tion (4.9) of the γ matrices, the helicity operator in the laboratory frame

$$h_0(p)=\frac{1}{2}\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{|\mathbf{p}|}=\frac{\frac{1}{2}}{|\mathbf{p}|}\begin{pmatrix} p^3 & p_- & 0 & 0\\ p_+ & -p^3 & 0 & 0\\ 0 & 0 & p_3 & p_-\\ 0 & 0 & p_+ & -p^3\end{pmatrix}, \quad (B1)$$

where[20] $p_\pm=p^1\pm ip^2$. Then the operator which measures helicity from a reference frame moving in the z direction with a velocity $v_z=-\tanh(\omega)$ is[21]

$$h_\omega(p)=\exp(-\tfrac{1}{2}\omega\gamma^0\gamma^3)h_0(q)\,\exp(\tfrac{1}{2}\omega\gamma^0\gamma^3), \quad (B2)$$

where

$$q^\mu=\Lambda(\omega)^\mu{}_\nu p^\nu,$$

$$\Lambda(\omega)^\mu{}_\nu=\begin{pmatrix}\cosh\omega & 0 & 0 & \sinh\omega\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\omega & 0 & 0 & \cosh\omega\end{pmatrix}.$$

[20] In this appendix all quantities are referred to the ordinary coordinate system. We omit the carets.

[21] Cf. J. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 18ff.

We compute

$$h_\omega(p)=\frac{\frac{1}{2}}{|\mathbf{q}|}\begin{pmatrix} q^3 & e^{-\omega}q_- & 0 & 0\\ e^{\omega}q_+ & -q^3 & 0 & 0\\ 0 & 0 & q^3 & e^{\omega}q_-\\ 0 & 0 & e^{-\omega}q_+ & -q_3\end{pmatrix}. \quad (B3)$$

Now let $\omega\to\infty$. Then

$$|\mathbf{q}|,\ q^3\to\tfrac{1}{2}e^\omega(p^0+p^3)=2^{-1/2}e^\omega\eta$$

and

$$h_\omega(p)\to h_\infty(p)=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0\\ \sqrt{2}p_+/\eta & -1 & 0 & 0\\ 0 & 0 & 1 & \sqrt{2}p_-/\eta\\ 0 & 0 & 0 & -1\end{pmatrix}. \quad (B4)$$

We can now verify that the spinors listed in (4.51) are eigenstates of $h_\infty(p)$:

$$h_\infty(p)u(p,\pm\tfrac{1}{2})=\pm\tfrac{1}{2}u(p,\pm\tfrac{1}{2}),$$
$$h_\infty(p)v(p,\pm\tfrac{1}{2})=\mp v(p,\pm\tfrac{1}{2}). \quad (B5)$$

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Model Amplitudes Containing Arbitrary Trajectories, Nondegenerate Daughters, and Duality*

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Model amplitudes are discussed which can accommodate nonlinear trajectories, satisfy duality, and have a nondegenerate spectrum of daughter resonances.

CONSIDERABLE activity has been generated recently by Veneziano's suggestion[1] of an elegant approximate form for scattering amplitudes which embodies many of the features thought to be possessed by real scattering amplitudes. This expression is constructed from functions of real linear Regge trajectories and correlates a rich resonance spectrum with Regge asymptotic behavior in a crossing-symmetric manner. On the other hand, not only is it restricted to linear trajectories, but also it fails to accommodate Regge behavior for real energies (except in an average sense) or finite-width resonances. Moreover, the requirement of factorized resonance residues demands a degeneracy of secondary resonances at $J-n$ (J being the spin of the leading resonance at a particular mass value) that increases very rapidly with n.[2] It is, therefore, of interest to explore alternative model amplitudes which contain additional desirable properties not possessed by the Veneziano formula while retaining most of its virtues.

The fundamental reason for both the requirement of linear trajectories and the high degeneracy in the Veneziano model is that the resonance residues are polynomials in the trajectory functions. The degeneracy is introduced in the transformation from the momentum transfer variable to the cosine of the scattering angle. We propose a model which can accommodate nonlinear rising trajectories with right-hand cuts and even left-

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

hand cuts if desired.[3] Each resonance residue is a function of the external masses and the energy but is independent of the crossed-channel trajectory function. At each mass there is a single Toller pole or, in other words, a nondegenerate sequence of daughter poles.[4] Any model with a single trajectory function and the correct analyticity, but no fixed singularities, must contain at least one Toller pole if it is to describe an arbitrary scattering process. Thus, our model with a single Toller pole can be considered to contain a minimal set of daughters. Moreover, those features of the Veneziano model, like the connection between exotic resonances and exchange degeneracy which are reflections of duality and are derivable from duality diagram considerations, are retained in our model.[5] We therefore ignore isospin complications in this paper.

