Bjorken Limit in Perturbation Theory

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We present detailed calculations illustrating the breakdown of the Bjorken limit in perturbation theory, in the "gluon" model of strong interactions. To second order in the gluon-ferrmion coupling constant in the scalar, pseudoscalar, and vector coupling models, we calculate the Bjorken-limit commutator of a pair of currents of arbitrary (vector, axial-vector, scalar, pseudoscalar, tensor) type. To fourth order in the coupling, in the scalar- and pseudoscalar-gluon models, we determine the leading logarithmic behavior of the SU_3-antisymmetric part of the vector-vector commutator. In the body of the paper we present the main results and discuss their various features and implications. The computational details are relegated to two appendices.

I. HISTORICAL INTRODUCTION

EQUAL-TIME current commutators have come to play a central role in particle physics. In his famous papers of 1961 and 1964, Gell-Mann[1] proposed that the time components of the vector and axial-vector octet currents satisfy a simple $SU_3\otimes SU_3$ algebra. The exploitation of this postulate by the "infinite-momentum" and "low-energy theorem" methods has led to important predictions, which agree well with experiment.[2] The beauty of these "classical" current-algebra methods is that they depend only on the postulated commutation relations together with such weak dynamical assumptions as pion-pole dominance and unsubtracted dispersion relations. They are independent of more detailed (and therefore, more dubious) dynamical assumptions. The experimental successes thus provide a strong argument that any future theory of the hadrons must incorporate the $SU_3\otimes SU_3$ time-component current algebra.

This requirement, of course, does not uniquely specify a model of the hadrons—there are many possible field-theoretic models which satisfy the Gell-Mann hypothesis. In an attempt to narrow the selection, attention has been turned recently to the study of the space-component–space-component commutators, which can be used to distinguish between models which have the same time-component algebra. The problem of finding experimental tests of the space-space algebra is made difficult by the fact that the "classical" current-algebra methods of infinite-momentum limits and low-energy theorems cannot be made to apply in this case. However, in 1966 Bjorken[3] pointed out that the asymptotic behavior of a time-ordered product of two currents is simply related to the equal-time commutator of the currents,

$$\lim_{q_0\to i\infty;\,\mathbf{q}\text{ fixed}}\int d^4x\, e^{iq\cdot x}T(J_{(1)}(x)J_{(2)}(0))$$
$$=iq_0^{-1}\int d^4x\, e^{iq\cdot x}\delta(x^0)[J_{(1)}(x),J_{(2)}(0)]+O(q_0^{-2}).\quad(1)$$

Equation (1) has been extensively applied to the study of space-space current commutators, leading to a new class of *asymptotic sum rules*.[4] These sum rules have testable experimental consequences in inelastic electron and neutrino scattering reactions and important implications in the theory of radiative corrections to hadronic β decay.

In all of the applications of Eq. (1), an important assumption is made: It is *assumed* that the equal-time commutator appearing on the right-hand side of Eq. (1) is the same as the "naive commutator" obtained by straightforward use of canonical commutation relations and equations of motion. That this is a questionable assumption was pointed out by Johnson and Low,[5] who independently discovered Eq. (1). They studied this equation in a simple perturbation-theory model, in which the currents couple through a fermion triangle loop to a scalar, pseudoscalar, or vector meson. They found that in most cases the results obtained by explicit evaluation of the left-hand side of Eq. (1) differ from those calculated from naive commutators by well-defined extra terms. Because of special features of the triangle graph model, however, these extra terms did not directly invalidate the applications of Eq. (1) mentioned above.

Recently, we have reported a more realistic perturba-

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics 1, 63 (1964).
[2] For a survey, see S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[3] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[4] For a survey, see lectures by J. D. Bjorken, in *Selected Topics in Particle Physics, Proceedings of the International School of Physics "Enrico Fermi," Course XLI*, edited by J. Steinberger (Academic, New York, 1968).
[5] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. Nos. **37–38**, 74 (1966). Important early work on the validity of the Bjorken limit, in the context of the Lee model, has also been done by J. S. Bell, Nuovo Cimento **47A**, 616 (1967).

tion-theory calculation,[6] which showed that for commutators of space components with space components, the Bjorken limit and the naive commutator do differ by terms which modify all of the principal applications of Eq. (1). In other words, *asymptotic sum rules derived from the naive space-space commutators fail in perturbation theory*. One is, of course, still free to postulate that nonperturbative effects conspire to make the asymptotic sum rules valid when all orders of perturbation theory are summed, but the need for this assumption means that asymptotic sum rules do not just give a test of the space-space algebra, but involve deep dynamical considerations as well.

In our previous work, we considered only vector and axial-vector current commutators in the quark model with a massive vector "gluon," to second order in the gluon-fermion coupling constant g_r.[7] In the present paper, we extend our results to arbitrary (vector, axial-vector, scalar, pseudoscalar, tensor) currents in the quark models with vector-, scalar-, or pseudoscalar-coupled gluon. Working to second order in g_r, we obtain results analogous to those found previously in the more restricted case. In addition, for the vector-vector commutator in the scalar- and pseudoscalar-gluon models, we obtain the leading logarithmic part of the g_r^4 term. In Sec. II we summarize our results and in Sec. III we discuss briefly their significance. To facilitate reading, all computational details are relegated to Appendices.

