Stability of the Schwarzschild Metric

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The stability of the Schwarzschild exterior metric against small perturbations is investigated. The perturbations superposed on the Schwarzschild background metric are the same as those given by Regge and Wheeler, consisting of odd- and even-parity classes, and with the time dependence $\exp(-ikt)$, where k is the frequency. An analysis of the Einstein field equations computed to first order in the perturbations away from the Schwarzschild background metric shows that when the frequency is made purely imaginary, the solutions that vanish at large values of r, conforming to the requirement of asymptotic flatness, will diverge near the Schwarzschild surface in the Kruskal coordinates. Since the background metric itself is finite at this surface, the above behavior of the perturbation clearly contradicts the basic assumption that the perturbations are small compared to the background metric. Thus perturbations with imaginary frequencies that grow exponentially with time are physically unacceptable, and hence the metric is stable. Perturbations with real values of k representing gravitational waves are also examined. It is shown that the only nontrivial stationary perturbation that can exist is one that is due to the rotation of the source, which is given by the odd perturbation with the angular momentum $l=1$. The significance of solutions with complex frequencies is pointed out, as is the lack of a theorem (completeness of the eigenfunction) for the even-parity case to parallel the Sturm-Liouville theory, which is applicable to the odd-parity case. Such a theorem would be required to convert the computations indicating stability as given here into a fully rigorous stability theorem.

I. INTRODUCTION

IN recent years the phenomenon of gravitational collapse as described within the framework of classical general relativity has been a topic of great interest.[1] The geometry surrounding a collapsing, nonradiating, spherical object is given by the Schwarzschild exterior metric.[2] To an external observer, a spherical mass configuration that has collapsed into the Schwarzschild horizon in the remote past is represented by the Schwarzschild exterior geometry extending from $r=2m$ to spatial infinity. Granted that physical phenomena do exist that ensure the possible formation of such a collapsed spherical mass and the consequent production of the Schwarzschild exterior space-time down to $r=2m$, the question naturally arises whether such an object can continue to exist. Unless the collapsed spherical configuration, or, equivalently, the Schwarzschild empty-space metric, is proved to be stable against small perturbations, one cannot continue to treat them as entities that nature can allow to exist. *A priori* there is no definite reason to believe that the Schwarzschild space-time does represent a stable configuration, and in the present work we prove formally that it is in fact stable against small perturbations.

The stability of the Schwarzschild metric was originally studied by Regge and Wheeler[3] in 1957, but the problem remained far from solved at that time. Their work presented the standard method of decomposing any given perturbation on a spherically symmetric background metric into its normal modes using tensor spherical harmonics. On the other hand, the main factor that prevented the problem from being solved was the lack of a suitable way to formulate and apply the boundary condition at the Schwarzschild surface $r=2m$. More specifically, since the background metric expressed in the usual Schwarzschild coordinates exhibits an apparent singularity at $r=2m$, it was impossible to judge whether any divergence shown by the perturbation at this surface was real or only a spurious effect caused by improper choice of coordinates. The discovery of the Kruskal coordinates[4] has since then remedied the situation. In these coordinates the background metric is singularity free down to the point $r=0$ and the surface $r=2m$ no longer displays the apparent pathologies that were originally present. The correct way to solve the problem is to carry out the perturbation analysis entirely in the Kruskal reference frame, which, however, would be a difficult task, since in the Kruskal coordinates the metric does not display its time independence in a simple way. Equivalently, we solve for the perturbations employing the Schwarzschild coordinates, transform the solutions thus found to the Kruskal coordinates, and study their behavior, which would now be free from effects due to improper choice of coordinates. Secondly, the differential equations for the radial factors of the perturbations contained errors as they appeared in the literature. The

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[1] For some recent discussions of gravitational collapse see C. W. Misner in *Astrophysics and General Relativity*, edited by M. Chretien, S. Deser, and J. Goldstein (Gordon and Breach, New York, 1969), Vol. 1, p. 113; K. S. Thorne, in *High-Energy Astrophysics*, edited by C. DeWitt, E. Schatzman, and P. Véron (Gordon and Breach, London, 1967), Vol. III, Chap. 8.

[2] G. D. Birkhoff, *Relativity and Modern Physics* (Harvard U. P., Cambridge, Mass., 1923).

[3] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1903 (1957).

[4] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).

correct set of these differential equations has been derived by Edelstein and the present author.[5] With the help of these corrected equations, and by employing the Kruskal coordinates to examine the behavior of the perturbations near the Schwarzschild surface, it has been possible to establish the stability of the Schwarzschild metric.

In Sec. II we outline the approach to the problem of stability taken by Regge and Wheeler and indicate the modifications necessary in applying the boundary conditions. In Sec. III the Kruskal coordinates are introduced and their role in the analysis of the perturbations is described. In Sec. IV we take up the problem of stability as originally formulated by Regge and Wheeler. The question raised here is the same as that raised by those authors: With the time dependence of the perturbations as $\exp(-ikt)$, where k is the frequency, are purely imaginary frequencies that make the perturbations grow exponentially with time admissible? By analyzing the linearized Einstein field equations for the perturbations, and examining the asymptotic behavior of their solutions in Kruskal coordinates, we prove that such perturbations are physically unacceptable and hence that the Schwarzschild metric is stable. In the course of our discussion of stability we also study perturbations with real frequencies corresponding to gravitational waves. Section V is devoted to stationary perturbations with $k=0$. It will be shown that the only nontrivial stationary perturbation is due to the rotation of the source. In Sec. VI we touch upon the perturbations with complex frequencies and, finally, indicate the work that is yet to be done concerning the problem of stability. Throughout this paper only the immediately relevant field equations for perturbations are cited. The details regarding these field equations can be found in Ref. 5.

II. REGGE-WHEELER APPROACH TO STABILITY OF SCHWARZSCHILD METRIC

The Schwarzschild metric is written in its usual form as

$$ds^2=-(1-2m/r)dt^2+(1-2m/r)^{-1}dr^2 +r^2(d\theta^2+\sin^2\theta d\phi^2)=g_{\mu\nu}dx^\mu dx^\nu, \quad (2.1)$$

with

$$x^0=t, \quad x^1=r, \quad x^2=\theta, \quad x^3=\phi,$$

and

$$c=G=1.$$

This corresponds to the initial time-independent equilibrium configuration. The problem is, then, if this metric is perturbed, whether the perturbations will execute undamped oscillations about the equilibrium state represented by the Schwarzschild background (stability) or will grow exponentially with time (instability). The analysis of the problem proceeds in the following steps.