We introduce the concept of compensating functions γ_{ij} and $\tilde{\gamma}_{ij}$. These are functions of the two complex variables α_i and α_j with the following properties: (i) γ_{ij} and $\tilde{\gamma}_{ij}$ are symmetric functions of α_i and α_j; (ii) when $\alpha_i=n$, a positive integer, $\gamma_{ij}=\alpha_j$ and $\tilde{\gamma}_{ij}=n$; (iii) for $\arg\alpha_i\neq0$, $|\alpha_i|\to\infty$, α_j fixed but not a positive integer, $\gamma_{ij}\to\alpha_i$ and $\tilde{\gamma}_{ij}\to\alpha_j$. Convenient representations of γ_{ij} and $\tilde{\gamma}_{ij}$ are

$$\gamma_{ij}=[\alpha_i f(\alpha_j)+\alpha_j f(\alpha_i)]/[f(\alpha_i)+f(\alpha_j)], \quad (1)$$

$$\tilde{\gamma}_{ij}=[\alpha_i f(\alpha_i)+\alpha_j f(\alpha_j)]/[f(\alpha_i)+f(\alpha_j)], \quad (2)$$

where $f(n)=\infty$ and $xf(x)\to0$ as $|x|\to\infty$ and $\arg x\neq0$. A possible choice of the function $f(x)$ is $[\Psi_m(-x)]^{1/\beta}$, where $\Psi_m(z)$ is the mth-order polygamma function and $m>\beta>2$. Using the asymptotic behavior of $\Psi_m(z)$, we find that in the limit $|\alpha_i|\to\infty$, $\gamma_{ij}=\alpha_i-\epsilon_i$ and $\tilde{\gamma}_{ij}=\alpha_j+\epsilon_i$, where

$$\epsilon_i=\frac{(\alpha_i-\alpha_j)f(\alpha_i)}{f(\alpha_i)+f(\alpha_j)}\sim\frac{\exp[i(\pi/\beta)][(m-1)!]^{1/\beta}}{[\Psi_m(-\alpha_j)]^{1/\beta}\alpha_i^{m/\beta-1}}.$$

We restrict m to be even.

We first construct the invariant amplitude for $\pi\pi\to\pi\omega$ scattering. As in the Veneziano model, it is given by a sum of three terms[1]:

$$A(s,t,u)=M(s,t)+M(s,u)+M(t,u), \quad (3)$$

where

$$M(s,t)=\beta\left(-\frac{m_\pi^2}{s_0}\right)^{\tilde{\gamma}_{st}-1}\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\gamma_{st})}C_{\tilde{\gamma}_{st}}^{(1)\prime}(\omega_u). \quad (4)$$

The poles come from the Γ functions in the numerator, and the properties of the compensating function ensure the absence of double poles. When $\alpha_s=J$, $\gamma_{st}=\alpha_t$ and the α_t dependence cancels out of the poles residue. $C_{\tilde{\gamma}_{st}}^{(1)\prime}(\omega_u)$ is the derivative of a Gegenbauer function[6] with respect to its argument ω_u which is, in turn, defined for $\pi\pi\to\pi\omega$ to be

$$\omega_u=(2m_\pi^2-u)/2m_\pi^2. \quad (5)$$

In fact, ω_u is just one of the Toller variables.[4] Since the residue at a pole $\alpha_s=J$ is proportional to the derivative of the Gegenbauer polynomial $C_J^{(1)}(\omega_u)$, we have just a single Toller pole.[4] The expansion[7]

$$C_J^{(1)\prime}(\omega_u)=\frac{m_\pi^2}{p_{\pi\pi}(s)p_{\pi\omega}(s)}\sum_{l=0}^{J}\Gamma_i^{lJ}(s)\Gamma_f^{lJ}(s)P_l'(z_s) \quad (6)$$

indicates that the leading resonance of spin J is accompanied by a set of nondegenerate daughter resonances with positive residues. The residue of the pole in the su term of (3) is proportional to $C_J^{(1)\prime}(\omega_t)$, which, when added to (6), guarantees that only odd angular momentum poles are present in (3). Note that α_t need not be a linear function of t; it need not even be real. If α_s passes through $J+i\delta$, only the partial waves up to J resonate. There are no ancestors. Parenthetically we remark that in the Veneziano model, adding an imaginary part to the trajectories in the region of their right-hand cuts but keeping them linear below threshold introduces finite-width resonances but no ancestors. In both models, partial waves higher than J are proportional to $i\delta$ and do not resonate.