II. RESULTS

We consider a simple, renormalizable model of the strong interactions, consisting of an SU_3 triplet of spin-$\frac{1}{2}$ particles ψ bound by the exchange of an SU_3-singlet massive "gluon." We assume that the gluon couples to the fermions by either scalar, pseudoscalar, or vector coupling. In order to treat simultaneously commutators involving vector (axial-vector, scalar, ...) currents, we introduce the abbreviated notation

$$J_{(1)}=\bar{\psi}\gamma_{(1)}\psi, \qquad J_{(2)}=\bar{\psi}\gamma_{(2)}\psi,$$
$$\gamma_{(1)}=\gamma_\mu\lambda^a(\gamma_\mu\gamma_5\lambda^a, \lambda^a, \ldots), \tag{2}$$
$$\gamma_{(2)}=\gamma_\nu\lambda^b(\gamma_\nu\gamma_5\lambda^b, \lambda^b, \ldots),$$

according to whether the first or second current is a vector (axial-vector, scalar, ...) current. The naive equal-time commutator of the two currents is

$$\delta(x^0-y^0)[J_{(1)}(x), J_{(2)}(y)]=\delta^4(x-y)\bar{\psi}(x)C\psi(x),$$
$$C=\gamma_0[\gamma_0\gamma_{(1)}, \gamma_0\gamma_{(2)}]=\gamma_{(1)}\gamma_0\gamma_{(2)}-\gamma_{(2)}\gamma_0\gamma_{(1)}. \tag{3}$$

We wish to compare the Bjorken-limit commutator with the naive commutator, in the special case in which Eqs. (1) and (3) are sandwiched between fermion states. To do this, we calculate the renormalized current-fermion scattering amplitude $\tilde{T}_{(1)(2)}{}^*(p, p', q)$ in the limit $q_0\to i\infty$, and compare the coefficient of the q_0^{-1} term with the renormalized vertex $\tilde{\Gamma}(C; p, p')$ of the naive commutator. The asterisk on $\tilde{T}_{(1)(2)}{}^*$ indicates that it is the full covariant scattering amplitude, which differs from the renormalized T product, $\tilde{T}_{(1)(2)}(p, p', q)$, by a "seagull" term $\tilde{\sigma}_{(1)(2)}(p, p', q)$ which is a polynomial in q_0,

$$\tilde{T}_{(1)(2)}{}^*(p, p', q)=\tilde{\sigma}_{(1)(2)}(p, p', \mathbf{q})+\tilde{T}_{(1)(2)}(p, p', q). \tag{4}$$

Identity of the Bjorken limit and naive commutators would mean that

$$\lim_{q_0\to i\infty;\,\mathbf{q},p,p' \text{ fixed}} \tilde{T}_{(1)(2)}(p, p', q)=q_0^{-1}\tilde{\Gamma}(C; p, p') +O(q_0^{-2}\ln q_0). \tag{5}$$

In the calculation which follows, we test the validity of Eq. (5) in perturbation theory.[8]

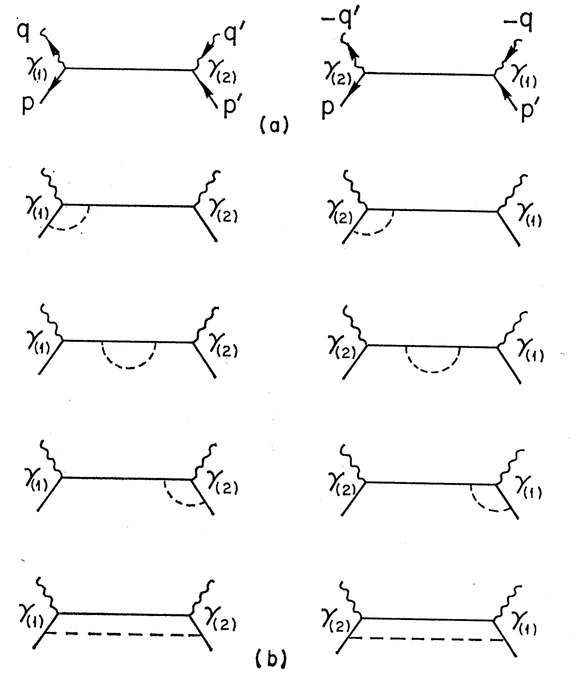

FIG. 1. (a) Lowest-order current-fermion scattering diagrams. (b) Diagrams obtained from the lowest-order ones by insertion of a single virtual gluon.

A. Second Order

To second order in the gluon-fermion coupling constant g_r, there are two classes of diagrams which contribute to $\tilde{T}_{(1)(2)}{}^*$. The diagrams of the first class, illustrated in Fig. 1, consist of the lowest-order current-

[6] S. L. Adler and W.-K. Tung, Phys. Rev. Letters **22**, 978 (1969). See also R. Jackiw and G. Preparata, *ibid.* **22**, 975 (1969), who have independently arrived at similar conclusions.

[7] In Ref. 6 we denoted the coupling constant g_r by g. In the present work, g will always indicate a gluon (or its four-momentum).

[8] A general discussion of the mechanism responsible for Bjorken-limit breakdown has been given by W.-K. Tung, Phys. Rev. **188**, 2404 (1969). See also R. Jackiw and G. Preparata, *ibid.* **185**, 1929 (1969).

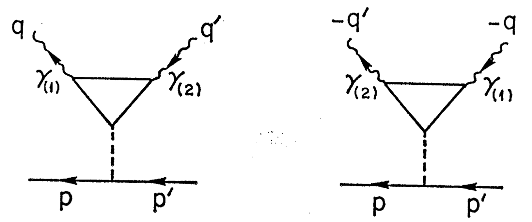

FIG. 2. Diagrams containing fermion triangles.

fermion diagrams and the second-order diagrams obtained from the lowest-order ones by insertion of a single virtual gluon. The diagrams of the second class, illustrated in Fig. 2, involve a fermion triangle diagram. We denote the contributions of these two classes to $\tilde{T}_{(1)(2)}{}^{*}$ by $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}$ and $\tilde{T}_{(1)(2)}{}^{*\text{triang}}$, respectively.

