} V_{j',n'\tau'}{}^{\alpha_{\gamma'}}. \quad (2.24)$$

The functions $\hat{b}_{m_a,n\tau}{}^{l\gamma}$ are related to the functions $b_{m_a,n\tau+}{}^{l\gamma}$ appearing in the modified $O(2,1)$ expansion (3.8b) and (A45) by

$$b_{m_a,n\tau+}{}^{l\gamma} = \frac{\Gamma[l+1+\tau(\alpha_\gamma-n)]\Gamma[l+1-\tau(\alpha_\gamma-n)]}{\Gamma(2l+2)}\, \hat{b}_{m_a,n\tau}{}^{l\gamma}, \quad (2.25)$$

which follows from Eq. (4.14) of CDM.

III. $O(2,1)$ AMPLITUDES AT $t=0$

We have shown in Sec. II that the $O(2,1)$ and $O(1,1)$ expansions of the production amplitudes are equivalent as asymptotic series in the parameters $\exp|\zeta|$, connected with the subenergies. At $t<0$ we have also defined, through the unitarity integral, the incomplete absorptive part $B_{m_a,n\tau}{}^{\gamma}(a;k,w)$, a function of the overall $O(2,1)$ transformation a, for the nth $O(1,1)$ "daughter" of a given angular momentum $\alpha_\gamma=\alpha_{\gamma u}+\alpha_{\gamma l}-\nu$, resulting from the addition of the upper and lower Regge-pole contributions. At $t=0$, the incomplete absorptive part can be defined either as a function of the $O(2,1)$ transformation a, or in terms of the $O(3,1)$ transformations $a_u=a_l\equiv\tilde{a}$. They are not the same function in different variables because they are constructed by splitting off different factors from the complete absorptive parts, depending upon whether they are derived from a factorized $O(1,1)$ or $O(2,1)$ expansion of the unitarity integrand. By using the explicit form of these expansions, we shall now derive a relation between the two incomplete absorptive parts, which eventually will give the relation between $O(2,1)$ and $O(3,1)$ partial-wave amplitudes.

Since $g_l=g_u\equiv g$ at $t=0$, the Clebsch-Gordan combination of upper and lower amplitudes is simple. Following CD, we assume

$$|M_{m_am_b}{}^{(N)}|^2 \sim \sum_{m_i,\gamma_i} D_{m_am_0}{}^{j_a}(r_a) R_{m_0}{}^{\alpha_{\gamma 1}}(t_1) \times \tilde{a}_{m_0m_1}{}^{-\alpha_{\gamma 1}-1}(g_1) R_{m_1}{}^{\gamma_1\gamma_2}(t_1,t_2)\cdots R_{m_{N+1}}{}^{\gamma_{N+1}b}(t_{N+1}) D_{m_{N+1}m_b}{}^{j_b}(r_b), \quad (3.1)$$

and we define the incomplete absorptive part $B_{m_am}{}^{\gamma}(\tilde{a},t)$ as in CD, by removing the last factors RD in the unitarity integral. If we compare the above expansion of the unitarity integrand with that obtained directly from (2.9) using (2.18), we see, by matching terms in the asymptotic expansion, that

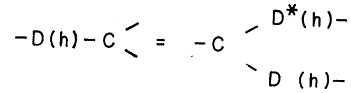

FIG. 4. The property of the Clebsch-Gordan coefficients used in the text.

$$\delta_{lp'} R_{p'}{}^{\gamma\gamma'} = \sum_{l_u,l_l,p_u',p_l',m} C(\alpha_u,\alpha_l,\nu;l_u,l_l,l) \times (G_{l_ump_u'})^* G_{l_lmp_l'} C'(\alpha_u',\alpha_l',\nu';p_u',p_l',p'), \quad (3.2)$$

where the factor $\delta_{lp'}$ follows from (2.11) and (2.19) (helicity conservation), α_u is short for $\alpha_{\gamma u}$, etc., and γ is short for $\{\gamma_u,\gamma_l,\nu\}$.

The $O(1,1)$ expansion of (3.1) can be obtained from (2.12) and (2.13), and we get

$$|M_{m_am_b}{}^{(N)}|^2 \sim \sum_{\gamma_i,n_i,\tau_i} \exp(-im_a\phi_a)\, \bar{U}_{m_a,n_1\tau_1}{}^{\alpha_{\gamma 1}}(k_1,w_1) \times \exp[\tau_1\zeta_1(\alpha_{\gamma 1}-n_1)]\theta(\tau_1\zeta_1) \times \bar{U}_{n_1\tau_1,n_2\tau_2}{}^{\gamma_1\gamma_2}(k_1,w_1,k_2,w_2) \exp[\tau_2\zeta_2(\alpha_{\gamma 2}-n_2)]\theta(\tau_2\zeta_2)\cdots \bar{U}_{n_{N+1}\tau_{N+1},m_b}{}^{\gamma_{N+1}b}(k_{N+1}w_{N+1}) \exp(-im_b\phi_b), \quad (3.3)$$

where

$$\bar{U}_{n\tau,n'\tau'}{}^{\gamma\gamma'} = \sum_p \tilde{D}_{n\tau,p}{}^{\alpha_\gamma}(h) R_p{}^{\gamma\gamma'} \tilde{D}_{p,n'\tau'}{}^{\alpha_{\gamma'}}(f'). \quad (3.4)$$

Equation (3.4) after substitution of (3.2) is to be compared with the expression (2.24) after substitution of (2.22), in the limit $t=0$. With the present procedure we first combine the upper and lower $a^{-\alpha-1}(f\zeta h)$ in the Clebsch-Gordan sequence and then factor the functions $\tilde{D}^\alpha(h)$ for asymptotic ζ's, whereas in Sec. II we performed the same operations in opposite order. The equivalence of the two procedures and the equality of respective $t=0$ residue functions and absorptive parts follow from the property of the coefficients C and C' schematically shown in Fig. 4. Hence we conclude that $\bar{U}_{n\tau,n'\tau'}=U_{n\tau,n'\tau'}$.

Looking at the expansions in Eqs. (3.1) and (3.3) with (3.4) in mind, we see that their equivalence implies that[20]

$$(\sin\theta)\tilde{B}_{m_am}{}^{\gamma}(a\theta h) \sim \sum_{n\tau} B_{m_a,n\tau}{}^{\gamma}(a)\tilde{D}_{n\tau,m}{}^{\alpha_\gamma}(h), \quad (3.5)$$

[20] $\tilde{B}_{m_am}(a\theta h)$ is the same function of $\tilde{a}=a\theta h$ as the CD incomplete absorptive part, but satisfies a different integral equation in which the variables q, θ, ζ replace $\tilde{q}$, $\tilde{\mu}$, $\tilde{\zeta}$, and $\tilde{\nu}$. The reason is that in the CD integral equation the integration over $\tilde{\mu}$ replaces the summation over intermediate helicities, whereas here it has been performed explicitly. The equivalence of the two equations can be proved by noting that, owing to helicity conservation, the equation satisfied by $\tilde{B}(a\theta h)$ in invariant under the substitution $h\to h\beta$, $h'\to h'\beta'$, $f'\to\beta^{-1}f'$, where β and β' are z rotations, and that an extra integration over β can therefore be added. This invariance permitted an arbitrariness with respect to z rotations in the definition of the CD frames.

where the factor $\sin\theta_i = k_i/(-t_i)^{1/2}$ comes from the phase space,[21] and use has been made of the relation [Eq. (2.8c)]

$$\tilde{a} = u_a^{-1} a\theta h \tag{3.6}$$

and of the fact that $u_a = I$ at $t=0$ for pairwise equal masses. Note that, if we parametrize

$$\tilde{a} = \tilde{r}\tilde{\eta}\tilde{g}, \quad \tilde{r}\in O(3), \quad \tilde{\eta}\equiv B_z(\tilde{\eta}), \quad \tilde{g}\in O(2,1),$$

$$a = \phi\eta\xi, \quad \phi\equiv R_z(\phi), \quad \eta\equiv B_x(\eta), \quad \xi\equiv B_y(\xi), \tag{3.7}$$

Eq. (3.5) is valid as a relation between asymptotic series in $e^{|\xi|}$. This follows from Eqs. (2.12) and (2.13), on which (3.3) is based, which are valid in the same asymptotic sense.