1. Perturbations. Regge and Wheeler have given the normal modes into which any arbitrary perturbation on a spherically symmetric background can be decomposed. The explicit forms of these modes will be given in Sec. IV. Here we note a few properties of these normal modes. These can be expressed in the form of products of four factors, each of which is a function of one of the coordinates t, r, θ, and ϕ; this separation is achieved by the use of generalized tensor spherical harmonics. Associated with any of these modes, we have the angular momentum l and its projection on the z axis M. For simplicity one can choose $M=0$, since the particular value of M chosen does not alter the final results. For any given value of l there are two independent classes of perturbation characterized by their parities $(-1)^l$ and $(-1)^{l+1}$ which are designated as the even- and odd-parity perturbations, respectively. Furthermore, a great degree of simplification in the form of the perturbation matrix can be achieved by making suitable gauge transformations which will reduce the general perturbation to the Regge-Wheeler canonical form which will have fewer matrix elements than the former. All calculations will be carried out in the canonical gauge. Finally, as has been mentioned before, the time dependence of the perturbations is given by $\exp(-ikt)$, since the background is independent of time.

2. Field equations. The next step in the analysis of the stability problem is to obtain the equations governing the above perturbations. Let us denote the Schwarzschild background metric by $g_{\mu\nu}$ and the superimposed perturbation by $h_{\mu\nu}$. The Einstein field equations for the Schwarzschild exterior metric are given by

$$R_{\mu\nu}(g)=0.$$

Here the Ricci tensor $R_{\mu\nu}$ has been computed from the Schwarzschild background metric $g_{\mu\nu}$ and this is indicated by the g in parentheses. For the perturbed space-time, the field equations would read

$$R_{\mu\nu}(g+h)=0,$$

where the computations are carried out using the total metric $g_{\mu\nu}+h_{\mu\nu}$. Here we have made the assumption that the perturbed space-time is still empty. Since the perturbations are assumed to be small so that the second- and higher-order terms in $h_{\mu\nu}$ can be neglected, the above equations can be expanded as

$$R_{\mu\nu}(g)+\delta R_{\mu\nu}(h)=0,$$

where $\delta R_{\mu\nu}(h)$ contain only the first-order terms in $h_{\mu\nu}$. Since $R_{\mu\nu}(g)=0$, the differential equations governing the perturbations are obtained from the equations $\delta R_{\mu\nu}(h)=0$. In order to compute $\delta R_{\mu\nu}$, the formulas

[5] L. A. Edelstein and C. V. Vishveshwara, Phys. Rev. D 1, 3514 (1970).

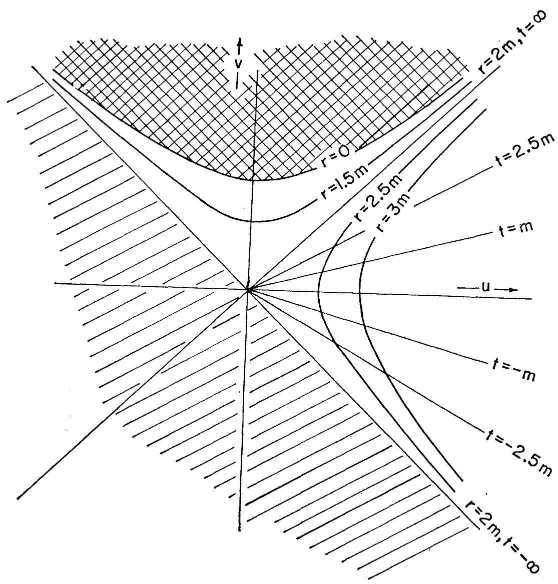

FIG. 1. Kruskal diagram for a spherical mass distribution that is assumed to have collapsed into the Schwarzschild horizon in the infinitely remote past. The world line of the collapsing surface has coalesced with the line $u=-v$. The hatched region $u+v<0$ should be replaced by an appropriate space-time geometry for the interior of this mass distribution. The unhatched region in the half-plane $u+v>0$ up to the hyperbola corresponding to the physical singularity $r=0$ represents the Schwarzschild empty space. The metric is not defined in the cross-hatched region.

given by Eisenhart[6] are employed:

$$\delta R_{\mu\nu}=-\delta\Gamma_{\mu\nu}{}^{\beta}{}_{;\beta}+\delta\Gamma_{\mu\beta}{}^{\beta}{}_{;\nu}. \tag{2.2}$$

Here the semicolons denote covariant differentiation, and the variation in the Christoffel symbol $\delta\Gamma_{\mu\nu}{}^{\beta}$ stands for the expression

$$\delta\Gamma_{\mu\nu}{}^{\beta}=\tfrac{1}{2}g^{\beta\alpha}(h_{\mu\alpha;\nu}+h_{\nu\alpha;\mu}-h_{\mu\nu;\alpha}). \tag{2.3}$$

The differential equations thus obtained will contain the frequency k as a constant parameter. As shown in Ref. 5, starting from these equations, which are coupled in the radial factors of the perturbations, a single second-order linear differential equation containing only one radial factor can be derived both in the odd- and the even-parity cases. The final task will then be to analyze this differential equation and see whether the frequency k can be purely imaginary or not. Before this can be done, we must formulate the proper boundary conditions for the perturbations.

3. Boundary conditions. The boundary conditions originally formulated by Regge and Wheeler are as follows: The two boundaries chosen are the spatial infinity and the Schwarzschild surface $r=2m$. A physically acceptable perturbation should be well behaved at both these points at the starting instant. In the first place, the perturbations were required not to diverge for large values of r. Secondly, in order to impose the boundary condition at $r=2m$, the Schwarzschild space was visualized as "inwardly unbounded" and considered as "one mouth of a wormhole, the other mouth of which emerged elsewhere." With this representation of the Schwarzschild space, it was demanded that one should be able to join the solution for $r>2m$ smoothly onto a solution in the other half of the tunnel. The authors claimed that the solution that vanished for large r also went to zero at the Schwarzschild radius, so that the above requirement could not be satisfied. However, as we shall see later, the solutions that go to zero at spatial infinity *do not* fall off at $r=2m$, but, on the contrary, diverge as expressed in the Schwarzschild coordinates. Moreover, as pointed out in the Sec. I, since the background metric itself contains an apparent singularity at $r=2m$, the behavior of a perturbation—divergent or otherwise—as expressed in the Schwarzschild coordinates is liable to be spurious and unphysical. We must therefore formulate and apply the boundary condition at $r=2m$ within the framework of a coordinate system which is singularity free at that point. We shall do this employing the Kruskal coordinates.