The first-class diagrams are evaluated by the standard technique of regulating the gluon propagator with a regulator of mass λ, which defines an *unrenormalized* amplitude $T_{(1)(2)}{}^{*\text{Compt}}$. To get the *renormalized* amplitude, one multiplies by the external fermion wave-function renormalization constant Z_2 and takes the limit $\lambda\to\infty$,

$$\tilde{T}_{(1)(2)}{}^{*\text{Compt}}=\lim_{\lambda\to\infty} Z_2 T_{(1)(2)}{}^{*\text{Compt}}. \tag{6}$$

In certain cases, as discussed below, this limit diverges logarithmically; in these cases, we take λ to be finite but very large, dropping terms which vanish as $\lambda\to\infty$ but retaining all terms which are proportional to $\ln\lambda^2$. The renormalized vertex

$$\tilde{\Gamma}(C;\,p,\,p')=\lim_{\lambda\to\infty} Z_2\Gamma(C;\,p,\,p')$$

is calculated by the same techniques from the diagram of Fig. 3. Finally, we take the limit $q_0\to i\infty$ in our expression for $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}$ and compare with $\tilde{\Gamma}(C;p,p')$, giving the results

$$\tilde{\sigma}_{(1)(2)}{}^{\text{Compt}}(p,\,p',\,q)=0, \tag{7a}$$

$$\begin{aligned}\lim_{q_0\to i\infty;\,\mathbf{q},p,p'\text{ fixed}} &\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p,\,p',\,q)\\ &=\lim_{q_0\to i\infty;\,\mathbf{q},p,p'\text{ fixed}} \tilde{T}_{(1)(2)}{}^{\text{Compt}}(p,\,p',\,q)\\ &=q_0^{-1}[\tilde{\Gamma}(C;\,p,\,p')+\Delta^{\text{Compt}}]+O(q_0^{-2}\ln q_0),\end{aligned} \tag{7b}$$

$$\begin{aligned}\Delta^{\text{Compt}}=(g_r^2/32\pi^2)\{&\ln(\lambda^2/|q_0|^2)[-\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_0\boldsymbol{\gamma}\gamma_0\gamma_{(2)}\\ &+\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}\\ &-\tfrac{1}{2}\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}-\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}\gamma_0\gamma_{(2)}]\\ &-\tfrac{3}{2}\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_0\boldsymbol{\gamma}\gamma_0\gamma_{(2)}-\tfrac{1}{2}\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma_0\boldsymbol{\gamma}\\ &+\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_0\gamma_{(2)}\gamma_0\boldsymbol{\gamma}+\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_0\gamma_{(2)}\\ &-\tfrac{1}{4}\gamma_{(1)}\gamma_0\boldsymbol{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}-\tfrac{1}{4}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}\gamma_0\gamma_{(2)}\\ &+\tfrac{1}{4}\boldsymbol{\gamma}[\gamma_\tau\gamma_{(1)}\gamma^\tau\gamma_{(2)}\gamma_0+\gamma_0\gamma_{(1)}\gamma_\tau\gamma_{(2)}\gamma^\tau]\boldsymbol{\gamma}\\ &-(1)\leftrightarrow(2)\}.\end{aligned} \tag{7c}$$

In Eq. (7), the notation $\boldsymbol{\gamma}\cdots\boldsymbol{\gamma}$ is a shorthand for $1\cdots1$ in the scalar-gluon case, $i\gamma_5\cdots i\gamma_5$ in the pseudoscalar-gluon case, and $(-\gamma_\rho)\cdots\gamma^\rho$ in the vector-gluon case. Some details of the calculation leading to Eq. (7) are given in Appendix A.

The second class diagrams (Fig. 2) have been calculated by Johnson and Low.[5] In our model, which has only SU_3-singlet gluons, these diagrams contribute only to the SU_3-singlet part of the commutator. Taking the Bjorken limit, and comparing with the bubble diagram contributions to $\tilde{\Gamma}(C;\,p,\,p')$ illustrated in Fig. 4, Johnson and Low find

$$\begin{aligned}\lim_{q_0\to i\infty;\,\mathbf{q},p,p'\text{ fixed}} &\tilde{T}_{(1)(2)}{}^{*\text{triang}}(p,\,p',\,q)\\ &=\tilde{\sigma}_{(1)(2)}{}^{\text{triang}}(p,\,p',\,q)+\lim_{q_0\to i\infty}\tilde{T}_{(1)(2)}{}^{\text{triang}}(p,\,p',\,q)\\ &=\tilde{\sigma}_{(1)(2)}{}^{\text{triang}}(p,\,p',\,q)+q_0^{-1}[\tilde{\Gamma}(C;\,p,\,p')^{\text{bubble}}+\Delta^{\text{triang}}]\\ &\qquad+O(q_0^{-2}\ln q_0).\end{aligned} \tag{8}$$

We will not exhibit the detailed form of Δ^{triang}, but only remark that *in all cases* Δ^{triang} vanishes when the three-momenta $\mathbf{q}$ and $\mathbf{q}'=\mathbf{q}+\mathbf{p}-\mathbf{p}'$ associated with the currents $J_{(1)}$ and $J_{(2)}$ vanish,

$$\Delta^{\text{triang}}\,|_{\mathbf{q}=\mathbf{q}'=0}=0. \tag{9}$$

[Equation (9) is true when the triplet of fermions ψ are degenerate in mass. Johnson and Low[5] also discuss the effect of mass splittings.] Thus, for the physically interesting case of the commutator of spatially integrated currents, the entire answer is given by Eq. (7). No cancellation between the SU_3-singlet part of Δ^{Compt} and Δ^{triang} is possible, and we conclude that the Bjorken limit and the naive commutator in our models differ in second-order perturbation theory.

To make contact with our previous work and with our fourth-order results, it is useful to write out two special cases of Eq. (7). We consider the commutator of two vector currents, taking $\gamma_{(1)}=\gamma_\mu\lambda^a$, $\gamma_{(2)}=\gamma_\nu\lambda^b$. In the vector-gluon case we find

$$\begin{aligned}\Delta^{\text{Compt}}=(g_r^2/16\pi^2)\{&2(g_{\mu\nu}-g_{\mu0}g_{\nu0})\gamma_0[\lambda^a,\,\lambda^b]\\ &+\tfrac{3}{2}(\gamma_\nu\gamma_0\gamma_\mu-\gamma_\mu\gamma_0\gamma_\nu)\{\lambda^a,\,\lambda^b\}\},\end{aligned} \tag{10}$$

in agreement with the result which we have reported in Ref. 6. In the scalar- and pseudoscalar-gluon cases we find

$$\begin{aligned}\Delta^{\text{Compt}}=(g_r^2/16\pi^2)\{&(g_{\mu\nu}-g_{\mu0}g_{\nu0})\gamma_0[\lambda^a,\,\lambda^b]\\ &-\tfrac{1}{2}(\gamma_\nu\gamma_0\gamma_\mu-\gamma_\mu\gamma_0\gamma_\nu)\{\lambda^a,\,\lambda^b\}[\ln(\lambda^2/|q_0|^2)-1]\}.\end{aligned} \tag{11}$$