Having derived the relation between incomplete absorptive parts, we now proceed to relate the partial-wave expansions. Consistently with the asymptotic meaning of (3.5) we shall perform some manipulations on the $O(2,1)$ and $O(3,1)$ decompositions in order (a) to express the left- and right-hand sides of (3.5) in terms of the residue functions $b_\alpha{}^{\lambda M}$ and $b_{\alpha,n\tau}{}^l$ which are the meaningful quantities in the asymptotic sense and which can be directly deduced from the diagonalized equations (B15) and (2.23), and (b) to extract the h dependence of the left-hand side consistently with the right-hand side.[22]

Problem (a) is solved in Appendix A, in which we prove that, for asymptotic $\tilde{g}$ and ξ,[23] we have [24]

$$\tilde{B}_{m_a m}(\tilde{a}) \sim \sum_{M,s} \int_0^{i\infty} d[\lambda]\, b_{\alpha s}{}^{\lambda M} \bar{D}_{j_a m_a,\alpha s m}{}^{\lambda M}(\tilde{a}), \tag{3.8a}$$

$$B_{m_a,n\tau}(a) \sim \sum_r \int d[l]\, b_{m_a,n\tau r}{}^l D_{m_a,\alpha_{n\tau} r}{}^l(\phi\eta) \times \exp(-\xi\alpha_{n\tau})\theta(\tau\xi), \tag{3.8b}$$

where we have dropped the index γ, we have defined $\alpha_{n\tau} \equiv -\tau(\alpha-n)$, and

$$\bar{D}_{j_a m_a,\alpha s m}{}^{\lambda M}(\tilde{a}) \equiv \sum_{m'} D_{j_a m_a;ls,m'}{}^{\lambda M}(\tilde{r}\tilde{\eta}) \times U_{m'}{}^{\alpha} a_{m'm}{}^{-\alpha-1}(\tilde{g})\, U_m{}^{-\alpha-1}, \tag{3.9}$$

and $s=\pm$ is the label of the two $O(2,1)$'s which occur in the reduction of the $O(3,1)$ group.[25]

Problem (b) is solved by noting that with the parametrization (3.7) in (3.6), when $|\xi|$ is asymptotic so is

$$\tilde{g} = \tilde{f}\xi h, \tag{3.10}$$

where $\tilde{f}\in O(2,1)$.[23] Then substituting Eqs. (2.12) and (2.13) into Eq. (3.9), we obtain the right h dependence in the form

$$\bar{D}_{j_a m_a,\alpha s m}{}^{\lambda M}(\tilde{a}) \sim \sum_{n,\tau} \tilde{D}_{j_a m_a;\alpha s,n\tau}{}^{\lambda M}(\phi\eta\theta) \times \exp(-\xi\alpha_{n\tau})\tilde{D}_{n\tau,m}{}^{\alpha}(h)\theta(\tau\xi), \tag{3.11}$$

where

$$\tilde{D}_{jm;\alpha s,n\tau}{}^{\lambda M}(b) \equiv \sum_{m'} D_{jm;\alpha s,m'}{}^{\lambda M}(b)\, V_{m',n\tau}{}^{\alpha} = 2\pi \lim_{\mu\to\alpha_{n\tau}} (-\tau)[(\mu-\alpha_{n\tau}) D_{jm;\alpha s,\mu+}{}^{\lambda M}(b)], \tag{3.12}$$

and the last equality follows from the definition of $V_{m',n\tau}{}^{\alpha}$ in Eq. (A50).

By making use of the group multiplication properties and noting that θ, K_y,[24] and ζ commute, we obtain

$$D_{j_a m_a;\alpha s,\mu+}{}^{\lambda M}(a\theta) = \sum_{s',r} \int d[l]\, D_{j_a m_a;ls',m_a}{}^{\lambda M}(I) \times D_{m_a,\mu r}{}^l(a) D_{ls',\mu r;\alpha s,\mu+}{}^{\lambda M}(\theta), \tag{3.13}$$

and going to the residues at the poles $\mu = \alpha_{n\tau}$, we get

$$\tilde{D}_{j_a m_a;\alpha s,n\tau}{}^{\lambda M}(\phi\eta\theta) = \sum_{s',r} \int d[l]\, [K_{m_a}(ls',j_a)]^* \times D_{m_a,\alpha_{n\tau} r}{}^l(\phi\eta) \tilde{D}_{ls',\alpha_{n\tau} r;\alpha s,n\tau}{}^{\lambda M}(\theta), \tag{3.14}$$

where the K function is written in Toller's[25] notation. We can now substitute (3.14) into (3.11), and then (3.11) and (3.8a), (3.8b) into (3.5) to get the final result

$$b_{m_a,n\tau}{}^l(k,w) = (\sin\theta) \sum_{M,s,s'} \int d[\lambda]\, [K_{m_a}{}^{\lambda M}(ls',j_a)]^* \times b_{\alpha s}{}^{\lambda M}(t) \tilde{D}_{ls',\alpha_{n\tau}+;\alpha s,n\tau}{}^{\lambda M}(\theta), \tag{3.15}$$

where $b_{m_a,n\tau}{}^l \equiv b_{m_a,n\tau+}{}^l$, and the $(-)$ amplitude can be obtained by the use of the conjugation properties of CDM [see Eq. (A1)]. An expression for $\tilde{D}^{\lambda M}(\theta)$ can be obtained from (3.12) and (C6).

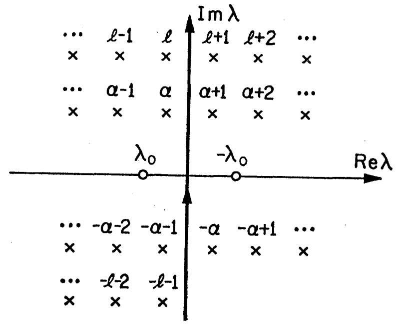

FIG. 5. Location of the poles in the λ plane for the integration of Eq. (3.15).

[21] The volume elements $dt\,\sinh\tilde{q}\,d^3g$ and $dk\,dw\,d\zeta$ are appropriate for the integral equations after the factors $(-t)^{1/2}$ and k are removed from the respective incomplete absorptive parts that come directly from the BCP expansion. For a given a, B and $\tilde{B}$ are therefore normalized in a different way; hence, the factor $\sin\theta$.

[22] Since h depends on the variables k, w, k', and w', this is needed in order to have a relation involving only one set of variables, k and w.

[23] Equations (3.8a) and (3.8b) are valid as asymptotic expressions in $\cosh\zeta$ and $e^{|\xi|}$, respectively, where $\tilde{g} = R_z(\tilde{\mu})B_x(\tilde{\zeta})R_z(\tilde{\nu})$.

[24] The notation is as follows (see Refs. 1–3): The $O(3,1)$ representation functions are labeled by (λ, M), which specifies the unitary representation, by (j,m) for the $O(3)$ basis, by (ls, m) or $(ls, \mu r)$ for the $O(2,1)$ basis. In the last case, l specifies the $O(2,1)$ representation for each $(s=\pm)$ of the two classes of $O(2,1)$ cosets which occur in the $O(3,1)$ group, and $(\mu\pm)$ refers to the two classes of $O(1,1)$ cosets in $O(2,1)$, with a given eigenvalue $(-i\mu)$ of K_y. $d[l]$ and $d[\lambda]$ are the relevant measures in the l and λ planes.

[25] A. Sciarrino and M. Toller, J. Math. Phys. 8, 1252 (1967).

Equation (3.15) solves the problem of connecting the solutions of the $t=0$ diagonalized equations with respect to $O(2,1)$ [Eq. (2.23)] and to $O(3,1)$ [Eq. (B15)]. When $b_{\alpha s}^{\lambda M}$ has Lorentz poles at $\lambda=\pm\lambda_0$,[26] the singularities of $b_{m_a,nr}^{\ l}$ come from the pinchings of the λ contour, and it is evident from Fig. 5 that they may occur at $l=\lambda_0-n-1$ and at the symmetric positions $l=-\lambda_0+n$. Actually, only the sequence $l=\lambda_0-n-1$ can occur in $b_{m_a,nr}^{\ l}$, because this amplitude is, according to CDM, analytic in the right-half l plane. It is possible to show that this is true for Eq. (3.15) by the methods of Appendix A, which are briefly summarized here, for the convenience of the reader.

Whenever we have a summation of the form

$$\sum_{s,M} \int d[\lambda] f_{ls,j}^{\ \lambda M} g_{j,ls}^{\ \lambda M}, \qquad (3.16)$$

where $f_{ls,j}^{\ \lambda M}$ transforms contravariantly under conjugation with respect to $O(3,1)$ (poles at $\lambda=-l+n$ and $\lambda=l+1+n$) and $g_{j,ls}^{\ \lambda M}$ covariantly [poles at $\lambda=l-n$ and $\lambda=-(l+1+n)$], we can replace the s summation above by

$$\beta_l^{\lambda M} g_{j,l+}^{\ \lambda M} (\beta_l^{-\lambda,-M} f_{l+,j}^{\ \lambda M} + \alpha_l^{-\lambda,-M} f_{l+,j}^{\ -\lambda,-M} U_j^{-\lambda,-M}), \qquad (3.17)$$

where $\alpha_l^{\lambda M}$, $\beta_l^{\lambda M}$, and $U_j^{\lambda M}$ are defined in Appendix A and $\beta_l^{\lambda M}$ has zeros in λ at the position of the poles of $g_{j,ls}^{\ \lambda M}$. It is clear, therefore, that the first term of (3.17) has no l-dependent poles in λ, while the second has poles only at $\lambda=l-n$ and $\lambda=-(l+1+n)$.

We now apply this result to the s' summation of Eq. (3.15), with $g^{\lambda M}\to K^{\lambda M}$ and $f^{\lambda M}\to b^{\lambda M}\tilde{D}^{\lambda M}$. We note that, in this case, $f_{j,l+}^{\ -\lambda,-M}$ has poles at $\lambda=\pm\lambda_0$ coming from $b_{\alpha s}^{\lambda M}$, and does not have the poles $\lambda=l-n$ in $\tilde{D}^{\lambda M}$ because of the $(+)$ character of the function $D_{M,\alpha+}^{\ l}$ occurring in its induction construction [see Eq. (C6)]. The first term of Eq. (3.17), which has no l-dependent poles in λ, cannot give rise to poles in l in the left-hand side of (3.15). The singularities in the λ plane of the second term of (3.17) are illustrated in Fig. 6. It is clear that the only possible pinchings are between $\lambda=-(l+1+n)$ and $\lambda=-\lambda_0$, and this implies the above-stated result.