[6] L. P. Eisenhart, *Riemannian Geometry* (Princeton U. P., Princeton, N.J., 1962), Chap. VI.

III. PERTURBATION ANALYSIS IN KRUSKAL COORDINATES

In the Kruskal coordinates, the Schwarzschild line element can be written as

$$ds^2=f^2(du^2-dv^2)+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{3.1}$$

with

$$f^2=(32m^2/r)\exp(-r/2m).$$

The relations between (r, t) and (u, v) are summarized by

$$(r/2m-1)\exp(r/2m)=u^2-v^2, \tag{3.2}$$
$$t/4m=\operatorname{arctanh}(v/u).$$

The coordinate singularity at $r=2m$ is completely removed and the metric remains finite down to the physical singularity $r=0$. The gravitational collapse of a spherical mass distribution as depicted on the Kruskal diagram has been described in detail by different authors.[1] Figure 1 here represents such a diagram for a spherical mass that is assumed to have collapsed in the infinitely remote past. The world line of the surface of the collapsing mass has coalesced with the line $u=-v$ and the empty-space geometry produced by the mass occupies the region $u+v>0$ up to the physical singularity $r=0$, beyond which, of course, the metric is not defined. The Schwarzschild exterior beyond $r=2m$ forms part of this region, i.e., the quadrant between the lines $u=\pm v$. We can now employ this picture in analyzing the perturbations on the Schwarzschild background.

Since the metric is regular everywhere in the Krushal coordinates, the correct way to analyze the perturbations is to study them in these coordinates. But the easiest way to obtain the perturbations that we want to study is to follow Regge and Wheeler in solving the equations $\delta R_{\mu\nu}=0$ in the Schwarzschild coordinates for r beyond $2m$. After the solutions are in hand, we then transform to the Kruskal coordinates for a critical study of them. These Kruskal transforms should be regular over the entire region that represents the empty space beyond $r=2m$ on the Kruskal diagram, i.e., the quadrant between the lines $u=\pm v$. However, at spatial infinity the u and v coordinates are not suitable for our purpose since they do not form a Lorentz frame, whereas we require the space to be asymptotically flat in this region. Therefore, we first select the solutions that fall off to zero for large values of r in the Schwarzschild coordinates, find their asymptotic forms near $r=2m$, and transform them to the Kruskal coordinates. We shall then study whether these transforms diverge or are well behaved. A perturbation giving rise to such a divergence—a divergence which can not be removed—is forbidden; if not, it is a perfectly valid perturbation that can have physical existence. We shall see that perturbations with purely imaginary frequencies do produce divergent Kruskal transforms. Hence such unstable perturbations do not exist and the Schwarzschild geometry is inherently stable against small perturbations.

IV. STABILITY OF SCHWARZSCHILD METRIC

In this section we shall solve the problem originally posed by Regge and Wheeler. The perturbations on the Schwarzschild metric can be represented by normal modes with time dependence $\exp(-ikt)$. Suppose one of these modes is superimposed upon the background metric at the initial moment $t=0$ and that it is regular everywhere in space. Can the frequency of this perturbation be purely imaginary? If so it will grow progressively in time, showing that the metric is inherently unstable. On the other hand, if it is found that the perturbation that is regular in all space at the initial instant must necessarily have real frequency, then the metric is stable. We shall show that this is precisely the situation. We emphasize the fact already mentioned in Sec. III: The regularity of the initial perturbation should be checked in the Kruskal coordinates in which the background metric itself is free of singularity. A study of the field equations and the asymptotic behavior of their solutions will show that this condition is not satisfied by perturbations with imaginary frequencies.

We shall first write down the most general representation of the perturbations as well as their canonical form. Next the Kruskal transforms corresponding to the canonical perturbations are given in order to examine later their asymptotic behavior near the Schwarzschild surface. The rest of the section will be devoted to the analysis of the first-order field equations for the perturbations corresponding to purely real and imaginary frequencies.

A. Perturbations in Schwarzschild Coordinates

The total perturbed metric may be written as

$$g^t{}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu},$$

where $g_{\mu\nu}$ is the Schwarzschild background metric and $h_{\mu\nu}$ is the small perturbation. We use the same perturbations $h_{\mu\nu}$ as originally given by Regge and Wheeler, retaining their notation. These fall into two distinct classes—odd and even—with parities $(-1)^{l+1}$ and $(-1)^l$, respectively, where l is the angular momentum of the particular mode. The most general form of the perturbations can be written as follows:

odd parity:

$$h_{\mu\nu}=\begin{bmatrix} 0 & 0 & -h_0(t,r)(\partial/\sin\theta\partial\phi)Y_l^M & h_0(t,r)(\sin\theta\partial/\partial\theta)Y_l^M \\ 0 & 0 & -h_1(t,r)(\partial/\sin\theta\partial\phi)Y_l^M & h_1(t,r)(\sin\theta\partial/\partial\theta)Y_l^M \\ \text{sym} & \text{sym} & h_2(t,r)(\partial^2/\sin\theta\partial\theta\partial\phi-\cos\theta\partial/\sin^2\theta\partial\phi)Y_l^M & \text{sym} \\ \text{sym} & \text{sym} & \tfrac{1}{2}h_2(t,r)(\partial^2/\sin\theta\partial\phi\partial\phi+\cos\theta\partial/\partial\theta-\sin\theta\partial^2/\partial\theta\partial\theta)Y_l^M & -h_2(t,r)(\sin\theta\partial^2/\partial\theta\partial\phi-\cos\theta\partial/\partial\phi)Y_l^M \end{bmatrix}; \tag{4.1}$$

even parity:

$$h_{\mu\nu}=\begin{bmatrix} (1-2m/r)H_0(t,r)Y_l^M & H_1(t,r)Y_l^M & h_0(t,r)(\partial/\partial\theta)Y_l^M & h_0(t,r)(\partial/\partial\phi)Y_l^M \\ H_1(t,r)Y_l^M & (1-2m/r)^{-1}H_2(t,r)Y_l^M & h_1(t,r)(\partial/\partial\theta)Y_l^M & h_1(t,r)(\partial/\partial\phi)Y_l^M \\ \text{sym} & \text{sym} & r[K(t,r)+G(t,r)(\partial^2/\partial\theta^2)]Y_l^M & \text{sym} \\ \text{sym} & \text{sym} & r^2G(t,r)(\partial^2/\partial\theta\partial\phi-\cos\theta\partial/\sin\theta\partial\phi)Y_l^M & r^2[K(t,r)\sin^2\theta+G(t,r)(\partial^2/\partial\phi^2+\sin\theta\cos\theta\partial/\partial\theta)]Y_l^M \end{bmatrix}. \tag{4.2}$$