B. Fourth Order

To fourth order in g_r, the number of diagrams contributing to $\tilde{T}_{(1)(2)}{}^{*}$ is so large that a direct calculation

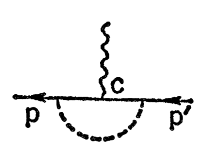

FIG. 3. Second-order correction to the vertex of the naive commutator C.

of the Bjorken limit, in analogy with our treatment of the second-order case, is prohibitively complicated. However, unitarity implies that the part of Δ proportional to $[\lambda^a, \lambda^b]$, and independent of the three-momenta $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{p}$, and $\mathbf{p}'$ and of the fermion mass m, is related to an integral over the longitudinal current-fermion inelastic total cross section.[6,9] Applying this connection to the commutator of vector currents in the scalar- and pseudoscalar-gluon cases, we have calculated the leading logarithmic contribution to the $[\lambda^a, \lambda^b]$ term in fourth order, with the result

$$\Delta=(g_{\mu\nu}-g_{\mu 0}g_{\nu 0})\gamma_0[\lambda^a,\lambda^b][(g_r^2/16\pi^2)+7(g_r^2/16\pi^2)^2 \times\ln(|q_0|^2)+g_r^4\times\text{const}]+(\text{terms symmetric in } a, b) + (\text{terms proportional to } \mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', \text{and } m). \quad (12)$$

Details of the unitarity relation and of the total cross-section calculation are outlined in Appendix B.

III. DISCUSSION

We proceed next to discuss a number of features of our results of Eqs. (7) and (10)–(12).

1. We begin by noting that to second order in g_r^2, Δ^{Compt} contains terms $\ln(\lambda^2/|q_0|^2)$ which diverge logarithmically both in the Bjorken limit $q_0\to i\infty$ and in the infinite-cutoff limit $\lambda\to\infty$. It is easy to see that the $\ln\lambda^2$ divergences result from a mismatch between the multiplicative factors needed to make $T_{(1)(2)}{}^{*\text{Compt}}(p, p', q)$ and $\Gamma(C; p, p')$ finite (i.e., $\ln\lambda^2$ independent). As we recall, the *renormalized* quantities $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p, p', q)$ and $\tilde{\Gamma}(C; p, p')$ are obtained from $T_{(1)(2)}{}^{*\text{Compt}}(p, p', q)$ and $\Gamma(C; p, p')$ by multiplying by the *wave-function renormalization* Z_2 and taking the limit $\lambda\to\infty$, keeping any residual $\ln\lambda^2$ dependence. On the other hand, the *finite* quantities $T_{(1)(2)}{}^{*\text{Compt}}(p, p', q)^{\text{finite}}$ and $\Gamma(C; p, p')^{\text{finite}}$ are obtained by multiplying by appropriate vertex and propagator renormalization factors which completely remove the $\ln\lambda^2$ dependence,

$$\Gamma(C;p,p')^{\text{finite}}=Z(C)\Gamma(C;p,p'),$$
$$T_{(1)(2)}{}^{*\text{Compt}}(p,p',q)^{\text{finite}}=Z(\gamma_{(1)})Z(\gamma_{(2)})Z_2^{-1} \times T_{(1)(2)}{}^{*\text{Compt}}(p,p',q). \quad (13)$$

In general, the vertex renormalizations $Z(C)$, $Z(\gamma_{(1)})$, and $Z(\gamma_{(2)})$ are not equal to each other or to Z_2. If we write

$$Z(C)=1+\Lambda(C),$$
$$Z_2=1+\Lambda_2, \quad (14)$$

then we find, to second order, that

$$\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p,p',q)=T_{(1)(2)}{}^{*\text{Compt}}(p,p',q)^{\text{finite}} +[2\Lambda_2-\Lambda(\gamma_{(1)})-\Lambda(\gamma_{(2)})] \quad (15)$$
$$\times\left[\gamma_{(1)}\frac{1}{\gamma\cdot p+\gamma\cdot q}\gamma_{(2)}+\gamma_{(2)}\frac{1}{\gamma\cdot p-\gamma\cdot q'}\gamma_{(1)}\right],$$
$$\tilde{\Gamma}(C;p,p')=\Gamma(C;p,p')^{\text{finite}}+[\Lambda_2-\Lambda(C)]C.$$

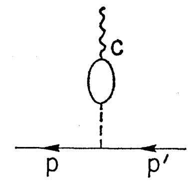

FIG. 4. Self-energy diagram which makes a second-order correction to the SU_3-singlet part of C.

Using the fact that finite quantities on the left- and right-hand sides of Eq. (7b) must match up, we see that

$$\Delta^{\text{Compt}}=[\Lambda_2+\Lambda(C)-\Lambda(\gamma_{(1)})-\Lambda(\gamma_{(2)})]C+\text{finite}, \quad (16)$$

confirming that the $\ln\lambda^2$ dependence in Δ^{Compt} results from a mismatch between the multiplicative renormalization factors on the left- and right-hand sides of Eq. (7b). To check Eq. (16) directly, we note from Eq. (A10) that

$$\Lambda(C)C=(g_r^2/32\pi^2)\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau C\gamma^\tau\boldsymbol{\gamma}\ln\lambda^2,$$
$$\Lambda_2\gamma_0=(g_r^2/32\pi^2)\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_0\gamma^\tau\boldsymbol{\gamma}\ln\lambda^2, \quad (17)$$

which allows us to rewrite the square bracket in Eq. (16) in the form

$$(g_r^2/32\pi^2)\ln\lambda^2\{\gamma_{(1)}\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_0\gamma^\tau\boldsymbol{\gamma}\gamma_{(2)}-\gamma_{(2)}\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_0\gamma^\tau\boldsymbol{\gamma}\gamma_{(1)}$$
$$+\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau[\gamma_{(1)}\gamma_0\gamma_{(2)}-\gamma_{(2)}\gamma_0\gamma_{(1)}]\gamma^\tau\boldsymbol{\gamma}$$
$$-\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}\gamma_0\gamma_{(2)}+\gamma_{(2)}\gamma_0\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}$$
$$-\gamma_{(1)}\gamma_0\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}+\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\}, \quad (18)$$

with the four lines coming from Λ_2, $\Lambda(C)$, $-\Lambda(\gamma_{(1)})$, and $-\Lambda(\gamma_{(2)})$, respectively. A little algebra then shows that Eq. (18) is indeed identical with the $\ln\lambda^2$ part of Eq. (7c).