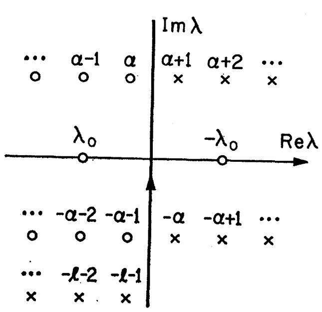

FIG. 6. Location of the poles in the λ plane of Eq. (3.15) after substitution of Eq. (3.17). We have shown with small circles the poles coming from $b_{\alpha s}^{\ l}$, and with crosses all other poles.

The Regge-pole eigenfunctions, which can be calculated by this method in terms of the Lorentz-pole eigenfunction, are useful, in principle, to obtain dynamical quantities such as derivatives of the Regge family at $t=0$. Note finally that the relation between the total absorptive parts which follows from (3.15) is model independent and is, of course, the same as the one obtained from the general group-theoretical analysis.[27]

ACKNOWLEDGMENTS

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APPENDIX A: MODIFIED COMPLETENESS RELATIONS

1. Conjugation Properties in Noncompact Basis

It has been shown by CDM that the $(r=-)$ partial-wave projection of $B_{0,\mu}^{\ l}$ is a linear combination of the $(r=+)$ function and the function with l replaced by $-l-1$, through the relation[28]

$$B_{0,\mu-}^{\ \ l}=\alpha_\mu^{\ l} B_{0,\mu+}^{\ \ l}+\beta_\mu^{\ l} B_{0,\mu+}^{\ \ -l-1}\Gamma(-l)/\Gamma(l+1), \quad (A1)$$

where

$$\alpha_\mu^{\ l}=-\cos\pi\mu/\cos\pi l,$$
$$\beta_\mu^{\ l}=[\pi/\Gamma(-\mu-l)\Gamma(\mu-l)\cos\pi l]. \qquad (A2)$$

Equation (A1) is a consequence of the equivalence of the representations D^l and D^{-l-1}, which, in the mixed basis, may be expressed as follows:

$$D_{m,\mu r}^{\ \ -l-1}(g)=\sum_{r'} (U_m^{\ l})^* D_{m,\mu r'}^{\ \ l}(g)\gamma_{r',\mu r}^{\ \ l}, \quad (A3)$$

where, as in Sciarrino and Toller,[25]

$$U_m^{\ l}=\Gamma(l+1+m)/\Gamma(-l+m) \qquad (A4)$$

and γ is a unitary matrix in the r basis,

$$\gamma_\mu^{\ l}=\begin{pmatrix}\gamma_{1\mu}^{\ l} & \gamma_{2\mu}^{\ l}\\ \gamma_{2\mu}^{\ l} & \gamma_{1\mu}^{\ l}\end{pmatrix}. \qquad (A5)$$

By inspection, one finds

$$\alpha_\mu^{\ l}=(-\gamma_{1\mu}^{\ l}/\gamma_{2\mu}^{\ l})^*, \qquad \beta_\mu^{\ l}=(1/\gamma_{2\mu}^{\ l})^*. \qquad (A6)$$

With $g=I$ in (A3), the expression reads

$$K_{m,\mu r}^{\ \ -l-1}=\sum_{r'} (U_m^{\ l})^* K_{m,\mu r'}^{\ \ l}\gamma_{r',\mu r}^{\ \ l}, \qquad (A7)$$

[26] We assume $\lambda_0<0$, so that the completeness relation is properly convergent [cf. (A33)]. Note that, because of the conventions of Toller (Ref. 22) and MM[1] for the λ plane, $b_{\alpha+}^{\ \lambda M}$ has a pole at $\lambda=-\lambda_0$, and is well behaved in the left-half λ plane.

[27] M. Toller, Nuovo Cimento **53A**, 671 (1968).

[28] The restriction $\eta>0$ in CDM can be removed by choosing the appropriate sheet in continuing to $\eta<0$. However, (A2) and (A5) need slight modifications for fermion representations.

where

$$K_{m,\mu r}{}^{l}=(K_{\mu r,m}{}^{l})^{*}=D_{m,\mu r}{}^{l}(I).\qquad\text{(A8)}$$

Analogous relations hold for the $O(3,1)$ group in the noncompact $O(2,1)$ basis, where the equivalence is between the representations (λ, M) and $(-\lambda,-M)$. We shall show that this equivalence leads to a simplification of the $O(3,1)$ completeness relation in this basis. Instead of (A3) we have

$$D_{jn;lsm}{}^{-\lambda,-M}(a)=\sum_{s'}(U_j{}^{\lambda M})^{*}D_{jn;ls'm}{}^{\lambda M}(a)\Gamma_{s'ls}{}^{\lambda M},\qquad\text{(A9)}$$

where Toller has defined

$$U_j{}^{\lambda M}=\prod_{i=|M|}^{j}\frac{i-\lambda}{i+\lambda},\qquad\text{(A10)}$$

and Γ is unitary in the s basis.[29] Sciarrino and Toller[25] have discussed the function

$$K_m{}^{\lambda M}(l,s;j)=[D_{jm;lsm}{}^{\lambda M}(I)]^{*}\qquad\text{(A11)}$$

and have obtained the identities

$$K_{-m}{}^{\lambda,-M}(l,s;j)=(-)^{M-m}K_m{}^{\lambda M}(l,s;j),\qquad\text{(A12a)}$$

$$K_m{}^{\lambda M}(l,-;j)=(-)^{j-m}K_m{}^{\lambda,-M}(l,+;j),\qquad\text{(A12b)}$$

$$K_m{}^{\lambda M}(-l-1,s;j)=U_m{}^{-l-1}U_{sM}{}^{l}K_m{}^{\lambda M}(l,s;j),\qquad\text{(A12c)}$$

which follow directly from the integral representation

$$K_m{}^{\lambda M}(l,\pm;j)=\tfrac{1}{2}(2s_a+1)^{1/2}\int_1^{\infty}(\cosh\zeta)^{\lambda-1}\times r_{Mm}{}^{j}(\theta^{\pm})d_{\pm M,m}{}^{-l-1}(\zeta)\,d\cosh\zeta,\qquad\text{(A13)}$$

where

$$\tan\tfrac{1}{2}\theta^{+}=\tanh\tfrac{1}{2}\zeta,\qquad\cot\tfrac{1}{2}\theta^{-}=\tanh\tfrac{1}{2}\zeta.\qquad\text{(A14)}$$

In addition the K function has orthogonality properties, implied by the identification

$$K_m{}^{\lambda M}(l,s;j)\leftrightarrow\langle l,s,m\,|\,jm\rangle,\qquad\text{(A15)}$$

which follow from group multiplication properties of the D functions. We shall make use of the property

$$\sum_j K_m{}^{\lambda M}(l,s;j)[K_m{}^{\lambda M}(l',s';j)]^{*}=\delta_{ss'}\delta(l,l').\qquad\text{(A16)}$$

Combining (A9), (A11), and (A16), we obtain the following expression for Γ:

$$\delta(l,l')\Gamma_{s'ls}{}^{\lambda M}=\sum_j[K_m{}^{-\lambda,-M}(l,s;j)]^{*}\times K_m{}^{\lambda M}(l',s';j)U_j{}^{\lambda M}.\qquad\text{(A17)}$$

From (A12a) and (A12b) it then follows that

$$\Gamma_l{}^{\lambda M}=\begin{pmatrix}\Gamma_{1l}{}^{\lambda M} & (-)^{2\epsilon}\Gamma_{2l}{}^{\lambda M}\\ \Gamma_{2l}{}^{\lambda M} & (-)^{2\epsilon}\Gamma_{1l}{}^{\lambda M}\end{pmatrix},\qquad\text{(A18)}$$

where, as in Sciarrino and Toller,

$$(-)^{2\epsilon}=(-)^{2M}.$$

Furthermore, from (A12),

$$\Gamma_{al}{}^{\lambda,-M}=(-)^{2\epsilon}\Gamma_{al}{}^{\lambda M}\quad\text{for }a=1,2,$$

$$\Gamma_{1,-l-1}{}^{\lambda M}=(-)^{2\epsilon}\Gamma_{1l}{}^{\lambda M},\qquad\text{(A19)}$$

$$\Gamma_{2,-l-1}{}^{\lambda M}=\Gamma_{2l}{}^{\lambda M}.$$

If we define

$$\alpha_l{}^{\lambda M}=-(\Gamma_{1l}{}^{\lambda M}/\Gamma_{2l}{}^{\lambda M})^{*},\qquad\beta_l{}^{\lambda M}=(1/\Gamma_{2l}{}^{\lambda M})^{*},\qquad\text{(A20)}$$

so that

$$[D_{jn;l-,m}{}^{\lambda M}(a)]^{*}=\alpha_l{}^{\lambda M}[D_{jn;l+,m}{}^{\lambda M}(a)]^{*}+\beta_l{}^{\lambda M}[D_{jn;l+m}{}^{-\lambda,-M}(a)]^{*}(U_j{}^{\lambda M})^{*},\qquad\text{(A21)}$$

then from (A19) we conclude that

$$\alpha_l{}^{\lambda,-M}=\alpha_l{}^{\lambda M},\qquad\beta_l{}^{\lambda,-M}=(-)^{2\epsilon}\beta_l{}^{\lambda M},$$

$$\alpha_{-l-1}{}^{\lambda M}=(-)^{2\epsilon}\alpha_l{}^{\lambda M},\qquad\beta_{-l-1}{}^{\lambda M}=\beta_l{}^{\lambda M}.\qquad\text{(A22)}$$