Here Y_l^M are spherical harmonics with angular momentum l and z component M, and "sym" indicates that $h_{\mu\nu}=h_{\nu\mu}$. As has been mentioned earlier, one can specialize to the case with $M=0$ without altering the physics of the situation. The time dependence of the perturbations is given by $\exp(-ikt)$, the constant k being the frequency. Further, by suitable gauge transformations the number of elements in the perturbation matrix can be reduced, thereby obtaining the canonical form for the perturbations given as follows:

odd parity:

$$h_{\mu\nu}=\begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ \text{sym} & \text{sym} & 0 & 0 \end{bmatrix} \exp(-ikt)[\sin\theta(\partial/\partial\theta)]P_l(\cos\theta); \tag{4.3}$$

even parity:

$$h_{\mu\nu}=\begin{bmatrix} H_0(1-2m/r) & H_1 & 0 & 0 \\ H_1 & H_2(1-2m/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2K & 0 \\ 0 & 0 & 0 & r^2K\sin^2\theta \end{bmatrix} \exp(-ikt)P_l(\cos\theta). \tag{4.4}$$

Here $P_l(\cos\theta)$ is the Legendre polynomial with angular momentum l.

B. Transformations to Kruskal Coordinates

The relations between the Schwarzschild and the Kruskal coordinates have been summarized in Sec. III. Using the tensor transformation law, the canonical perturbations in the Schwarzschild coordinates can be transformed to obtain the corresponding perturbations in the Kruskal coordinates. Since the angular coordinates are common to both of the above frames, the transforms represent the canonical perturbations in the Kruskal coordinates with the Schwarzschild time and radial coordinates mixed through u and v. The Kruskal transforms can be obtained by a straightforward computation, and the transforms involving r and t are given below. Components like h_{22}, etc., that involve only angular coordinates are the same in both systems.

$$\begin{aligned} h^k{}_{00}&=f^2(r)(u^2-v^2)^{-1}[u^2(1-2m/r)^{-1}h^s{}_{00} \\ &\qquad +v^2h^s{}_{11}(1-2m/r)-2uvh^s{}_{01}], \\ h^k{}_{11}&=f^2(r)(u^2-v^2)^{-1}[v^2(1-2m/r)^{-1}h^s{}_{00} \\ &\qquad +u^2h^s{}_{11}(1-2m/r)-2uvh^s{}_{01}], \\ h^k{}_{01}&=f^2(r)(u^2-v^2)^{-1}\{(u^2+v^2)h^s{}_{01} \\ &\qquad -uv[(1-2m/r)^{-1}h^s{}_{00}+(1-2m/r)h^s{}_{11}]\}, \\ h^k{}_{03}&=4m(u^2-v^2)^{-1}[uh^s{}_{03}-v(1-2m/r)h^s{}_{13}], \\ h^k{}_{13}&=-4m(u^2-v^2)^{-1}[vh^s{}_{03}-u(1-2m/r)h^s{}_{13}], \end{aligned} \tag{4.5}$$

where the superscripts s and k refer to the Schwarzschild and Kruskal coordinates, respectively. For future reference we give the following relations:

$$\exp(r^*/2m)=(u^2-v^2),$$

where r^* is defined by

$$r^*/2m=r/2m+\ln(r/2m-1) \tag{4.6a}$$

and

$$\exp(t/2m)=(u+v)/(u-v), \tag{4.6b}$$

since t is given by

$$t/2m=2\tanh^{-1}(v/u)=\ln[(u+v)/(u-v)].$$

C. Odd-Parity Perturbations

We now examine the stability of the Schwarzschild metric against the odd perturbations. The two cases, $l>1$ and $l=1$, will have to be studied separately since the field equations are not the same in these two cases.

Case 1: $l>1$. For the angular momentum $l>1$, the field equations lead to the "wave equation" [the time dependence of the perturbations is $\exp(-ikt)$]

$$d^2Q/dr^{*2}+(k^2-V_{\text{eff}})Q=0, \tag{4.7a}$$

where

$$Q=(h_1/r)(1-2m/r),$$
$$V_{\text{eff}}=(1-2m/r)[l(l+1)/r^2-6m/r^3], \tag{4.7b}$$

and r^* is as defined in (4.6a). The radial function h_0 can be found from the equation

$$h_0=(i/k)(d/dr^*)(rQ). \tag{4.7c}$$

The coordinate r^* ranges from $-\infty$ to $+\infty$, corresponding to the range of r from $2m$ to $+\infty$. The effective potential V_{eff} is real and positive everywhere and vanishes at $r^* = \pm\infty$, i.e., at the boundaries.

First consider the solutions with purely imaginary k, which will give rise to unstable perturbations, i.e., perturbations that grow exponentially with time. Set $k = i\alpha$, where α is real and positive, so that the time dependence of the perturbations becomes $\exp(\alpha t)$. Then Eqs. (4.7a) and (4.7c) read

$$d^2Q/dr^{*2} = (\alpha^2 + V_{\text{eff}})Q \tag{4.8a}$$

and

$$h_0 = (1/\alpha)(d/dr^*)(rQ). \tag{4.8b}$$

The asymptotic solutions of (4.8) for Q as r approaches infinity and $2m$ are given by

$$Q_\infty \sim \exp(\pm\alpha r), \qquad Q_{2m} \sim \exp(\pm\alpha r^*).$$

Since we require that the perturbation fall off to zero for large values of r, we choose

$$Q_\infty \sim \exp(-\alpha r).$$

But, if Q is taken to be positive, Eq. (4.8a) shows that d^2Q/dr^{*2} never becomes negative within the range of r from $2m$ to ∞, and hence the solution that goes to zero at spatial infinity cannot be matched to the one that goes to zero at $r = 2m$, so that asymptotic solution near $r = 2m$ has to be

$$Q_{2m} = A \exp(-\alpha r^*),$$

where A is a constant. Using this solution, the radial function h_0 in the neighborhood of $r = 2m$ is readily obtained as

$$h_0 = -2mAQ_{2m}.$$

Substituting the above solutions for h_0 and h_1, we find that the perturbations in the Kruskal coordinates near the surface $u = v$ would be (angular dependence has been suppressed)

$$h^k{}_{03} = 8m^2A(u^2 - v^2)^{-1}(u+v)\exp(-\alpha r^*)e^{\alpha t}$$
$$= 8m^2A(u-v)^{-(2m\alpha+1)}[(u+v)/(u-v)]^{2m\alpha}$$
$$= 8m^2A(u+v)^{2m\alpha}(u-v)^{-(4m\alpha+1)}.$$