If $s\to\infty$ with t fixed, we have $|\alpha_s|\to\infty$, $\gamma_{st}=\alpha_s-\epsilon_s$, and $\tilde{\gamma}_{st}=\alpha_t+\epsilon_s$. Owing to the ϵ factors, the asymptotic behavior is not of a pure Toller pole unless ϵ vanishes faster than any power:

$$M(s,t)\approx\beta(-m_\pi^2/s_0)^{\alpha_t-1}\Gamma(1-\alpha_t)C_{\alpha_t}^{(1)\prime}(\omega_u)+O(\epsilon_s). \quad (7)$$

The terms proportional to ϵ_s do not have the form of Regge poles; however, using the polygamma form of the compensating functions, we find that ϵ_s vanishes like $(\alpha_s)^{1-m/\beta}$. Thus, by increasing m, we can shift this background part of the asymptotic behavior further to the left in terms of singularities in the complex angular momentum plane. Inasmuch as there are singularities other than simple Regge poles in the left-hand angular momentum plane, the presence of such terms in our model amplitude is not disturbing. It may even make the amplitude more realistic.

Since $u\to-\infty$ as $s\to\infty$ with t fixed, the $M(t,u)$ term in (3) approaches (7) with ω_u replaced by ω_s. When

[3] Other crossing-symmetric models which are direct generalizations of the Veneziano model have been developed by M. A. Virasoro, Phys. Rev. **177**, 2309 (1969); S. Mandelstam, *ibid.* **183**, 1374 (1969); D. D. Coon, *ibid.* **186**, 1422 (1969); M. Suzuki, Phys. Rev. Letters **23**, 205 (1969).

[4] M. Toller, CERN Reports Nos. Th. 770 and 780, 1967 (unpublished); G. Domokos, Phys. Rev. **159**, 1387 (1967); M. H. Rubin, *ibid.* **162**, 1551 (1967).

[5] J. Rosner, Phys. Rev. Letters **22**, 689 (1969); H. Harari, *ibid.* **22**, 562 (1969).

[6] *Higher Transcendental Functions*, edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vol. 1, p. 178.

[7] The $\Gamma^{Jl}(s)$ are related to the analytic continuation of the irreducible nonunitary representations of $SL(2,C)$: K. Bitar and G. Tindle, Phys. Rev. **165**, 1835 (1968).

these two terms are added together, we find

$$A(s,t,u)=\frac{2\pi\beta\alpha_t}{\Gamma(\alpha_t)}\frac{[1-\exp(-i\pi\alpha_t)]}{\sin\pi\alpha_t}\left(\frac{s}{s_0}\right)^{\alpha_t-1}$$
$$-2\beta\alpha_t\Gamma(1-\alpha_t)\left(\frac{s}{s_0}\right)^{\alpha_t-1}\left\{\epsilon_s\left[\ln\left(\frac{s}{\alpha_s s_0}\right)+\frac{2}{\alpha_t}\right]\right.$$
$$\left.-\exp(-i\pi\alpha_t)\,\epsilon_u\left[\ln\left(\frac{s}{\alpha_u s_0}\right)+\frac{2}{\alpha_t}\right]\right\},\quad(8)$$

where we have inserted the asymptotic form of the Gegenbauer function. We have explicitly displayed the leading non-Regge term.

The requirement that the *su* term vanish in this limit places additional restrictions on the compensating functions. We have not investigated these in detail, but note that if we use the polygamma function form with m even, then in the limit that the trajectory function has only right-hand cuts $\alpha_s\rightarrow Re^{i\phi}$, $\alpha_u\rightarrow -R$, $R\rightarrow\infty$,

$$\gamma_{su}\rightarrow iR\exp(i\phi/2)\,\frac{\sin[\frac{1}{2}\phi(1+m/\beta)+\pi m/2\beta]}{\cos(\frac{1}{2}\phi m/\beta+\pi m/2\beta)},$$

and

$$[\Gamma(1-\alpha_s)\Gamma(1-\alpha_u)/\Gamma(1-\gamma_{su})]\rightarrow R^{-R\eta},$$

where

$$\eta=\tan\tfrac{1}{2}\phi-\tan[m(\phi+m)/2\beta].$$

Hence, the *su* amplitude can be made to vanish faster than any power for trajectories which rise faster than logarithmically and for suitably chosen values of m and β. Thus our model has the correct asymptotic behavior, again without the requirement of linear, real trajectories. Our model differs from the Veneziano model in that we introduce an arbitrary scale factor s_0 in (4). The scale factor in the Veneziano model is just the inverse of the slope of the linear trajectory function. For nonlinear trajectories, there does not exist such a natural choice of scale factor. However, there is a single s_0 for any given set of reactions related by crossing.