In order to continue the foregoing relations to values of l and λ corresponding to nonunitary representations, we make the replacement $l\to -l-1$, $\lambda\to-\lambda$ in expressions involving the complex conjugation of the functions K, Γ, α, and β. Thus

$$(\alpha_l{}^{\lambda M})^{*}=\alpha_{-l-1}{}^{-\lambda,M},\qquad(\beta_l{}^{\lambda M})^{*}=\beta_{-l-1}{}^{-\lambda,M}.\qquad\text{(A23)}$$

To conclude our summation of general properties of the functions α and β we observe that the unitarity of Γ implies the following important relations[30]:

$$\alpha_{-l-1}{}^{-\lambda,M}=-\alpha_l{}^{\lambda M},\qquad 1-(\alpha_l{}^{\lambda M})^2=\beta_l{}^{\lambda M}\beta_{-l-1}{}^{-\lambda,M}.\qquad\text{(A24)}$$

We now propose to show that

$$1-(\alpha_l{}^{\lambda M})^2=\beta_l{}^{\lambda M}\beta_{-l-1}{}^{-\lambda M}=0\qquad\text{(A25)}$$

whenever $\lambda=\pm(l+1+n)$ or $\lambda=\pm(-l+n')$, i.e., at the "kinematical" poles of the representation functions. We show in Sec. 2 of this appendix that this property of α produces the necessary cancellation of the kinematical poles in the $O(3,1)$ completeness relation. The proof of (A25) makes use of the fact[25] that $K_m{}^{\lambda M}(l,s;j)$ has poles at $\lambda+l=n$ and $\lambda-l-1=n$ [see (A13)] but no other poles which move in λ as a function of l. Sciarrino and Toller have defined the residues at these poles

$$\lim_{l\to\lambda-n-1}(l-\lambda+n+1)K_m{}^{\lambda M}(l,+;j)\equiv W_{jm}{}^{\lambda Mn},\qquad\text{(A26)}$$

from which it follows [see (A12c)] that

$$\lim_{l\to-\lambda+n}(l+\lambda-n)K_m{}^{\lambda M}(l,+;j)=-U_m{}^{-\lambda+n}U_M{}^{\lambda-n-1}W_{jm}{}^{\lambda Mn}.\qquad\text{(A27)}$$

From (A9), (A11), (A12b), and (A18) we obtain the

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[29] That $\Gamma^{\lambda M}$ must be independent of m may be verified by putting $a\to ag$ in (A4) and using the irreducibility of D^l.

[30] Explicit calculation from (A9) gives, for $M=0$, $\alpha_l{}^{\lambda 0}=\sin\pi l/\sin\pi\lambda$, and $\beta_l{}^{\lambda 0}=-\Gamma(\lambda)\Gamma(\lambda+1)/[\Gamma(l+1+\lambda)\Gamma(\lambda-l)]$. The properties (A23)–(A25) can be explicitly verified for these expressions.

relation

$$K_m^{-\lambda,-M}(l,+;j)=U_j^{\lambda M}[K_m^{\lambda M}(l,+,j)\Gamma_{1,-l-1}^{-\lambda,M}+(-)^{j-m}K_m^{\lambda,-M}(l,+,j)\Gamma_{2,-l-1}^{-\lambda,M}]. \quad (A28)$$

If we require that Eq. (A28) be consistent with the left-hand side having poles at $-\lambda-l-1=n$ and $-\lambda+l=n$ it follows that $\Gamma_{1,-l-1}^{-\lambda,M}$ or $\Gamma_{2,-l-1}^{-\lambda,M}$ or both must have poles at these points. Let us denote the residues by $r_{1n}^{-\lambda M}$ and $r_{2n}^{-\lambda M}$ in each case. With these definitions we write the residue of (A28) for both signs of M at $-\lambda-l-1=n$:

$$W_{jm}^{-\lambda,-M,n}=U_j^{\lambda M}[K_m^{\lambda M}(l,+,j)r_{1n}^{-\lambda,M}+(-)^{j-m}K_m^{\lambda,-M}(l,+,j)r_{2n}^{-\lambda,M}],$$

$$W_{jm}^{-\lambda,M,n}=(-)^{2\epsilon}U_j^{\lambda M}[K_m^{\lambda,-M}(l,+,j)r_{1n}^{-\lambda,M}+(-)^{j-m}K_m^{\lambda,M}(l,+,j)r_{2n}^{-\lambda,M}], \quad (A29)$$

where we have used (A10) and (A19). Sciarrino and Toller have given the identity

$$W_{jm}^{\lambda,-M,n}=(-)^{j-m+n}W_{jm}^{\lambda Mn}. \quad (A30)$$

With this identity and the orthogonality property (A16) [with (A12b)] we conclude that

$$r_{1n}^{-\lambda,M}/r_{2n}^{-\lambda,M}=(-)^n(-)^{2\epsilon}. \quad (A31)$$

Consequently, from the definition (A15),

$$\lim_{l\to\lambda+1+n}(\alpha_l^{\lambda M})^2=\lim_{l\to\lambda+1+n}(\Gamma_{1,-l-1}^{-\lambda,M}/\Gamma_{2,-l-1}^{-\lambda,M})^2=1. \quad (A32)$$

By the same methods one may verify that (A32) holds for $-\lambda+l=n$. From (A23) and (A24) it follows that (A31) is also valid in all cases when $\lambda\to-\lambda$, which proves (A25).

2. Asymptotic $O(3,1)$ Decomposition in Noncompact Basis

In this part we derive an expansion of the incomplete absorptive part in terms of $O(3,1)$ representation functions, which is asymptotic in the sense of Sec. II. From our final formula (A41) one may also obtain a simple expression for the leading $O(3,1)$ pole contribution.

Into the completeness relation for $O(3,1)$ partial waves,

$$\tilde{B}_{m_am}(a)=\sum_{M,s}\int_0^{i\infty}d[\lambda]\int d[l]\,B_{ls,m}^{\lambda M}D_{j_am_a;ls,m}^{\lambda M}(a), \quad (A33)$$

we substitute the following identities, derived from Eq. (A21) and the definition of $B^{\lambda M}$:

$$B_{l-,m}^{\lambda M}=\alpha_l^{\lambda M}B_{l+,m}^{\lambda M}+\beta_l^{\lambda M}B_{l+,m}^{-\lambda,-M}(U_{j_a}^{\lambda M})^*,$$

$$D_{j_am_a;l-,m}^{\lambda M}=(\alpha_l^{\lambda M})^*D_{j_am_a;l+,m}^{\lambda M}+(\beta_l^{\lambda M})^*D_{j_am_a;l+,m}^{-\lambda,-M}U_{j_a}^{\lambda M}. \quad (A34)$$

The integrand then reads, schematically,

$$\{B_{l+}^{\lambda M}D_{l+}^{\lambda M}[1+\alpha_l^{\lambda M}(\alpha_l^{\lambda M})^*]+B_{l+}^{\lambda M}D_{l+}^{-\lambda,-M}U^{\lambda M}\alpha_l^{\lambda M}(\beta_l^{\lambda M})^*\}+\{B_{l+}^{-\lambda,-M}D_{l+}^{-\lambda,-M}\beta_l^{\lambda M}(\beta_l^{\lambda M})^*+B_{l+}^{-\lambda,-M}D_{l+}^{\lambda M}(U^{\lambda M})^*(\alpha_l^{\lambda M})^*\beta_l^{\lambda M}\}. \quad (A35)$$

By making use of (A22) and (A23), one may readily verify that the terms grouped in the first set of curly brackets are the same as those in the second after replacing (λ,M) by $(-\lambda,-M)$. Consequently, we extend the limits of integration over λ, and keep only the first two terms in (A35):

$$\tilde{B}_{m_am}(a)=\sum_M\int_{-i\infty}^{+i\infty}d[\lambda]\left\{\int_{-1/2}^{-1/2+i\infty}d[l]+\sum_{l=k+}\right\}\times B_{l+,M}^{\lambda M}\beta_l^{-\lambda,M}[\beta_l^{\lambda M}D_{j_am_a;l+,m}^{\lambda M}(a)+U_{j_a}^{\lambda M}D_{j_am_a;l+,m}^{-\lambda,-M}(a)\alpha_l^{\lambda M}]. \quad (A36)$$