In deciding whether the Schwarzschild metric is stable, we start with a perturbation which is regular everywhere in space at $t = 0$ and see whether such a perturbation will grow with time. The above perturbation was chosen to be regular at spatial infinity. However, at $t = 0$ near the Schwarzschild surface, it would have the Kruskal transform (set $v = 0$)

$$h^k{}_{03}(t=0) = 8m^2Au^{-(2m\alpha+1)}.$$

By choosing u small ($u \to 0$) this perturbation can be made as large as we wish, i.e., the perturbation diverges as $u \to 0$, whereas the background metric remains finite. This clearly contradicts the assumption that the perturbation is small compared to the background. Such a perturbation is unacceptable and hence can not exist. Thus perturbations with imaginary k that grow exponentially with time are ruled out.

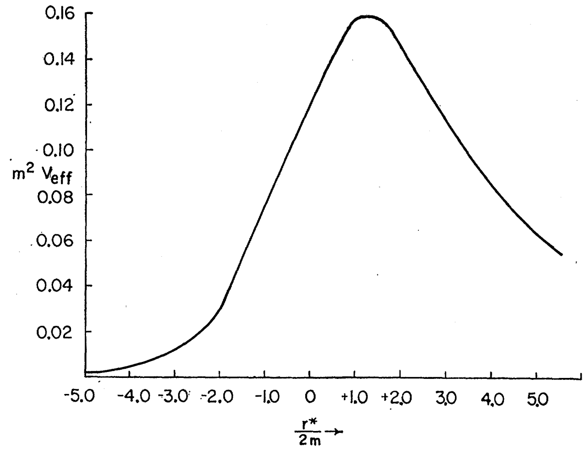

FIG. 2. Effective potential $V_{\text{eff}} = (1-2m/r)[l(l+1)/r^2 - 6m/r^3]$ for the odd perturbation of angular momentum $l = 2$ is plotted against $r^*/2m = r/2m + ln(r/2m - 1)$. The peak of the potential is at $r \simeq 3.3m$.

Next let us consider the solutions corresponding to real frequencies. From Eq. (4.7a) and the plot of V_{eff} shown in Fig. 2, we see that the asymptotic solutions near $r = 2m$ are

$$Q_{2m} = A\exp(\pm ikr^*), \qquad r^* \to -\infty$$

where A is a constant. The two solutions correspond to the outgoing and incoming waves. Consider the ingoing solution $Q_{2m} = A\exp(-ikr^*)$. Then we have

$$h_1 = 2mA(1-2m/r)^{-1}\exp(-ikr^*)$$

and

$$h_0 = 2mA\exp(-ikr^*),$$

so that

$$h^k{}_{03} = 8m^2A(u+v)^{-1}\exp[-ik(r^*+t)]$$
$$= 8m^2A(u+v)^{-1}(u+v)^{-4ikm}. \tag{4.9}$$

Since we finally take the real part of the perturbation, the function $(u+v)^{-4ikm}$ contributes only a rapidly oscillating function near $u = -v$. Nevertheless, the singularity due to $(u+v)^{-1}$ appears to be serious. This can easily be remedied by building wave packets out of the monochromatic waves. For instance, let us choose purely ingoing waves at $r^* \to -\infty$, as in Eq. (4.9) above, but form them into a packet by using for $A = A(k)$ the Fourier transform of a function $f(\xi) = \int A(k) \times \exp(-ik\xi)dk$ which vanishes for $\xi < 0$. Then, when integrated over k, Eq. (4.9) reads

$$h^k{}_{03} = 8m^2(u+v)^{-1}f[4m\ln(u+v)]. \tag{4.10}$$

There is then no singularity from the $(u+v)^{-1}$ factor, since f is nonzero only when $u+v > 1$. Equation (4.10) gives the asymptotic form for $r^* \to -\infty$ of this perturbation; for $r^* \to +\infty$ it can be examined in the Schwarzschild coordinates and will typically contain both outgoing and ingoing parts, but is evidently regular. Thus, perturbations containing purely ingoing waves for r near $2m$ give wave packets which are regular everywhere in the Kruskal geometry. These are stable perturbations and are physically acceptable.

For real frequencies k, we found that the regularity conditions at $r=2m$ could be satisfied by packets of waves which were purely ingoing at $r^* \to -\infty$, provided that these packets were bounded away from the line $u=-v$. They could, of course, also (time reversal) be satisfied by packets which were purely outgoing at $r^* \to -\infty$, provided that these packets were bounded away from the line $u=v$. Superpositions of these two regular types of packets would also be regular. The physics of the situation makes the first type a natural form to consider. Matzner,[7] in considering the scattering of scalar waves from the Schwarzschild "singularity," imposes the boundary condition of only ingoing waves at the Schwarzschild surface. This is possible because the effective potential in that problem has a peak at $r=\frac{4}{3}(2m)$ and vanishes exponentially as $r^* \to -\infty$, so that there is no backscatter for even small negative values of r^*. The situation is exactly the same in case of the gravitational waves. Our V_{eff} has a peak at about $r=3m$ and goes to zero like $\exp(r^*/2m)$ as $r^* \to -\infty$. Hence the same boundary condition as in the case of Matzner's calculations could be imposed here too. This boundary condition is assumed by Edelstein[8] in calculating the gravitational radiation due to a point mass revolving around a larger spherical mass which produces the Schwarzschild background metric. One wishes to impose this boundary condition to define a problem in which all the radiation is being generated by sources outside $r=2m$, and none is due to the matter which collapsed a long time in the past to produce the Schwarzschild background field.

Case 2: $l=1$. The case for $l=1$ is completely different from that for $l>1$. The perturbation in the Ricci tensor δR_{23} reduces identically to zero, since the angular factor multiplying the radial equation in δR_{23} is given by

$$[\cos\theta(d/d\theta)-\sin\theta(d^2/d\theta^2)]P_l(\cos\theta),$$

which vanishes for $l=1$. The equation $\delta R_{13}=0$ yields the relation between h_0 and h_1

$$h_1=(i/k)r^2(d/dr)(h_0/r^2), \tag{4.11}$$

where the time dependence of the perturbations is retained as $\exp(-ikt)$. When the above relation is substituted into δR_{03}, it reduces again to zero, giving no new information. Thus the set of field equations gives rise to a single relation between h_0 and h_1. We now show that this relation enables us to transform away both h_{03} and h_{13} by a gauge transformation that leaves the other components of the perturbation unchanged.

