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Mass Dependence in the Dirac Quark Model*

GERALD ROSEN
Drexel Institute of Technology, Philadelphia, Pennsylvania 19104
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The dependence of an eigenvalue hadron mass m on a constituent quark or antiquark mass $M_{(s)}$ must be such that $|\partial m/\partial M_{(s)}|<1$ if the dynamics of a system composed of n' quarks and $(n-n')$ antiquarks is described by an n-particle Dirac Hamiltonian. By evoking the quasi-independent quark-model approximation, we obtain $\partial m_N/\partial M\simeq 2.378$ for the rate at which the nucleon mass would change theoretically subject to a simultaneous increase in the three constituent strangeness-zero quark masses M. In the case of a potential with Coulomb-Schwinger homogeneity under scale transformations, we find a linear mass relation without appeal to the quasi-independent quark-model approximation. The latter mass relation yields correct mass values for the parallel-spin hadrons with $L=0$ (the $J^P=1^-$ mesons and $J^P=\frac{3}{2}^+$ baryons) and exhibits consistency in the $|S|=1$ quark-antiquark mass M' to $S=0$ quark-antiquark mass M ratio: $M'/M=1.345\pm(0.005)$.

LET us suppose that the dynamics of a system composed of n' quarks and $(n-n')$ antiquarks is described by a Hamiltonian of the form

$$H=\sum_{s=1}^{n}(\boldsymbol{\alpha}_{(s)}\cdot\mathbf{p}_{(s)}+\beta_{(s)}M_{(s)})+V^{(n',n-n')},\tag{1}$$

with the potential

$$V^{(n',n-n')}=V^{(n',n-n')}(\mathbf{r}_{(1)},\ldots,\mathbf{r}_{(n)},\boldsymbol{\alpha}_{(1)},\ldots,\boldsymbol{\alpha}_{(n)},\beta_{(1)},\ldots,\beta_{(n)})\tag{2}$$

a real function of the spatial coordinates and Dirac matrices associated with the n particles. The potential (2) is supposed to be independent of the $M_{(1)},\ldots,M_{(n)}$, and differences in the latter quark-antiquark masses are supposed to account for the main mass splitting that is evident in $SU(3)$ groupings of hadrons. Then for a bound state represented by a 4^n-component wave function $\psi=\psi_{a_1\ldots a_n}(\mathbf{r}_{(1)},\ldots,\mathbf{r}_{(n)})$, the dependence of the hadron mass

$$m=(\psi,H\psi)/(\psi,\psi)\tag{3}$$

on the constituent particle masses is such that

$$\frac{\partial m}{\partial M_{(s)}}=\left(\psi,\frac{\partial H}{\partial M_{(s)}}\psi\right)\bigg/(\psi,\psi)=(\psi,\beta_{(s)}\psi)/(\psi,\psi),\tag{4}$$

because the terms in $\partial\psi/\partial M_{(s)}$ cancel out in computing the partial derivatives of (3) by virtue of the Schrödinger equation $H\psi=m\psi$. It follows immediately from (4) and the Schwarz inequality with $\beta_{(s)}{}^2=1$ that

$$|\partial m/\partial M_{(s)}|<1.\tag{5}$$

Thus, a theoretical change in a quark or antiquark mass manifests a smaller absolute change in the hadron mass, a result suggested previously by nonrelativistic and/or spin-zero particle theory.[1-3]

An estimate for the final member in Eq. (4) can be obtained by evoking the quasi-independent quark-model approximation.[4] The latter approximation gives

$$(\psi,\beta_{(s)}\psi)/(\psi,\psi)\cong(\hat{\psi},\beta\hat{\psi})/(\hat{\psi},\hat{\psi})\tag{6}$$

in terms of a one-particle Dirac wave function $\hat{\psi}=\hat{\psi}_a(\mathbf{r})$ solution to the equation

$$[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M_{(s)}+V_{(s)}-E_{(s)}]\hat{\psi}=0,\tag{7}$$

with

$$\sum_{s=1}^{n}E_{(s)}=m$$

and $V_{(s)}=V_{(s)}(\mathbf{r},\beta)$ an effective spherically symmetric scalar potential associated with the over-all average force on the s quark or antiquark. Since the ground-state solution to (7) takes the form

$$\hat{\psi}=\begin{pmatrix}\phi_{(s)}(r)\chi\\ ig_{(s)}(r)\boldsymbol{\sigma}\cdot\mathbf{r}\chi\end{pmatrix},\tag{8}$$

$\chi\equiv$ a constant two-component spinor

in a representation with $\beta=\mathrm{diag}[1,1,-1,-1]$, the second member of (6) is

$$(\hat{\psi},\beta\hat{\psi})/(\hat{\psi},\hat{\psi})=1-3\delta_{(s)},\tag{9}$$

where

$$\delta_{(s)}\equiv\tfrac{2}{3}\int|g_{(s)}(r)|^2r^2d^3r\Big/\int\{|\phi_{(s)}(r)|^2+|g_{(s)}(r)|^2r^2\}d^3r\tag{10}$$

is the quasi-independent quark-model parameter[4] for the s quark or antiquark. Combining the exact equations (4) and (9) via the approximation (6), we obtain

$$\partial m/\partial M_{(s)}\cong 1-3\delta_{(s)}.\tag{11}$$

If we specialize Eq. (1) to the baryons by putting $n'=n=3$ and assume that $M_{(1)}\cong M_{(2)}\cong M_{(3)}\equiv M$ and $V_{(1)}\cong V_{(2)}\cong V_{(3)}$ in Eq. (7) for the proton state, then irrespective of the magnitude of M or the form of $V^{(3,0)}$ in Eq. (1), we have $\delta_{(1)}\cong\delta_{(2)}\cong\delta_{(3)}\equiv\delta$ related to the magnetic moment of the proton by the relation[4]

$$\mu_p=3(1-\delta)\ \text{proton magnetons.}\tag{12}$$

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[1] S. Tanaka, Progr. Theoret. Phys. (Kyoto) **35**, 975 (1966).
[2] M. Hirayama, Progr. Theoret. Phys. (Kyoto) **36**, 1219 (1966).
[3] L. J. Tassie and D. B. Lichtenberg, Australian J. Phys. **19**, 599 (1966).
[4] P. N. Bogolioubov, Ann. Inst. Henri Poincaré **8**, 163 (1968).