The presence of terms which diverge as $\ln|q_0|^2$ in Eq. (7c) indicates that, *in the general case, the Bjorken limit does not exist.* The fact that the $\ln|q_0|^2$ and $\ln\lambda^2$ terms occur in the combination $\ln(\lambda^2/|q_0|^2)$ means that, *to second order, the existence of the Bjorken limit is directly connected with the matching of renormalization factors on the left- and right-hand sides of Eq.* (7b): When the renormalization factors match, the Bjorken limit exists; when the factors do not match, the Bjorken limit diverges.[10] Unfortunately, we shall see that this simple connection does not hold in higher orders of perturbation theory.

To interpret the divergence of the Bjorken limit, we note that the renormalized T product can be written as[3]

$$\tilde{T}_{(1)(2)}(p,p',q)=\int_{-\infty}^{\infty}dq_0'\frac{\rho(p,p',\mathbf{q},q_0')}{q_0-q_0'}, \quad (19)$$

[9] F. J. Gilman, Phys. Rev. **167**, 1365 (1968).

[10] This was first noted by A. I. Vainshtein and B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **6**, 917 (1967) [Soviet Phys. JETP Letters **6**, 341 (1967)]. These authors conjectured that when the renormalization factors match, the Bjorken limit and naive commutator agree. Our calculations show that this conjecture is invalid.

TABLE I. Cases involving vector (V) and axial-vector (A) octet currents with finite Bjorken limit in second order.

Model	Current $J_{(1)}$	Current $J_{(2)}$	Piece of current $\bar{\psi}C\psi$
Vector gluon	V or A	V or A	V or A
Scalar or pseudo-	V	V	V
scalar gluon	V	A	A
	A	V	A

where the spectral function ρ is defined by

$$\bar{u}(p)\rho(p,p',\mathbf{q},q_0)u(p') = (2\pi)^3\sum_N \langle p|J_{(1)}|N\rangle\langle N|J_{(2)}|p'\rangle\delta^4(q+p-N) - (2\pi)^3\sum_N \langle p|J_{(2)}|N\rangle\langle N|J_{(1)}|p'\rangle\delta^4(q+N-p'). \quad (20)$$

Provided that the spectral function does not oscillate an infinite number of times[11] (and it cannot have this kind of pathological behavior in perturbation theory), when the Bjorken limit of $\tilde{T}_{(1)(2)}(p, p', q)$ exists it is equal to the integral

$$\bar{u}(p)\int_{-\infty}^{\infty} dq_0'\,\rho(p,p',\mathbf{q},q_0')u(p') = (2\pi)^3\sum_N \langle p|J_{(1)}|N\rangle\langle N|J_{(2)}|p'\rangle\delta^3(\mathbf{q}+\mathbf{p}-\mathbf{N}) - (2\pi)^3\sum_N \langle p|J_{(2)}|N\rangle\langle N|J_{(1)}|p'\rangle\delta^3(\mathbf{q}+\mathbf{N}-\mathbf{p}'), \quad (21)$$

which is just the usual sum-over-intermediate-states definition of the commutator. Conversely, in the cases in which the Bjorken limit diverges logarithmically, the integral and sum in Eq. (21) must diverge logarithmically.

2. There are a number of interesting cases in which the renormalization factors do match, and hence the Bjorken limit exists in second order. We have enumerated in Table I all examples of this type in which all of the currents involved, $J_{(1)}$, $J_{(2)}$, and $\bar{\psi}C\psi$, are either vector or axial-vector octet currents. Specific formulas for Δ^{Compt} in the case when $J_{(1)}$ and $J_{(2)}$ are vector currents were given in Eqs. (10) and (11) above. (To obtain the corresponding formulas when $J_{(1)}$ and/or $J_{(2)}$ are axial-vector currents, in the vector-gluon case, one simply multiplies from the left or right by γ_5 according to the scheme shown in Table II.)

The remarkable result that emerges from these examples is that, *even when the Bjorken limit exists in second order, it does not agree with the naive commutator* (that is, Δ^{Compt} is finite but nonzero). According to our previous discussion, this means that the Bjorken limit agrees with the spectral function integral of Eq. (21), but the naive commutator does not. Most of the principal applications of the Bjorken limit technique for space-component–space-component commutators *assume* the identity of the Bjorken limit and naive commutator, and therefore, according to our results, break down in perturbation theory. Further details of this breakdown are given in Ref. 6.

3. From an inspection of Eq. (10) and Table II, we see that in the vector-gluon case, for all commutators involving vector and axial-vector currents, Δ^{Compt} vanishes when either $\mu=0$ or $\nu=0$. This result can be deduced directly from the Ward identity[10] satisfied by $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p, p', q)$, which, in the case when $J_{(1)}$ and $J_{(2)}$ are both vector currents, states that

$$\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p,p',q)\,|_{\gamma(1)=q^\mu\gamma_\mu\lambda^a,\gamma(2)=\gamma_\nu\lambda^b} = \tilde{\Gamma}([\lambda^a,\lambda^b]\gamma_\nu;p,p'). \quad (22)$$

Multiplying by q_0^{-1} and taking the limit $q_0\rightarrow i\infty$ gives immediately

$$\lim_{q_0\rightarrow i\infty}\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p,p',q)\,|_{\gamma(1)=\gamma_0\lambda^a,\gamma(2)=\gamma_\nu\lambda^b} = q_0^{-1}\tilde{\Gamma}([\lambda^a,\lambda^b]\gamma_\nu;p,p')+O(q_0^{-2}\ln q_0), \quad (23)$$

confirming our explicit calculation. A similar derivation holds in the cases involving axial-vector currents, provided that the divergence of the axial-vector current is "soft,"[12] as it is in the vector-gluon case. We thus see that the breakdown of the Bjorken limit which we have found is consistent with the constraints imposed by Ward identities. Therefore all of the results of the Gell-Mann time-component algebra, which are derived directly from the Ward identities, remain valid.[13]

4. We turn next to the order g_r^4 result of Eq. (12), which gives the $VV\rightarrow V$ commutator in the scalar- and pseudoscalar-gluon models (the second line in Table I). We see that *even though the renormalization factors match, the Bjorken limit in this case diverges in fourth*

TABLE II. Substitutions to get axial-vector current results in the vector-gluon case.