Consider the infinitesimal coordinate transformation

$$x'^\alpha=x^\alpha+\xi^\alpha, \qquad \xi^\alpha \ll x^\alpha.$$

Then $h_{\mu\nu}$ changes to $h'_{\mu\nu}=h_{\mu\nu}-\delta h_{\mu\nu}$, where $\delta h_{\mu\nu}=\xi_{\mu;\nu}+\xi_{\nu;\mu}$. Choose $\xi_\mu=(0,0,0,\xi_3)$ with

$$\xi_3=(i/k)h_0\exp(-ikt)\sin\theta(\partial/\partial\theta)P_1(\cos\theta).$$

Then one can readily show that $\delta h_{03}=h_{03}$, so that $h'_{03}=0$. Similarly, $\delta h_{13}=h_{13}$. The other perturbation components are unaltered. For instance, $\delta h_{23}=0$, as a result of the angular factor being identically zero. Therefore, the perturbations with $l=1$ can be transformed away irrespective of the nature or value assigned to the frequency k (except $k=0$, which we will deal with later), a particular case being that with purely imaginary k. This completes the proof of stability for the odd perturbations.

D. Even-Parity Perturbations

In Sec. IV A we wrote down the most general form of the perturbation matrix. For $l>1$ there are seven independent radial functions, which, in the canonical gauge, reduce to only four, namely, H_0, H_2, H_1, and K. The field equations yield the relation $H_0=H_2=H$. This is no longer true for $l=1$. Nevertheless, as can be seen readily from the perturbation matrix, there are only six independent radial functions in the general form when $l=1$. This affords an additional degree of freedom while making the gauge transformation which can be utilized to *impose* the condition $H_0=H_2=H$. The rest of the computations to obtain the radial equations follow as in the case $l>1$, and all the equations appearing in Ref. 5 hold for both $l=1$ and $l>1$. In what follows we prove stability for only $l>1$. The same method can also be adopted for $l=1$.

For the sake of simplicity, we remove the factor $2m$ from the equations, which are rather complicated, by defining

$$x=r/2m, \qquad x^*=r^*/2m=x+\ln(x-1), \qquad \bar{k}=2mk.$$

Also, let

$$S=H_1/r.$$

The second-order differential equation for S is given in Ref. 5. Dropping the bar on $\bar{k}$ [time dependence of the perturbations is $\exp(-ikt)$], the asymptotic forms of this equation, and the corresponding solutions, are found to be

$$d^2S_\infty/dx^2+k^2S_\infty=0, \quad S_\infty \sim \exp(\pm ikx) \qquad \text{for } x\to\infty,$$
$$d^2S_1/dx^{*2}+2(dS_1/dx^*)+(k^2+1)S_1=0, \tag{4.12}$$
$$S_1 \sim \exp[(-1\pm ik)x^*] \qquad \text{for } x\to 1.$$

As in the odd case, set $k=i\alpha>0$. Then the asymptotic solutions will be

$$S_\infty \sim \exp(\pm\alpha x), \qquad S_1 \sim \exp[(-1\pm\alpha)x^*].$$

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[7] R. A. Matzner, J. Math. Phys. 9, 163 (1968).

[8] L. A. Edelstein, Ph.D. thesis, University of Maryland, Department of Physics and Astronomy, 1970 (unpublished).

To ensure asymptotic flatness at spatial infinity, we choose $S_\infty \sim \exp(-\alpha x)$. In the case of S_1 the solution $\exp[(\alpha-1)x^*]$ goes to zero as $x^* \to -\infty$ for $\alpha > 1$ (the case $\alpha = 1$ will be discussed separately). We investigate whether the two asymptotic solutions $S_\infty \sim \exp(-\alpha x)$ and $S_1 \sim \exp[(\alpha-1)x^*]$ can be joined to each other. If this is possible, the function S should have a stationary point between $x=1$ and $x=\infty$, i.e., at some point in this range $dS/dx^*=0$ and $d^2S/dx^{*2}<0$, assuming S to be positive in this neighborhood. We shall now show that such a point does not exist. Setting $dS/dx^*=0$, the differential equation can be expressed as

$$d^2S/dx^{*2} = [N(x)/D(x)]S,$$

where

$$D(x) = (1-1/x)^{-1}[(4/x)(1-1/x) + 2(l-1)(l+2)(1-1/x) + 2\alpha^2 x^2 - 1/2x^2],$$

which is readily seen to be positive for all values of x from 1 to ∞, and

$$N(x) = \alpha^2 D(x) - \left[8\alpha^2 + 2l(l+1)\frac{1}{x^3} + 4(l-1)(l+2)\frac{1}{x^2} + \frac{12}{x^3}\right]\left(1-\frac{3}{2x}\right) + \frac{2}{x^2}\left(1-\frac{3}{2x}\right)D(x) + \frac{l(l+1)}{x^2}\left(1-\frac{1}{x}\right)D(x). \quad (4.13)$$

It can be shown that $N(x)>0$ throughout the range $x>1$, although this is not obvious from the expression for $N(x)$. Our proof of this fact[9] is rather cumbersome, requiring separate consideration of the regions $1<x<\frac{3}{2}$ and $x>\frac{3}{2}$, and a careful grouping of terms. Since both $D(x)$ and $N(x)$ are positive functions, the second derivative of S, d^2S/dx^{*2}, has the same sign as S for all x, and therefore S has no stationary point. We conclude that the solution going to zero for large values of x has the asymptotic behavior $S_1 \sim \exp[-(\alpha+1)x^*]$ near the Schwarzschild surface. Next, in order to show that this solution gives rise to divergent perturbations in the Kruskal coordinates, we compute the asymptotic solution of the radial function H near $r=2m$.