We now proceed to shift the contour of integration in l in (A36) so as to collect the residues at the input poles of $B_{l+,m}^{\lambda M}$ at $l=\alpha_\gamma$ and $l=-\alpha_\gamma-1$. If we write $a=r\eta g$, where $r\in O(3)$, $g=R_z(\mu)B_x(\zeta)R_z(\nu)\in O(2,1)$, and η is a z boost, then

$$D_{j_am_a;ls,m}^{\lambda M}(a)=\sum_{m'}D_{j_am_a,lsm'}^{\lambda M}(r\eta)D_{m'm}^l(g), \quad (A37)$$

and we seek an expansion of $\tilde{B}_{m_am}(r\eta g)$ as an asymptotic series in $\cosh\zeta$. Following Toller,[15,25] we first write

$$D_{m'm}^l(g)=a_{m'm}^l(g)+U_{m'}^la_{m'm}^{-l-1}(g)U_m^{-l-1} \quad (A38)$$

and then substitute (A37) into (A36). By making use of identities for the reflection $l\to-l-1$, we obtain

$$\tilde{B}_{m_am}(a)=\sum_M\int_{-i\infty}^{+i\infty}d[\lambda]\left\{\int_{-1/2-i\infty}^{-1/2+i\infty}d[l]+\sum_{l=k\pm}\right\}\times B_{l+,m}^{\lambda M}\beta_l^{-\lambda M}[\beta_l^{\lambda M}\bar{D}_{j_am_a;l+,m}^{\lambda M}(a)+U_{j_a}^{\lambda M}\bar{D}_{j_am_a;l+,m}^{-\lambda,-M}(a)\alpha_l^{\lambda M}], \quad (A39)$$

where

$$\bar{D}_{jm;ls,m}^{\lambda M}(a)=\sum_{m'}D_{jm;ls,m'}^{\lambda M}(r\eta)\times U_{m'}^la_{m'm}^{-l-1}(g)U_m^{-l-1}. \quad (A40)$$

The partial-wave amplitude $B_{l+,m}^{\lambda M}$ has a pole on the left-hand side at $l=\alpha$ in addition to λ-dependent kinematical poles. The $\bar{D}$ function contributes additional λ-dependent poles and also contains "nonsense" poles in l arising from the a function in (A40).[15] However, when the l contour is shifted to the left, the first term in the curly brackets in (A39) contributes a residue only at $l=\alpha$. The kinematical poles are canceled because of (A25), and the nonsense poles are canceled in the usual way by the contributions of the discrete series. The kinematical poles in the second term are not all canceled. However, the residues of these poles are regular in λ and, since $D_{l+}^{-\lambda,-M}B_{l+}^{\lambda M}$ vanishes exponentially as $\text{Re}\lambda \to -\infty$,[31] they give vanishing contributions to the λ integral.

The upshot of this analysis is that, asymptotically in g, we may simply replace the integration over l and summation over the discrete series by the pole contribution at $l=\alpha$, as follows:

$$\tilde{B}_{m_a m}(a) \sim \sum_{M,s} \int_0^{+i\infty} d[\lambda]\, b_{\alpha s}{}^{\lambda M} \bar{D}_{j_a m_a;\alpha s,m}{}^{\lambda M}(a)$$

$$= \sum_M \int_{-i\infty}^{+i\infty} d[\lambda]\, b_{\alpha+}{}^{\lambda M} \beta_\alpha{}^{-\lambda M} \times \{\beta_\alpha{}^{\lambda M} \bar{D}_{j_a m_a;\alpha+,m}{}^{\lambda M}(a) + \alpha_\alpha{}^{\lambda M} \bar{D}_{j_a m_a;\alpha+,m}{}^{-\lambda,-M}\}. \quad \text{(A41)}$$

Equation (A41) is also suitable for obtaining, for η large, the asymptotic contributions coming from dynamical singularities in λ of $B^{\lambda M}$. Shifting the λ contour to the right[31] for the first term and to the left for the second, we see that the α-dependent poles in λ of the first term are canceled as before, whereas the second term has no such poles on the left and no dynamical poles in $b_+{}^{\lambda M}$, either. Except for an additional complication, one would replace the integral over λ by a sum over the residues at the Lorentz singularities of $b_{\alpha+}{}^{\lambda M}$ of the first term only. The complication is that in general it is possible that $\alpha^{\lambda M}$ has extra poles in λ [32] that are absent in $b_{\alpha s}{}^{\lambda M}$. It can be shown, from arguments based on the absence of such poles in $D_{l-}{}^{\lambda M}$ and $b_{\alpha-}{}^{\lambda M}$ in (A34) that, should such extra poles occur, they must be canceled by contributions from similar poles in $\beta_\alpha{}^{\lambda M}$ in the first term in the curly brackets. This circumstance has a precedent in the Mandelstam-Sommerfeld-Watson transform.[32–34]

3. O(1, 1) Decomposition of O(2, 1) Representation Functions

Note first that manipulations analogous to those of Sec. 2 of this appendix can be performed in the case of the $O(2,1)$ expansion of $B_{m,n\tau}(a)$. If we parametrize

$$a \equiv \phi\eta\xi, \quad \phi \equiv R_z(\phi), \quad \eta \equiv B_x(\eta), \quad \xi \equiv B_y(\xi), \quad \text{(A42)}$$

and substitute the conjugation relations (A1) and (A3) into the $O(2,1)$ expansion

$$B_{m,n\tau}(a) = \sum_r \int_{-1/2}^{-1/2+i\infty} d[l] \times \int_{-i\infty}^{+i\infty} (-i)\, d\mu\, B_{n\tau,\mu r}{}^l D_{m,\mu r}{}^l(a), \quad \text{(A43)}$$

we get

$$B_{m,n\tau}(a) = \int_{-1/2-i\infty}^{-1/2+i\infty} d[l] \int_{-i\infty}^{+i\infty} (-i)\, d\mu\, B_{n\tau,\mu+}{}^l \beta_\mu{}^{-l-1} \times [\beta_\mu{}^l D_{m,\mu+}{}^l(\phi\eta) + \alpha_\mu{}^l D_{m,\mu+}{}^{-l-1}(\phi\eta)\, U_m{}^l] e^{-\mu\xi}. \quad \text{(A44)}$$

The first term in the integrand has the l-dependent poles in the μ plane canceled by the factor $\beta_\mu{}^l \beta_\mu{}^{-l-1}$, whereas the second term still has the poles $\mu = \pm(-l+n)$, but their residues are analytic and well behaved in the right-half l plane, so they give vanishing contribution to the integral. Therefore, the asymptotic series of (A44) in $e^{|\xi|}$ is simply obtained by picking up the contribution at the "dynamical" pole

$$\mu = -\tau(\alpha - n) \equiv \alpha_{n\tau},$$

which is nonvanishing only when $\tau\xi > 0$. We have finally

$$B_{m,n\tau}(a) \sim \sum_r \int d[l]\, b_{m,n\tau r}{}^l D_{m,\alpha_{n\tau} r}{}^l(\phi\eta) \times \exp(-\alpha_{n\tau}\xi)\theta(\tau\xi), \quad \text{(A45)}$$

where $b_{m,n\tau r}{}^l$ are the residues of $B_{n\tau,\mu r}{}^l$ at the poles $\mu = \alpha_{n\tau}$.

We want now to obtain the $O(1,1)$ decomposition of the function $\bar{a}_{mm'}{}^{-\alpha-1}(\zeta)$, which occurs in the production amplitudes (2.22) and (3.1) as single Regge-pole contribution. The main purpose is to prove factorization at the $O(1,1)$ poles.

We start from the relation

$$D_{mm'}{}^l(\zeta) = \sum_r \int_{-i\infty}^{+i\infty} (-i)\, d\mu\, K_{m,\mu r}{}^l e^{-\mu\zeta} K_{\mu r,m'}{}^l, \quad \text{(A46)}$$

which follows from the definition (A8) of the transformation functions $K_{m,\mu r}{}^l$. After substitution of the conjugation relation (A7) in the form (A1), the right-hand side of Eq. (A46) can be written in the form

$$\bar{a}_{mm'}{}^l + U_m{}^l \bar{a}_{mm'}{}^{-l-1} U_{m'}{}^{-l-1}, \quad \text{(A47)}$$

where

$$\bar{a}_{mm'}{}^{-l-1} \equiv \int_{-i\infty}^{+i\infty} (-i)\, d\mu\, (\beta_\mu{}^{-l-1} K_{m,\mu+}{}^{-l-1} - \alpha_\mu{}^l K_{m,\mu+}{}^l U_m{}^{-l-1}) e^{-\mu\zeta} \beta_\mu{}^l K_{\mu+,m'}{}^{-l-1}. \quad \text{(A48)}$$

Note that the first term in the integrand has no l-dependent poles in the μ plane, as usual. The second term has only the poles $\mu = \pm(l-n)$ coming from $K_{m,\mu+}{}^l$. By displacing the μ contour either to the right or to the left according to whether $\zeta \gtrless 0$, and neglecting

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[31] We follow here the conventions of Sciarrino and Toller (Ref. 25) and MM[1] for the sign of λ in the induction construction of the representation in the mixed basis. This implies that $D_{l+}{}^{-\lambda,-M}$ is well behaved in the left-half λ plane. Note that this convention is opposite to the usual one for the l plane.

[32] The explicit expression given in Ref. 30 has such poles at integer values of λ. They correspond to the half-integer values in the l plane. At such values $b_{\alpha+}{}^{\lambda M}$ has a symmetry analogous to the Mandelstam (Ref. 33) symmetry, and called "gemel symmetry" by Gatto and Menotti (Ref. 34).

[33] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 254 (1962).