Current $J_{(1)}$	Current $J_{(2)}$	Change in Eq. (10)
V	V	none
A	V	$\Delta^{\text{Compt}}\rightarrow-\gamma_5\Delta^{\text{Compt}}$
V	A	$\Delta^{\text{Compt}}\rightarrow\Delta^{\text{Compt}}\gamma_5$
A	A	$\Delta^{\text{Compt}}\rightarrow-\gamma_5\Delta^{\text{Compt}}\gamma_5$

[11] This pathological case is discussed in detail by R. Brandt and J. Sucher, Phys. Rev. Letters **20**, 1131 (1968).

[12] See Ref. 2, pp. 257–260, for a discussion of soft divergences.

[13] There is one exception to this statement, which arises when Ward identity anomalies are present. See S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969); S. L. Adler and W. A. Bardeen, *ibid.* **182**, 1517 (1969).

order. We note, however, that the divergence behaves as $g_r^4 \ln|q_0|^2$, whereas in fourth order, terms behaving like $g_r^4(\ln|q_0|^2)^2$ could in principle be present. On the basis of this behavior and our second-order results, we make the following *conjecture: When the renormalization factors needed to make $T_{(1)(2)}^*(p, p', q)$ and $\Gamma(C; p, p')$ finite are the same, the Bjorken limit in order 2n of perturbation theory contains no terms $g_r^{2n}(\ln|q_0|^2)^n$, but begins in general with terms $g_r^{2n}(\ln|q_0|^2)^{n-1}$.*

We have only calculated results for the scalar- and pseudoscalar-gluon models because these models have the simple property that, when the unitarity method of Appendix B is used, each individual intermediate state makes a contribution behaving at worst as $g_r^4 \ln|q_0|^2$. The situation in the vector-gluon model is more complicated, since here the individual intermediate states contain terms behaving as $g_r^4(\ln|q_0|^2)^2$, as well as terms $g_r^4 \ln|q_0|^2$. If our conjecture is correct, the $g_r^4(\ln|q_0|^2)^2$ terms from the various intermediate states in the vector-gluon case must add up to zero. We have not checked whether this happens; it would clearly be worth doing.

5. As mentioned in Sec. I, one can try to save asymptotic sum rules by postulating that nonperturbative effects conspire to make asymptotic sum rules valid when all orders of perturbation theory are summed. A simple way that this could happen would be if our order g_r^2 terms in Δ^{Compt} were the lowest-order terms in an expression

$$A \exp[-Bg_r^2 \ln|q_0|^2], \qquad B>0 \tag{24}$$

which damps to zero as $q_0 \to i\infty$. However, examination of our fourth-order result in Eq. (12) shows that exponentiation gives

$$\frac{g_r^2}{16\pi^2}+7\left(\frac{g_r^2}{16\pi^2}\right)^2 \ln|q_0|^2 \approx \frac{g_r^2}{16\pi^2} \times\exp\left(7\,\frac{g_r^2}{16\pi^2}\ln|q_0|^2\right)+O(g_r^6), \tag{25}$$

which blows up exponentially rather than damping. In other words, the simple damping mechanism of Eq. (24) cannot be correct, although our fourth-order calculation obviously cannot rule out more complicated damping mechanisms.

6. In Eqs. (A14) and (A15) of Appendix A, we indicate that when the Bjorken limit $q_0 \to i\infty$ is taken *before* letting the regulator mass λ go to infinity, one obtains just the naive commutator. Thus, it is tempting to try to "save" asymptotic sum rules by prescribing that, instead of using renormalized perturbation theory (limit $\lambda \to \infty$ taken first), one should always work with the unrenormalized quantities, with λ very large but finite.[14] We will now argue, however, that this is a spurious resolution of the difficulty. Let us consider the sum rule, derived in Appendix B, connecting the $[\lambda^a, \lambda^b]$ term in Eq. (11) with an integral over the longitudinal current-nucleon cross section $L^-(q^2, \omega)$, with $\omega = -q^2/p\cdot q$. In the renormalized ($\lambda \to \infty$) theory, where there is Bjorken-limit breakdown, we find *to second order* that

$$\lim_{q_0\to i\infty} \tilde{T}_{(1)(2)}^{*\text{Compt}} = q_0^{-1}\tfrac{1}{2}[\lambda^a, \lambda^b] \times[\gamma_\mu\gamma_0\gamma_\nu+\gamma_\nu\gamma_0\gamma_\mu+2(g_{\mu\nu}-g_{\mu0}g_{\nu0})\gamma_0 f] +(\text{term symmetric in } a, b)+O(q_0^{-2}\ln q_0), \tag{26}$$

$$f=\lim_{q^2\to-\infty} 2\int_0^2 d\omega\, L_{n+g}^-(q^2,\omega), \tag{27}$$

where the subscript $n+g$ is a reminder that to second order we need only retain the single neutron plus gluon intermediate-state contribution in calculating L^-. Our explicit calculation shows that

$$f=g_r^2/16\pi^2,$$
$$\lim_{q^2\to-\infty} L_{n+g}^-(q^2,\omega)=(g_r^2/64\pi^2)\omega, \tag{28}$$

in agreement with Eq. (27). As we noted in Ref. 6, Eq. (27) indicates that the breakdown of the Bjorken limit in Eq. (26) is essentially the same phenomenon as the breakdown of the Callan-Gross relation,[15] which states that