The two radial functions H_1 and H are related to each other by the equation

$$\left[\frac{d^2H_1}{dx^{*2}} + \frac{d}{dx^*}\left(\frac{H_1}{x^2}\right) + \alpha^2 H_1\right] - 2\alpha\left[\frac{dH}{dx^*} + \frac{1}{2x^2}H\right] = 0. \quad (4.14)$$

Since

$$S = H_1/r \sim \exp[-(\alpha+1)x^*]$$

near $x=1$, we assume the asymptotic forms

$$H_1 = A\exp[-(\alpha+1)x^*] \qquad H = B\exp[-(\alpha+1)x^*],$$

where A and B are constants. Substituting these in Eq. (4.14), we find

$$B = -A,$$

so that $H=-H_1$ near $x=1$. Then the perturbation in the Kruskal coordinates is given, for instance (angular dependence suppressed), by

$$\begin{aligned} h^k{}_{00} &= f^2(r)(u^2-v^2)^{-1}[(u^2+v^2)H - 2uvH_1]e^{\alpha t} \\ &= f^2(r)(u^2-v^2)^{-1}(u+v)^2 H e^{\alpha t} \\ &= Bf^2(r)(u-v)^{-2(\alpha+1)}. \end{aligned} \quad (4.15)$$

At $t=0$ $(v=0)$, the Kruskal transform will be

$$h^k{}_{00} = Bf^2(r)u^{-2(\alpha+1)},$$

which is divergent as $u \to 0$. Hence the perturbation is unacceptable.

Case $\alpha=1$. In this case the asymptotic forms of the differential equation for S and the corresponding solutions are

$$d^2S_\infty/dx^{*2} - S_\infty = 0, \qquad S_\infty \sim e^{\pm x} \qquad \text{for } x\to\infty,$$

$$d^2S_1/dx^{*2} + 2(dS_1/dx^*) = 0, \quad (4.16)$$

i.e.,

$$dS_1/dx^* + 2S_1 = C \qquad \text{for } x \to 1,$$

where C is a constant. Hence the asymptotic solutions are

$$S_1 \sim \exp(-2x^*), \qquad S_1 \sim \tfrac{1}{2}C.$$

If $S_\infty \sim e^{-x}$ can be matched to $S_1 \sim$const, there should be a region where $dS/dx^*<0$, $d^2S/dx^{*2}<0$ for $S>0$, which can be shown to be impossible, as in the case $\alpha>1$. Again, the alternative solution $S_1 \sim \exp(-2x^*)$ can be shown to produce divergent Kruskal transforms at the initial instant.

Thus we have shown that perturbations with $l>1$ and purely imaginary frequencies are physically unacceptable since they are divergent even at the initial moment. This can be shown to be true for perturbations with $l=1$ by the same method as above. Moreover, Campolattaro and Thorne[10] have shown that the perturbations corresponding to $l=1$ can be altogether removed by suitable gauge transformations for both real and imaginary values of k. So this perturbation is only a coordinate effect and has no physical existence. We conclude then that the Schwarzschild metric is stable against even perturbations, as it was against the odd ones. This completes the proof of stability against small oscillations.

E. Even Perturbations with Real Frequencies

We have already given the asymptotic solutions near the Schwarzschild surface for real frequencies. The outgoing and incoming waves correspond to solutions of the form $\exp[(ik-1)x^*]$ and $\exp[-(ik+1)x^*]$, respectively. Also, when we evaluate the asymptotic forms of the radial function H in these two cases, we find $H=-H_1$ for the outgoing waves, and $H=H_1$ for the incoming waves. From this information we readily ob-

[9] For details, see C. V. Vishveshwara, Ph.D. thesis, University of Maryland, Department of Physics and Astronomy, 1968 (unpublished).

[10] A. Campolattaro and K. S. Thorne, Astrophys. J. 159, 847 (1970).

tain the Kruskal transform

$$h^k{}_{00}=Af^2(r)(u-v)^{-2}(u-v)^{4mik} \qquad \text{(outgoing)},$$

$$h^k{}_{00}=Af^2(r)(u+v)^{-2}(u+v)^{-4mik} \qquad \text{(incoming)}.$$

The singular behavior of these at $u=v$ or $u=-v$ is exactly similar [but of one higher order since we have the term $(u\pm v)^{-2}$ here] to that for odd-parity perturbations. Once again, as in the odd case, wave packets can be built by superposition of solutions with the whole range of real frequencies and the divergence removed.

V. STATIONARY PERTURBATIONS

In deriving the field equations, the time dependence was assumed to be $\exp(-ikt)$, so that, as they appear in Ref. 5, the time derivatives are replaced by a multiplying factor $(-ik)$. If the perturbations are independent of time, the corresponding field equations are obtained by setting $k=0$. In what follows we shall work with these equations and study their solutions.

A. Odd Perturbations

Case (i): $l=1$ (rotational perturbation). For $l=1$ and $k=0$, the perturbations of the Ricci tensor δR_{23} and δR_{13} both reduce identically to zero; in the first instance, it is the consequence of the vanishing of the angular factor and, in the second, it is because of the condition $k=0$. The equation $\delta R_{03}=0$ yields the differential equations

$$d^2h_0/dr^2=(2/r)h_0. \qquad (5.1)$$

The solution that falls off to zero for large values of r is given by

$$h_0=c/r, \qquad (5.2)$$

where c is a constant. Then

$$h_{03}=(c/r)\sin^2\theta, \qquad (5.3)$$

which can be clearly identified with the rotational perturbation by comparing it with the weak-field approximation.[11] Moreover, the solution is acceptable down to the Schwarzschild surface $r=2m$, since a gauge transformation performed on the angular coordinate ϕ makes the corresponding Kruskal perturbation regular at $u=v$, and leaves the other components of the perturbation unchanged. Make the gauge transformation

$$\phi'=\phi+c(2m)^{-3}t.$$

In the new coordinates the perturbation is given by

$$h_{03}=(c\sin^2\theta/8m^3)(1-2m/r)(r^2+2mr+4m^2), \qquad (5.4)$$

with the corresponding perturbed Kruskal line element

$$dS^2=f^2(r)(du^2-dv^2)+r^2(d\theta^2+\sin^2\theta d\phi^2)$$
$$+c\exp(-r/2m)(r^2+2mr+4m^2)$$
$$\times\sin^2\theta\, d\phi(udv-vdu)/mr, \qquad (5.5)$$

which is identical with the expression derived by Brill and Cohen.[12] The above gauge transformation does not affect the other components of the perturbation. We observe that h_{13} or h_1 was not determined by the field equations. Once again, whatever value we assign to h_1, it can be transformed away by choosing the appropriate gauge that does not change the other elements of the perturbation matrix.