The invariant amplitude defined by (3) and (4) has the usual dynamical branch points from the trajectory functions. In addition, since the Gegenbauer functions are cut from $-\infty\leq\omega\geq -1$,[6] there is a branch point in $M(s,t)$ at $u=4m_\pi^2$. This coincides with the normal threshold branch point. Thus, in the s-channel resonance region, $M(t,u)$ in (3) acts as a complex, nonresonating background term. In addition, we have a series of unphysical cuts arising from the zeros of $\Psi_m(-\alpha_j)$ (m even). Just as the resonances whose position is given by $\Psi_m(-\alpha_j)=\infty$ lie on unphysical sheets, so do the branch points $\Psi_m(-\alpha_j)=0$, $\Psi_m(-\alpha_j)=\infty$. (For m even, these branch points occur for real values of α.) Furthermore, the first branch point lies above the first resonance in energy. Indeed, the choice $f_s=[\Gamma(-\alpha_s)/\Gamma(\lambda-\alpha_s)]^{1/\beta}$ enables us to place the first branch point at an arbitrarily high energy. This series of cuts ensures that any essential singularities arising from the zeros of $f(\alpha_i)+f(\alpha_j)$ occur on unphysical sheets. In fact, these singularities are reached only by passing through the cut in $[\Psi_m(-\alpha_j)]^{1/\beta}$, which in turn is located on the unphysical sheet of the full scattering amplitude.

As a second application of our ideas, we consider the scattering of four spinless particles of arbitrary masses. K-π and π-π scattering are special cases of this example. As mentioned above, we ignore isospin considerations since they are identical in our model and the Veneziano model. The invariant amplitude is the appropriate sum of terms of the form

$$M(x,y)=\beta\left(-\frac{mm'}{s_0}\right)^{\tilde\gamma_{xy}}\frac{\Gamma(1-\alpha_x)\Gamma(1-\alpha_y)}{\Gamma(1-\gamma_{xy})}\,C_{\tilde\gamma_{xy}}^{(1)}(\omega_z),\quad(9)$$

where
$$\omega_z=(m^2+m'^2-z)/2mm',$$

and the masses m and m' are chosen so that the branch point of the Gegenbauer function $C_{\tilde\gamma_{xy}}^{(1)}(\omega_z)$ at -1 corresponds to the lowest threshold in the z channel. Again β is a constant and s_0 an arbitrary scale factor. Equation (9) is written with the first poles in the x and y channels occuring at $J=1$. By an obvious modification of the compensating functions, (9) could describe a process where the first poles in the x and y channel occur at different J. $M(x,y)$ has both poles and Regge asymptotic behavior. Since the Gegenbauer function $C_j(\omega_z)$ has an expansion similar to (6) in terms of Legendre polynomials, there will be a sequence of nondegenerate daughters with positive widths at each mass value.

We have constructed a model of a crossing-symmetric scattering amplitude which, like the Veneziano model, satisfies duality[8] but which can accommodate arbitrary rising trajectories. There are no degenerate daughter poles. If nothing else, this model shows that resonance-Regge pole duality as used in finite-energy sum rules is insufficient to determine the trajectory function.[9] Satellite amplitudes can be constructed, as in the Veneziano model, by changing the argument of the Γ functions in (4). However, the satellite terms are introduced at the expense of increasing the degeneracy of the secondary poles. This model is more complicated to analyze in terms of angular momentum content than the Veneziano model. Our amplitude does not satisfy the Adler self-consistency condition for processes involving pions.[10] However, as discussed by Yellin,[11] this aspect of the Veneziano model may be spurious. Ultimately the choice between various models has to be made on the basis of comparison with experiment. The freedom in constructing the compensating functions and choosing the trajectory functions makes the model proposed here inherently more difficult to test. However, its existence implies that the Veneziano model is far more restrictive than the general physical concepts upon which it is supposedly based.

[8] There seems to be no universal definition of duality, but the observation by D. B. Lichtenberg, R. G. Newton, and E. Predazzi [Phys. Rev. Letters **22**, 1215 (1969)] does not appear to be applicable to our model.

[9] A. R. Swift and R. W. Tucker, Phys. Rev. **186**, 1553 (1969).

[10] C. Lovelace, Phys. Letters **28B**, 265 (1968).

[11] J. Yellin, UCRL Report No. 18637, 1968 (unpublished).