$$\lim_{q^2\to-\infty} L_{n+g}^-(q^2,\omega)=0. \tag{29}$$

Let us now consider the analogs of Eqs. (26)–(29) in the regulated (λ-finite) theory. Since, in order g_r^2, matrix elements are always *linear* in the gluon propagator, to obtain the regulated matrix element in order g_r^2 we simply subtract from the renormalized matrix element the corresponding expression with the gluon mass μ^2 replaced by the regulator mass λ^2. Since f in Eq. (28) is independent of μ^2, we find that Eqs. (26)–(28) become

$$\lim_{q_0\to i\infty} T_{(1)(2)}^{*\text{Compt}} = q_0^{-1}\tfrac{1}{2}[\lambda^a, \lambda^b](\gamma_\mu\gamma_0\gamma_\nu+\gamma_\nu\gamma_0\gamma_\mu) +(\text{term symmetric in } a, b)+O(q_0^{-2}\ln q_0), \tag{30}$$

$$0=\lim_{q^2\to-\infty} 2\int_0^2 d\omega\, L_{\text{tot}}^-(q^2,\omega);$$
$$\lim_{q^2\to-\infty} L_{\text{tot}}^-(q^2,\omega)=0, \tag{31}$$

with

$$L_{\text{tot}}^-(q^2,\omega)=L_{n+g}^-(q^2,\omega)-L_{n+g}^-(q^2,\omega)\,|_{\mu^2\to\lambda^2}. \tag{32}$$

As expected, in the regulated theory the Bjorken limit is normal and the Callan-Gross relation is satisfied. However, a disturbing problem arises when we examine in detail exactly *how* the Bjorken limit is satisfied. Let us suppose that the regulator mass λ is much larger than

[14] This point of view has been advocated by C. R. Hagen, Phys. Rev. **188**, 2416 (1969).

[15] C. G. Callan and D. J. Gross, Phys. Rev. Letters **22**, 156 (1969).

the fermion mass m and the gluon mass μ,

$$\lambda^2 \gg m^2, \qquad \lambda^2 \gg \mu^2, \tag{33}$$

and let us consider, for fixed ω, two ranges of values of $-q^2$,

$$\text{range 1:} \quad \mu^2, m^2 \ll -q^2 < \xi/(2\omega^{-1}-1),$$
$$\text{range 2:} \quad \xi/(2\omega^{-1}-1) \leq -q^2, \qquad \xi=(\lambda+m)^2-m^2. \tag{34}$$

The dividing point between the two ranges is just the threshold for regulator particle production. For $-q^2<\xi/(2\omega^{-1}-1)$, we have $(q+p)^2<(\lambda+m)^2$, and regulator particle production is forbidden. Thus, in range 1, the second term in Eq. (32) vanishes,

$$L_{n+g}^{-}(q^2,\omega)\,|_{\mu^2\to\lambda^2}=0, \tag{35}$$

while the first term has its asymptotic value

$$L_{n+g}^{-}(q^2,\omega)\approx(g_r^2/64\pi^2)\omega, \tag{36}$$

and the Callan-Gross limit is *not* satisfied. In range 2, we have $(q+p)^2 \geq (\lambda+m)^2$, and regulator production is allowed; for $-q^2 \gg \lambda^2/(2\omega^{-1}-1)$, the second term in Eq. (32) attains the same asymptotic value as Eq. (36), and the Callan-Gross limit is satisfied. Thus, we see that *in the regulator theory, the Callan-Gross limit is satisfied only in a region in which $-q^2$ is big on a scale determined by λ^2, and then only by virtue of the unphysical, negative contribution of regulator production to the total longitudinal cross section.* We conclude that the regulator theory does not afford a satisfactory resolution of the breakdown of the Bjorken limit in perturbation theory.

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APPENDIX A: CALCULATION OF Δ^{Compt}

In this appendix we outline the calculation leading to Eq. (7) in the text. We recall that $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}$ is defined as the contribution to the current-fermion scattering amplitude of the diagrams shown in Fig. 1, consisting of the lowest-order current-fermion diagrams and the second-order diagrams obtained from the lowest-order ones by insertion of a single virtual gluon. We may write

$$\tilde{T}_{(1)(2)}{}^{*\text{Compt}}(p,p',q)=\tilde{\Gamma}(\gamma_{(1)};p,p+q)\tilde{S}(p+q)$$
$$\times\tilde{\Gamma}(\gamma_{(2)};p+q,p')+\tilde{\Gamma}(\gamma_{(2)};p,p-q')\tilde{S}(p-q')$$
$$\times\tilde{\Gamma}(\gamma_{(1)};p-q',p')+B_{(1)(2)}(p,p',q), \tag{A1}$$

where $\tilde{S}$ and $\tilde{\Gamma}$ are the renormalized propagator and vertex functions and where $B_{(1)(2)}$ denotes the sum of the two box diagrams on the fifth line of Fig. 1. We shall calculate $\tilde{T}_{(1)(2)}{}^{*\text{Compt}}$ in the limit $q_0\to i\infty$ and isolate the coefficient of the q_0^{-1} term. This is to be compared with the matrix element of the naive commutator between fermion states, given by $\tilde{\Gamma}(C;p,p')$.