[34] R. R. Gatto and P. Menotti [Phys. Letters **28B**, 668 (1969); **29B**, 592 (1969)] have studied this symmetry in the case $\alpha=0$ where $\alpha_0{}^{\lambda 0}=0$, and therefore the poles at the integers do not appear in our expression. When $\alpha \neq 0$, the absence of such poles in $b_{\alpha-}{}^\lambda$ can be used in Eq. (A34) to prove the gemel symmetry very easily.

the background integrals, we get

$$U_m{}^l\tilde{a}_{mm'}{}^{-l-1}U_{m'}{}^{-l-1}\sim\sum_{n\tau}V_{m,n\tau}{}^l \times\exp[\tau\zeta(l-n)]\theta(\tau\zeta)W_{n\tau,m'}{}^l, \quad (A49)$$

where

$$V_{m,n\tau}{}^l\equiv 2\pi\lim_{\mu\to-\tau(l-n)}(-\tau)[\mu+\tau(l-n)]K_{m,\mu+}{}^l,$$

$$W_{n\tau,m'}{}^l\equiv[-\alpha_\mu{}^l\beta_\mu{}^lK_{\mu+,m'}{}^{-l-1}U_{m'}{}^{-l-1}]_{\mu=-\tau(l-n)}. \quad (A50)$$

Since the asymptotic expansion (A49) contains only the powers $(\exp|\zeta|)^{l-n}$, we can identify $\tilde{a}^l$ with Toller's a^l, and we finally get

$$\tilde{a}_{mm'}{}^{-l-1}\equiv U_m{}^la_{mm'}{}^{-l-1}U_{m'}{}^{-l-1} =\sum_{n\tau}V_{m,n\tau}{}^l\exp[\tau\zeta(l-n)]\theta(\tau\zeta)W_{n\tau,m'}{}^l, \quad (A51)$$

which exhibits the factorization at the $O(1,1)$ pole contributions.

For convenience of the reader, we quote finally the result[35]

$$K_{m,\mu\tau}{}^l=[\Gamma(-l+\mu)\Gamma(-l-\mu)/2\pi2^l\Gamma(-2l)] \times\exp[i\pi(l+m-\mu)/2]F(-l-\tau m,-l+\mu;-2l;2+i0). \quad (A52)$$

APPENDIX B: $O(3, 1)$ DIAGONALIZATION OF $t=0$ EQUATION

We present here a simplified form for the diagonalized $t=0$ equation of Mueller and Muzinich.[1] The simplification parallels the method of CDM for the $t<0$ equation.

Rather than using the Andrews-Gunson E function[36] for the BCP amplitude, we prefer to use the $\tilde{a}$ function (2.10) of the text. We require the $O(2,1)$ decomposition of Toller's a function, which reads

$$a_{mm'}{}^{-l_1-1}(g)=\int_{-1/2}^{-1/2+i\infty}d[l_2]\,K_{mm'}(l_1,l_2)D_{mm'}{}^{-l_2-1}(g) +\sum_{k\pm}K_{mm'}(l_1,k\pm)D_{mm'}{}^{k\pm}(g) \qquad \text{for } \mathrm{Re}l_1=-\tfrac{1}{2}-\epsilon, \quad (B1)$$

where the a function and the D functions for the continuous and discrete series have been given by Toller.[15] From Andrews and Gunson's formulas (3.3), (2.1), and (7.12) we find that with $d[l]\equiv\eta(l)dl$ for $m\geq m'$,

$$K_{mm'}(l_1,l_2)=[2\pi i\eta(l_1)]^{-1}\frac{\Gamma(l_1+m'+1)}{\Gamma(l_1+m+1)} \times\frac{\Gamma(m+l_2+1)}{\Gamma(m'+l_2+1)}\frac{(2l_1+1)}{(l_2-l_1)(l_1+l_2+1)}, \quad (B2)$$

[35] Equation (A52) is obtained from the complex conjugate of Eq. (A20) of CDM after multiplication by the phase factor $\exp(\frac{1}{2}i\pi m)$. The reason is that Mukunda's convention for the $O(2)$ basis differs from Toller's. We are now using Toller's basis, whereas we used Mukunda's basis in Eqs. (A19)–(A20) of CDM. [See N. Mukunda, J. Math. Phys. 8, 2210 (1967).]

[36] M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).

and for $m\leq m'$,

$$K_{mm'}(l_1,l_2)=K_{-m,-m'}(l_1,l_2).$$

A property of the K function which we will find useful may be deduced from the orthogonality relationship between two D functions and the expression given by Toller:

$$D_{mm'}{}^l(g)=a_{mm'}{}^l(g)+U_m{}^la_{mm'}{}^{-l-1}(g)U_{m'}{}^{-l-1}, \quad (B3)$$

where

$$U_m{}^l=\Gamma(l+m+1)/\Gamma(m-l).$$

That property is

$$\lim_{\epsilon\to0+}K_{mm'}(l_1-\epsilon,l_2)+U_m{}^{-l_1-1}K_{mm'}(-l_1-1-\epsilon,l_2)U_{m'}{}^{l_1} =\delta(l_1,l_2), \quad (B4)$$

where $\mathrm{Im}l_1>0$ and $\mathrm{Im}l_2>0$, and $\mathrm{Re}l_1=\mathrm{Re}l_2=-\frac{1}{2}$.

We have defined

$$\int_{-1/2}^{-1/2+i\infty}d[l_1]\,\delta(l_1,l_2)f(l_1)=f(l_2).$$

With this form for the decomposition of the a function and the form (3.1) for the unitarity integrand, the $t=0$ equation as diagonalized by MM[1] reads

$$b_{l'm's'}{}^{\lambda M}(t')={}_{(0)}b_{l'm's'}{}^{\lambda M}(t')+\sum_{ms}\left[\int d[l]+\sum_{l=k\pm}\right] \times\int_{-\infty}^0dt\,\sinh q\,\tilde{b}_{lms}{}^{\lambda M}(t)R_{m'}(t,t') \times d_{l's',ls,m'}{}^{\lambda M}(q^{-1})[-K_{mm'}(-\alpha-1,-l-1)], \quad (B5)$$

where

$$\tilde{b}_{lms'}{}^{\lambda M}(t)=\sum_n b_{l'n's'}{}^{\lambda M}(t)[-K_{nm}(-\alpha-1,-l-1)] \quad (B6)$$

is the amplitude of MM[1], and we have suppressed the γ index for the moment. Recall that [37]

$$d_{l'+,l+,m}{}^{\lambda M}(q^{-1})=\int_1^\infty d\cosh a[d_{+M,m}{}^{l'}(a)]^* \times(\cosh q+\sinh q\cosh a)^{\lambda-1}d_{+M,m}{}^l(a') \qquad \text{for } q>0, \quad (B7)$$

$$\cosh a'\equiv(\sinh q+\cosh q\cosh a)/(\cosh q+\sinh q\cosh a),$$

and similarly for the other representation functions (cf. MM[1]).

Because q is always positive, the d function in Eq. (B5) vanishes for $s'=+$ and $s=-$. As with the $t<0$ equation, the system of equations in s reads

$$b_+{}^{\lambda M}={}_{(0)}b_+{}^{\lambda M}+b_+{}^{\lambda M}K_{++}{}^{\lambda M},$$

$$b_-{}^{\lambda M}={}_{(0)}b_-{}^{\lambda M}+b_+{}^{\lambda M}K_{-+}{}^{\lambda M}+b_-{}^{\lambda M}K_{--}{}^{\lambda M}. \quad (B8)$$

As in Eq. (A1), one can make use of the equivalence of the representations (λ,M) and $(-\lambda,-M)$ [Eq. (A9)]

[37] We keep the conventions of Sciarrino and Toller and MM[1] for the sign of λ (see Ref. 26).

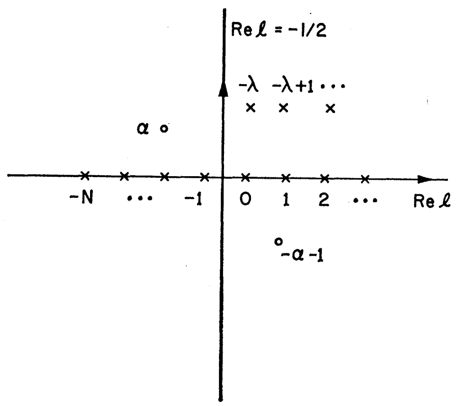

FIG. 7. Location of the poles in the l plane in the integration of Eq. (B13).

to reduce the second equation to the form

$$b_+^{-\lambda,-M} = {}_{(0)}b_+^{-\lambda,-M} + b_+^{-\lambda,-M} K_{++}^{-\lambda,-M}. \qquad \text{(B9)}$$

Therefore, only the first equation in (B8) is needed to determine the locations of the Lorentz poles. We shall henceforth restrict our attention to this equation.