Case (ii): $l>1$. The field equations for $k=0$ and $l>1$ reduce to

$$h_1=0$$

and

$$\frac{d^2h_0}{dr^{*2}}-\frac{2m}{r^2}\frac{dh_0}{dr^*}-\left[\frac{l(l+1)}{r^2}-\frac{4m}{r^3}\right]\left(1-\frac{2m}{r}\right)h_0=0. \qquad (5.6)$$

For large values of r this equation reads

$$d^2h_0/dr^2-[l(l+1)/r^2]h_0=0, \qquad (5.7)$$

which has the solutions

$$h_0\sim r^{-l} \quad\text{and}\quad r^{l+1}. \qquad (5.8)$$

We choose $h_0\sim r^{-l}$, which vanishes for large values of r. Near $r=2m$ $(r^*\to-\infty)$ the asymptotic form of Eq. (5.6) is

$$d^2h_0/dr^{*2}-(1/2m)(dh_0/dr^*)=0, \qquad (5.9)$$

which has the solutions

$$h_0\sim\exp(r^*/2m) \qquad h_0\sim\text{const}=c.$$

We cannot match the solution $\exp(r^*/2m)$, which vanishes at $r=2m$, to the solution r^{-l}, which vanishes for large r. If we could, there should be a point between $r=2m$ and $r=\infty$ where $dh_0/dr^*=0$ and $d^2h_0/dr^{*2}<0$, assuming $h_0>0$. From Eq. (5.6) it is evident that this is impossible for $l>1$. Hence, at $r=2m$ we are left with the solution $h_0=\text{const}=c$. The corresponding Kruskal transform will be

$$h^k{}_{03}=4m(u^2-v^2)^{-1}uh_0$$
$$=4mcu(u^2-v^2)^{-1}, \qquad (5.10)$$

which diverges both at $u=v$ and at $u=-v$. It can be shown that if we try to remedy this by gauge transformation the divergence will show up in h_{13} and, moreover, the perturbations produced by the gauge transformation will necessarily be functions of either t or ϕ or both. Hence stationary odd perturbations with $l>1$ do not exist.

B. Even Perturbations

Regge and Wheeler have shown that stationary perturbations of even parity with $l=0$ and $l=1$ represent, respectively, an infinitesimal addition to the Schwarzschild mass and a small displacement of the center of attraction.

For $l>1$ the field equations reduce to

$$H_0=H_2=H, \qquad H_1\equiv0, \qquad (5.11)$$

[11] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), Sec. 103.

[12] D. R. Brill and J. M. Cohen, Phys. Rev. **143**, 1011 (1966).

and

$$\frac{d^2H}{dr^{*2}}+\frac{2}{r}\left(1-\frac{2m}{r}\right)\frac{dH}{dr^*}-\left[\frac{4m^2}{r^4}+\frac{l(l+1)}{r^2}\left(1-\frac{2m}{r}\right)\right]H=0.$$

The asymptotic solutions are

$$H\sim r^{-l} \text{ and } r^{l+1} \quad \text{for } r\to\infty$$
$$\sim\exp(\pm r^*/2m) \quad \text{for } r\to 2m. \tag{5.12}$$

As in the odd case, the two solutions going to zero at the two boundaries cannot be matched. If we choose $H\sim r^{-l}$ at spatial infinity, we will be left with $H\sim\exp(-r^*/2m)$ near $r=2m$, and the corresponding Kruskal transform will be

$$h^k{}_{00}=f^2(r)(u^2-v^2)^{-1}(u^2+v^2)H$$
$$=f^2(r)(u^2+v^2)(u^2-v^2)^{-2}. \tag{5.13}$$

This divergence cannot be overcome by a gauge transformation and hence stationary perturbations of even parity with $l>1$ do not exist.

We may note that Doroshkevich *et al.*[13] studied the stationary perturbations on the Schwarzschild metric. They found that the even perturbations diverge near the Schwarzschild surface; however, they do not seem to have analyzed those perturbations in the Kruskal coordinates. Assuming the angular dependence to be specifically $\sin^2\theta$, they have derived the rotational perturbation (again not analyzed in the Kruskal coordinates); higher values of l were not considered.

VI. CONCLUSION

In the foregoing we have answered all the questions raised originally by Regge and Wheeler. This involved essentially a detailed analysis of perturbations on the Schwarzschild exterior with the frequency assuming values that are real, imaginary, and finally zero. A few words about one other possibility, namely, the frequency being complex, are in order.[9] This aspect can easily be studied in the case of the odd perturbations as they involve a simple differential equation. The imposition of the boundary conditions that the gravitational waves be entirely outgoing at $r=\infty$ and purely incoming at $r=2m$ leads at once to the requirement of complex frequencies with negative imaginary parts. This result is related to the phenomena of radiation damping and resonance scattering, and leads to a statement of causality. We can conceive of an initial curvature or a "wiggle" superimposed on the Schwarzschild empty space-time somewhere outside $r=2m$, which can act as a source of gravitational radiation. As the wiggle is smoothed out and the energy associated with it is carried away in the form of gravitational radiation, the radiation damping will make the frequency complex with its negative imaginary part reflecting the decay of the source. On the other hand, if the relevant phenomenon is the scattering of gravitational waves, the complex eigenvalues of the frequency imply poles in the scattering matrix and the consequent occurrence of resonances in the scattering cross section. If one considers the scattering of a wave packet, the analyticity of the scattering matrix in the upper half-plane of complex k (since $\mathrm{Im}k<0$, the poles lie only in the lower half-plane) ensures the causal propagation of the scattered wave packet after the incoming wave packet has reached the scattering center. This again is an indirect confirmation of the stability of the background metric. For, were the background not stable, it could generate a disturbance independent of the incoming wave packet, and this would be observed at a given point before the scattered wave packet had time to reach it.

We finally come to a crucial point regarding the proof of stability. Let us emphasize explicitly what we proved: Any *single* mode corresponding to *a particular* imaginary value of frequency k could not have existed at the initial instant. But, the metric is inherently stable only if it is stable against *any arbitrary* perturbation which is well behaved at $t=0$. A completely rigorous proof of stability then requires that any arbitrary well-behaved initial perturbation be a superposition of modes corresponding to real frequencies only, or, in other words, that the radial functions associated with real values of k form a complete set. The last requirement can in fact be proved in the case of the odd perturbations,[9] since the differential equation is in the well-known Sturm-Liouville self-adjoint form and since the effective potential is positive between $r=2m$ and $r=\infty$, and vanishes on the boundary. Unfortunately, this has not been possible in the case of the even perturbations, because the frequency (or k^2) does not appear linearly in the differential equation. Nevertheless, the close similarity in the asymptotic forms of the odd and even perturbations, combined with the lack of logical reasons for fundamentally different physical behavior of the two classes of perturbations, points to the possible existence of a rigorous proof of stability in the latter case as well. This can come forth only as a result of further work based perhaps on a different formalism.

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[13] A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksperim. i Teor. Fiz. **49**, 734 (1965) [Soviet Phys. JETP **22**, 122 (1966)].