Therefore, we find the empirical value $\delta=0.0691$ for the proton, and

$$\frac{\partial m_N}{\partial M}\cong\sum_{s=1}^{3}\frac{\partial m_N}{\partial M_{(s)}}\cong 3(1-3\delta)=2.378 \qquad (13)$$

for the rate at which the nucleon mass would change theoretically subject to an increase in the strangeness-zero quark mass.

It is uncertain whether the quasi-independent quark-model approximation is applicable to a quark-antiquark dynamical system governed by a Hamiltonian of the form (1). Moreover, because it is known that M is greater than about 10 BeV, and hence that $m_N/M\lesssim\frac{1}{10}$, the predicted numerical value (13) for $\partial m_N/\partial M$ is surprisingly large. The quasi-independent quark-model approximation yields essentially nonrelativistic dynamics, with the parameters defined by (10) small compared to unity, while it has been shown that familiar forms for the potential in Eq. (1) (Coulomb, Yukawa, or exponential) require extremely relativistic motion to produce the observed hadron masses.[5]

For a potential with Coulomb-Schwinger[6] homogeneity under scale transformations,

$$V^{(n',n-n')}(\lambda\mathbf{r}_{(1)},\ldots,\lambda\mathbf{r}_{(n)},\boldsymbol{\alpha}_{(1)},\ldots,\boldsymbol{\alpha}_{(n)},\beta_{(1)},\ldots,\beta_{(n)})$$
$$\equiv\lambda^{-1}V^{(n',n-n')}(\mathbf{r}_{(1)},\ldots,\mathbf{r}_{(n)},\boldsymbol{\alpha}_{(1)},\ldots,\boldsymbol{\alpha}_{(n)},\beta_{(1)},\ldots,\beta_{(n)}) \qquad (14)$$

for all real $\lambda>0$, we can establish a linear mass relation, without appeal to the quasi-independent quark-model approximation. It follows from (1), (14), and the stationary character of (3) subject to variations $\psi\to\psi+\delta\psi$ that[7]

$$(\psi,\sum_{s=1}^{n}\boldsymbol{\alpha}_{(s)}\cdot\mathbf{p}_{(s)}\psi)+(\psi,V^{(n',n-n')}\psi)=0 \qquad (15)$$

for a bound state, and thus (3) becomes

$$m=\sum_{s=1}^{n}M_{(s)}(\psi,\beta_{(s)}\psi)/(\psi,\psi). \qquad (16)$$

Hence, Eqs. (4) and (16) imply that the quantities $(\psi,\beta_{(s)}\psi)/(\psi,\psi)\equiv c_{(s)}$ are independent of the masses $M_{(1)},\ldots,M_{(n)}$ for a potential with the homogeneity property (14), and we obtain the linear mass relation

$$m=\sum_{s=1}^{n}c_{(s)}M_{(s)},\qquad |c_{(s)}|<1. \qquad (17)$$

Assuming that the $c_{(s)}$ are all equal in the case of hadrons with $L=0$ and all quark and antiquark spins parallel, we have $c_{(1)}=c_{(2)}\equiv c_m$ for the $J^P=1^-$ mesons, $c_{(1)}=c_{(2)}=c_{(3)}\equiv c_b$ for the $J^P=\frac{3}{2}^+$ baryons, and (17) yields the following empirically correct mass values for suitable c_m and c_b:

$$J^P=1^-\left\{\begin{array}{l} m_\rho=2c_mM=760\\ m_{K^*}=c_m(M+M')=890\\ m_\phi=2c_mM'=1020\end{array}\right\}\Rightarrow\frac{M'}{M}=1.34,$$

$$J^P=\tfrac{3}{2}^+\left\{\begin{array}{l} m_{N^*}=3c_bM=1240\\ m_{Y_1^*}=c_b(2M+M')=1385\\ m_{\Xi^*}=c_b(M+2M')=1530\\ m_\Omega=3c_bM'=1675\end{array}\right\}\Rightarrow\frac{M'}{M}=1.35. \qquad (18)$$

In (18), the mass values are expressed in MeV, M denotes the $S=0$ quark-antiquark mass, and M' denotes the $|S|=1$ quark-antiquark mass. Consistency of the M'/M ratio here encourages further investigation of the Dirac quark model and, in particular, potentials with the homogeneity property (14).

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[5] O. W. Greenberg, Phys. Rev. **147**, 1077 (1966).

[6] A magnetostatic potential with the homogeneity property (14) has been proposed by J. Schwinger [Science **165**, 757 (1969); **166**, 690 (1969)]. However, electromagnetic retardation effects may preclude a dynamical description based on a Hamiltonian of the form (1).

[7] G. Rosen, J. Math. Phys. **7**, 2066 (1966).