We now wish to present a scheme for shifting the l contour in Eq. (B5) so as to collect only those residues arising from the input Regge poles. As the equation now stands, we are prevented from doing this by the presence of λ-dependent "kinematical" poles in l in the function b, which lie on both sides of the contour. They appear at the same locations as the poles of $d_{l'+,l+,m}^{\lambda M}$ in l', which, from Eq. (B7), appear at

$$l' = -\lambda + n,$$
$$-l' - 1 = -\lambda + n \qquad \text{for } n = 0, 1, \ldots . \qquad \text{(B10)}$$

The two sets of poles are additive with respect to each other, as may be seen by substituting Eq. (B3) into Eq. (B7). This offers the possibility of writing b in Eq. (B5) as a sum of two terms, each of which has only one set of singularities in l. We define

$$\hat{b}_{\bar{l}m}^{\lambda M} = \left[\int_{-1/2}^{-1/2+i\infty} d[l] + \sum_{l=k\pm}\right] b_{lm+}^{\lambda M} K_{Mm}(\bar{l}, l) \qquad \text{for } \mathrm{Re}\,\bar{l} = -\tfrac{1}{2} - \epsilon. \qquad \text{(B11)}$$

This function has kinematical poles at $\bar{l} + \lambda = n$ but none at $-\bar{l} - 1 + \lambda = n$. Moreover, because of Eq. (B4),

$$b_{lm+}^{\lambda M} = \hat{b}_{lm}^{\lambda M} + U_M^{-l-1} \hat{b}_{-l-1,m}^{\lambda M} U_m^{l}. \qquad \text{(B12)}$$

Substituting Eq. (B12) into (B5) and the result into Eq. (B11), we obtain

$$\hat{b}_{l'm'}^{\lambda M}(t') = {}_{(0)}\hat{b}_{l'm'}^{\lambda M}(t') + \sum_m \left[\int_{-1/2-i\infty}^{-1/2+i\infty} d[l] + \sum_{l=k\pm}\right] \int_{-\infty}^{0} dt \sinh q \times b_{lm}^{\lambda M}(t) [-K_{mm'}(-\alpha-1, -l-1)] \times R_{m'}(t, t') \hat{d}_{l'lm'}^{\lambda M}(q^{-1}), \qquad \text{(B13)}$$

where

$$\hat{d}_{l'lm'}^{\lambda M}(q^{-1}) = \int_1^\infty d \cosh a \; a_{+M,m}^{-l'-1}(a) \, d_{+M,m}^{l}(a') \times (\cosh q + \sinh q \cosh a)^{\lambda-1}. \qquad \text{(B14)}$$

We have constructed $\hat{b}_{lm}^{\lambda M}$ so that it has λ-dependent poles in l at $-\lambda + n$ (i.e., in the right-half l plane) only. It also has poles and zeros contributed by a_{Mm}^{-l-1} in the separation (B12). These poles and zeros cancel poles and zeros in the weight function $\eta(l)$ in the usual way,[15] and the resultant l-plane singularity structure of the integrand in Eq. (B13) is indicated in Fig. 7. If we shift the contour to the left, we collect the residues at the nonsense poles at $l = -1, -2, \ldots, -N$.[38] These cancel the contributions of the discrete series, as usual, and we are left with the contribution from the "dynamical" pole at α. In terms of the values at the dynamical poles $\hat{b}_{\alpha\gamma m}^{\lambda M}$, the equation reads

$$\hat{b}_{\alpha\gamma'm'}^{\lambda M}(t') = {}_{(0)}\hat{b}_{\alpha\gamma'm'}^{\lambda M}(t') + \sum_{m,\gamma} \int_{-\infty}^{0} dt \sinh \tilde{q} \times \hat{b}_{\alpha\gamma m}^{\lambda M}(t) R_{m'}^{\gamma\gamma'}(t, t') \hat{d}_{\alpha\gamma'\alpha\gamma m'}^{\lambda M}(\tilde{q}^{-1}), \qquad \text{(B15)}$$

from which an m-independent equation for $\hat{b}_{\alpha\gamma}^{\lambda M} = \sum_m b_{\alpha\gamma m}^{\lambda M}$ can be obtained, having as the kernel

$$\sinh \tilde{q} \sum_m R_m^{\gamma\gamma'}(t, t') \hat{d}_{\alpha\gamma'\alpha\gamma m}^{\lambda M}(\tilde{q}^{-1}). \qquad \text{(B16)}$$

Owing to Eqs. (B12) and (B6), the residue functions $b_{\alpha s}^{\lambda M}$ appearing in the modified completeness relation (A41) are given by[39]

$$b_{\alpha\gamma+}^{\lambda M} = \sum_m [\hat{b}_{\alpha\gamma m}^{\lambda M} + U_M^{-\alpha\gamma-1} \hat{b}_{-\alpha\gamma-1,m}^{\lambda M} U_m^{\alpha\gamma}]. \qquad \text{(B17)}$$

APPENDIX C: REPRESENTATION FUNCTION NEEDED IN TEXT

We derive here an integral representation for an $O(3,1)$ representation function required in Eqs. (3.12) and (3.15) of the text. That function is the matrix element of a y rotation in the noncompact $O(2,1)$ basis:

$$D_{l's',\mu'r';ls,\mu r}^{\lambda M}(\theta) \equiv \langle l's', \mu'r' \,|\, \exp(-i\theta J_y) \,|\, ls, \mu r \rangle \equiv \delta(\mu' - \mu) d_{l's',ls;\mu r'r}^{\lambda M}(\theta). \qquad \text{(C1)}$$

The index $s = \pm$ represents the required doubling of the $O(2,1)$ basis and the index $r = \pm$ the analogous doubling of the $O(1,1)$ basis for the representations of $O(2,1)$.

The procedure for constructing matrix elements of the Lorentz group in the $O(2,1)$ basis by the method

[38] $N = \min(|m|, |M|)$ for $mM \geq 0$ and $N = 0$ for $mM \leq 0$ (see Ref. 15).

[39] In the $O(2,1)$ case, we were able to remove the kinematical poles from the incomplete absorptive part explicitly by factoring out a B function. Since we have not been able to do the same in the $O(3,1)$ case, we do not have an expression analogous to (2.25) relating $b_\alpha^{\lambda M}$ to $b_{\alpha+}^{\lambda M}$. Hence in practice one must substitute (B14) into (B17) to relate $b_{\alpha+}^{\lambda M}$ to $b_\alpha^{\lambda M}$, although we believe the relationship is not fundamentally a dynamical one.

of induced representations has been summarized nicely by MM[1], who give further references. We shall merely sketch those points which must be altered in their treatment for these special representations.

The parametrization of the $O(2,1)$ elements appropriate to the basis required is

$$g_s = \exp(-i\phi J_z)\exp(-ia_s K_x)\exp(-i\lambda K_y), \quad \text{(C2)}$$

where $0<\phi<4\pi$, $-\infty<a_s<+\infty$, and $-\infty<\lambda<+\infty$ spans the group. Note in particular that both signs of a_s are required here. The mapping on g induced by the rotation θ leaves λ and ϕ unchanged. The mapping on a_s is

$$s'\exp(a'_{s'}) = \frac{s\cos\frac{1}{2}\theta\exp(a_s) - \sin\frac{1}{2}\theta}{s\sin\frac{1}{2}\theta\exp(a_s)+\cos\frac{1}{2}\theta}. \quad \text{(C3)}$$

The mapping on the elements in the Hilbert space $\mathcal{H} = \mathcal{L}_2{}^M \oplus \mathcal{L}_2{}^{-M}$ is

$$U^{(\lambda,M)}[\exp(-i\theta J_y)]\{\phi_+(g_+),\phi_-(g_-)\} = \{\sum_s \chi_{+,s}{}^{(\lambda)}(\theta,a_s)\phi_s(g_+'), \sum_s \chi_{-,s}{}^{(\lambda)}(\theta,a_s)\phi_s(g_-')\}, \quad \text{(C4)}$$

where $g'_{s'} = \exp(-i\phi J_z)\exp(-ia'_{s'}K_x)\exp(-i\lambda K_y)$ for $a'_{s'}$ as defined in (C3). We have defined

$$\chi_{s's}{}^{(\lambda)}(\theta,a_s) = (s'\sin\theta\sinh a_s + s's\cos\theta)^{\lambda-1} \times\theta(s'\sin\theta\sinh a_s + s's\cos\theta). \quad \text{(C5)}$$

We use, as a basis for the Hilbert space $\mathcal{H}$, the representations of $O(2,1)$ in the mixed $O(2)\times O(1,1)$ basis, described in MM[2] and CDM:

$$\langle g_+, g_- \mid l+, \mu r\rangle = \{D_{M,\mu r}{}^l(g_+), 0\},$$
$$\langle g_+, g_- \mid l-, \mu r\rangle = \{0, D_{-M,\mu r}{}^l(g_-)\}.$$

In this basis we have, from (C4), the final result

$$d_{l's',ls;\mu'r'}{}^{\lambda M}(\theta) = \int_{-\infty}^{+\infty} d\sinh a_s\,[d_{s'M,\mu r'}{}^{l'}(a_s)]^* \times \chi_{s',s}{}^{(\lambda)}(\theta,a_s)\,d_{sM,\mu r}{}^l(a'_{s'}). \quad \text{(C6)}$$

APPENDIX D: AFS-TYPE MODEL AS EXAMPLE

It has been shown in CDM that the unitarity model of Fubini *et al.*[40] (AFS-type model) can be described easily with the three-dimensional BCP variables. Analogous treatment holds in the $t=0$ case. Since spinless particles are exchanged, the kernel of the multiperipheral equation is g-independent ($\alpha_\gamma=0$ throughout, and no Clebsch-Gordan coefficients are needed); and it contains the off-shell π-π cross section[3] $\tilde{A}_2(\cosh\tilde{q}) = A_2(\cosh q)$ as a factor replacing the δ function which appears in the single-ladder approximation.

The $t=0$ equation can be obtained from (B15)[41] noting that, apart from the factor $\tilde{A}_2(\cosh\tilde{q})$, we require

$$R_0{}^{\gamma\gamma'}\tilde{a}_{00}{}^{-\alpha_{\gamma'}-1}(g') \xrightarrow[\alpha_\gamma,\alpha_{\gamma'}\to 0]{} (t'-\mu^2)^{-2}, \quad \text{(D1)}$$

and since $\tilde{a}_{00}{}^{-\alpha-1}(g)\to 1$ as $\alpha\to 0$, we have

$$R_0{}^{\gamma\gamma'}(t,t')\to(t'-\mu^2)^{-2}. \quad \text{(D2)}$$

Substituting (D2) into (B15), and noting that

$$\hat{d}_{000}{}^{\lambda 0}(q^{-1}) = \int_1^\infty dx(\cosh q + x\sinh q)^{\lambda-1} = e^{\lambda q}/\lambda\sinh q, \quad \text{(D3)}$$

we get the equation

$$b^\lambda(t') = {}_{(0)}b^\lambda(t') + \int_{-\infty}^0 tdt\, b^\lambda(t)\,V^\lambda(t,t')\,(t'-\mu^2)^{-2}, \quad \text{(D4)}$$

where

$$V^\lambda(t,t') \equiv \int_{z_0(t,t')}^\infty \frac{\sinh\tilde{q}\, d\cosh\tilde{q}\,\tilde{A}_2(\cosh\tilde{q})e^{\lambda q}}{\lambda\sinh\tilde{q}},$$
$$z_0 \equiv (4\mu^2 - t - t')/2(tt')^{1/2}, \quad \text{(D5)}$$
$$\sinh\tilde{q}_0 = (s-m^2-t')/2m(-t')^{1/2},$$
$${}_{(0)}b^\lambda(t') \equiv \tilde{A}_2(\sinh\tilde{q}_0)\,(t'-\mu^2)^{-2}.$$

Note that $\hat{b}^\lambda = b^\lambda$, because $\hat{b}_{-1}{}^\lambda = 0$, since $a_{00}{}^0 = 0$.

The $O(3,1)$ expansion now reads

$$\tilde{B}(\tilde{a}) = \int_0^{i\infty} d[\lambda]\, b_s{}^\lambda D_s{}^\lambda(\tilde{a}) = \int_{-i\infty}^{+i\infty} d[\lambda]\, b^\lambda D_+{}^\lambda(\tilde{a}), \quad \text{(D6)}$$

where

$$b^\lambda \equiv b_+{}^\lambda, \qquad D_+{}^\lambda(a) \equiv D_{00;0+,0}{}^{\lambda 0}(a) = D_-{}^{-\lambda}(a). \quad \text{(D7)}$$

Since h in Eq. (3.5) is an $O(2,1)$ transformation and $\alpha=0$, the relation between the two incomplete absorptive parts is rather trivial[42]:

$$\tilde{B}(a\theta) = B(a), \quad \text{(D8)}$$

and the indices m and $n\tau$ are not needed. [More precisely,[43] $B(a)\equiv\sum_\tau B_{0\tau}(a)$ contains both (+) and (−) $O(1,1)$ poles.] The partial-wave amplitude $b^l = b_{0+,+}{}^l$ is then easily obtained, either by direct application of group theory to (D8), or from (3.15) in the limit $\alpha=0$. We have

$$b^l(k,w) = \sum_{s,s'} \int d[\lambda]\,(K_{ls',00}{}^{\lambda 0})^* b_s{}^\lambda(t)\, d_{ls's}{}^\lambda(\theta), \quad \text{(D9)}$$

where the relevant functions, according to (3.17), are[43]

$$d_{l+\,+}{}^\lambda(\theta) = D_{l+,0+;0+,0}{}^{\lambda 0}(\theta), \qquad d_{l+\,-}{}^\lambda(\theta) = d_{l+\,+}{}^\lambda(\pi-\theta). \quad \text{(D10)}$$

[40] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962).

[41] The $t=0$ equation can be obtained directly in a much simpler way [see S. Nussinov and J. Rosner, J. Math. Phys. **7**, 1670 (1966)]. Here we want simply to show how the $\alpha=0$ limit is reached with our formalism.

[42] The absence of the factor $\sin\theta$ of Eq. (3.5) is consistent with the form of Eq. (D4) and of Eq. (4.12) of CDM.

[43] From Eq. (A50) and (A52) one can verify that $V_{0,n_\tau}{}^0 = W_{0,n_\tau}{}^0 = \delta_{n0}$.

From Eq. (C6) we get the explicit expression[44]

$$d_{l+}{}^{+\lambda}(\theta)=\pi^{-1}\int_{\cot\theta}^{\infty} dx\, i^{l+1}Q_l(ix)\,(\sin\theta\, x-\cos\theta)^{\lambda-1}$$

$$=2^l\frac{\Gamma(\lambda)\Gamma(l+1)}{\Gamma(l+\lambda+1)}\frac{(\sin\theta)^l}{\sin\pi(\lambda-l)}C_{\lambda-l-1}{}^{l+1}(\cos\theta), \quad \text{(D11)}$$

where $C_n{}^\nu$ are the Gegenbauer functions.[45]

After the manipulations of the end of Sec. III, we can explicitly calculate the Regge-pole eigenfunctions $f_K(t,\theta)$ of the Kth daughter $l_K=\lambda_0-K-1$ corresponding to a given Lorentz pole of eigenfunction $f_0(t)$. The result is, apart from inessential factors,

$$f_K(t,\theta)\propto f_0(t)\frac{\Gamma(l_K+1)\,i^K}{\Gamma(\frac{1}{2}-\frac{1}{2}K)\Gamma(\frac{1}{2}-\frac{1}{2}K+\lambda_0)}\times(\sin\theta)^{l_K+1}C_K{}^{l_K+1}(\cos\theta). \quad \text{(D12)}$$

Note that the odd daughters are absent because, due to (D7) and (D10), b^l is even under $\theta\leftrightarrow\pi-\theta$ $(w\leftrightarrow -w)$. Note also that (D11) gives a result similar to the Bethe-Salpeter calculation[8] when the initial particles are put on-shell. The latter circumstance explains why only amplitudes even in w are obtained in this simple case.

[44] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, Eqs. 3.7 (31), 3.3 (13), and 3.15 (4).

[45] Reference 44, Sec. (3.15).

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Possible Extension of Minimal Current Algebra and Applications

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An attempt has been made to extend the minimal current algebra of Bjorken and Brandt starting from a gauge-field Lagrangian and including in it nonets of scalar and pseudoscalar fields and making use of canonical communtation relations both for spin-zero and spin-one fields. To apply it to the problem of weak-interaction divergences, we identify suitably normalized fields with weak currents and scalar and pseudoscalar densities introduced by Gell-Mann. As in the case of Bjorken and Brandt, we go to the limit $m_0\to 0$, $g_0\to 0$ such that $g_0/m_0{}^2=\text{const}\neq 0$, where m_0 and g_0 are masses and coupling constants of the Yang-Mills field. In the extended minimal algebra, the nonleptonic weak processes are free of all divergences to lowest order and of a class of leading divergences to all orders in the weak-coupling constant.

I. INTRODUCTION

THE minimal algebra of Bjorken and Brandt[1] has the particularly attractive feature that it makes the electromagnetic mass differences of hadrons finite to lowest order in the fine structure constant. It has been shown in Ref. 1, that this algebra can be obtained as a particular limit of the massive Yang-Mills theory, i.e., as $m_0\to 0$ and $g_0\to 0$ such that m_0/g_0 is nonzero and finite, where m_0 is the mass and g_0 is the coupling constant in the theory. Of course, one uses the field-current identity of Kroll, Lee, and Zumino.[2] The purpose of the present paper is to extend the minimal algebra to include the scalar and pseudoscalar densities defined by Gell-Mann. A convenient way to achieve this goal is to work with a Yang-Mills Lagrangian with the scalar and pseudoscalar fields as matter fields and go to the limit prescribed above. To this end, we first construct an $SU(3)\otimes SU(3)$ symmetric Lagrangian out of vector, axial-vector, scalar, and pseudoscalar fields. We then identify the vector and axial-vector fields with currents and scalar and pseudoscalar fields with corresponding densities introduced by Gell-Mann, Oakes, and Renner.[3] We assume canonical commutation relations for fields, and by the limiting procedure introduced above, we obtain a simpler set of commutation rules for currents and densities. We then apply the resulting commutation relations to study the problem of the leading divergences in weak interaction. We show that nonleptonic processes are finite to lowest order in the weak-coupling constant and are free of leading divergences to all orders. We also show that, to order G^2, there are no Λ^4 and $\Lambda^2\ln\Lambda$ divergences in $\Delta S=1$ processes, where G is the weakcoupling constant. It is obvious from the above that radiative corrections to nonleptonic decays are also free of leading divergences to order G.

II. ALGEBRA OF SCALAR AND VECTOR FIELDS

We start with the following Lagrangian in the simple case with $SU(2)$ symmetry:

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_B, \quad (1)$$

where $\mathcal{L}_0$ is $SU(2)$ symmetric and $\mathcal{L}_B$ is the symmetry-breaking part. We work in terms of a triplet of vector

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[1] J. Bjorken and R. Brandt, Phys. Rev. **177**, 2331 (1969).

[2] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967); T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

[3] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).