The renormalized vertex function $\tilde{\Gamma}$ for the current $\bar{\psi}\gamma_{(1)}\psi$ is given, to second order in g, by

$$\tilde{\Gamma}(\gamma_{(1)};p,p')=Z_2\Gamma(\gamma_{(1)};p,p')=Z_2\gamma_{(1)}+\Lambda(\gamma_{(1)};p,p'), \tag{A2}$$

with Z_2 the fermion wave-function renormalization and with $\Lambda(\gamma_{(1)};p,p')$ the usual unrenormalized second-order vertex part (arising from diagrams on the second and fourth lines of Fig. 1). Note that $\tilde{\Gamma}$ is obtained by multiplying the unrenormalized vertex function by the wave-function renormalization, with *no further subtractions or rescaling*. The renormalized propagator is given, to second order in g_r, by the usual expression[16]

$$\tilde{S}(p)^{-1}=Z_2S(p)^{-1}=Z_2(\gamma\cdot p-m_0)-\Sigma(p), \tag{A3}$$

with $\Sigma(p)$ the unrenormalized proper fermion self-energy part (arising from the diagrams on the third line of Fig. 1) and with $m_0=m+\delta m$ the fermion bare mass. Denoting the lowest-order current-fermion amplitude by $T_{(1)(2)}{}^{\text{Born}}$, we see that the first two lines on the right-hand side of Eq (A1) may be rewritten as

$$Z_2T_{(1)(2)}{}^{\text{Born}}+[\Lambda(\gamma_{(1)};p,p+q)(\gamma\cdot p+\gamma\cdot q-m)^{-1}\gamma_{(2)}$$
$$+\gamma_{(1)}(\gamma\cdot p+\gamma\cdot q-m)^{-1}(\delta m+\Sigma(p+q))$$
$$\times(\gamma\cdot p+\gamma\cdot q-m)^{-1}\gamma_{(2)}+\gamma_{(1)}(\gamma\cdot p+\gamma\cdot q-m)^{-1}$$
$$\times\Lambda(\gamma_{(2)};p+q,p')+((1)\leftrightarrow(2),q\leftrightarrow-q')]. \tag{A4}$$

According to Eq. (A2), the matrix element of the naive commutator is

$$Z_2C+\Lambda(C;p,p'). \tag{A5}$$

It is easy to see that, as $q_0\to i\infty$, the q_0^{-1} term of $Z_2T_{(1)(2)}{}^{\text{Born}}$ is precisely Z_2C. Our task is therefore reduced to comparing the q_0^{-1} term of

$$[\Lambda(\gamma_{(1)};p,p+q)(\gamma\cdot p+\gamma\cdot q-m)^{-1}\gamma_{(2)}+\cdots$$
$$+((1)\leftrightarrow(2),q\leftrightarrow-q')]+B_{(1)(2)}(p,p',q) \tag{A6}$$

with $\Lambda(C;p,p')$.

The unrenormalized self-energy and vertex parts Σ and Λ are calculated by the usual technique of introducing a meson regulator of mass λ, giving

$$\Sigma(p)=\frac{-g_r^2}{16\pi^2}\int_0^1 dx\,\boldsymbol{\gamma}(x\gamma\cdot p+m)\,\ln\boldsymbol{\gamma}\left[\frac{x(1-x)(-p^2+m^2)+x\lambda^2+(1-x)^2m^2}{x(1-x)(-p^2+m^2)+x\mu^2+(1-x)^2m^2}\right] \tag{A7}$$

[16] In this equation, $S(p)$ denotes the unrenormalized propagator in the presence of all interactions.

and

$$\Lambda(\gamma_{(1)};p,p')=\frac{-g_r^2}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\bigg\{\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}\ln\left[\frac{z\lambda^2-C(p,p',x,y)}{z\mu^2-C(p,p',x,y)}\right]$$
$$+\boldsymbol{\gamma}[(1-x)\gamma\cdot p-y\gamma\cdot p'+m]\gamma_{(1)}[(1-y)\gamma\cdot p'-x\gamma\cdot p+m]\boldsymbol{\gamma}\left[\frac{1}{C(p,p',x,y)-z\mu^2}-\frac{1}{C(p,p',x,y)-z\lambda^2}\right]\bigg\}, \quad (A8)$$

where

$$z=1-x-y,$$
$$C(p,p',x,y)=x(1-x)p^2+y(1-y)p'^2-2xyp\cdot p'-(x+y)m^2. \quad (A9)$$

In order to obtain the renormalized propagator and vertex from Eqs. (A2) and (A3), we must calculate the $\lambda\to\infty$ limit of the unrenormalized self-energy and vertex parts, dropping all terms which vanish in this limit but retaining powers of $\ln\lambda$. [Note that the $\lambda\to\infty$ limits of Σ and Λ are *not* the same as the *renormalized* self-energy and vertex parts $\tilde{\Sigma}$ and $\tilde{\Lambda}$, which are defined, in the $\lambda\to\infty$ limit, by $Z_2(\gamma\cdot p-m_0)-\Sigma(p)=\gamma\cdot p-m-\tilde{\Sigma}(p)$, $Z_2\gamma_{(1)}+\Lambda(\gamma_{(1)};p,p')=\gamma_{(1)}+\tilde{\Lambda}(\gamma_{(1)};p,p')$.] Taking the $\lambda\to\infty$ limit in Eqs. (A7)-(A9), we find

$$\lim_{\lambda\to\infty}\Sigma(p)=\frac{-g_r^2}{16\pi^2}\int_0^1 dx\ \boldsymbol{\gamma}(x\gamma\cdot p+m)\boldsymbol{\gamma}\ln\left[\frac{x\lambda^2}{x(1-x)(-p^2+m^2)+x\mu^2+(1-x)^2m^2}\right],$$

$$\lim_{\lambda\to\infty}\Lambda(\gamma_{(1)};p,p')=\frac{-g_r^2}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\ \bigg\{\tfrac{1}{2}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}\ln\left[\frac{z\lambda^2}{z\mu^2-C(p,p',x,y)}\right]$$
$$+\boldsymbol{\gamma}[(1-x)\gamma\cdot p-y\gamma\cdot p'+m]\gamma_{(1)}[(1-y)\gamma\cdot p'-x\gamma\cdot p+m]\boldsymbol{\gamma}\frac{1}{C(p,p',x,y)-z\mu^2}\bigg\}. \quad (A10)$$

Finally, taking infinite-momentum limits of these expressions, we find

$$\lim_{p_0\to i\infty}\lim_{\lambda\to\infty}\Sigma(p)=(-g_r^2/16\pi^2)\boldsymbol{\gamma}\gamma_0\boldsymbol{\gamma}p_0[\tfrac{1}{2}\ln(\lambda^2/|p_0|^2)+\tfrac{3}{4}]+O(\ln p_0),$$
$$\lim_{p_0\to i\infty}\lim_{\lambda\to\infty}\Lambda(\gamma_{(1)};p,p')=(g_r^2/16\pi^2)\{-\tfrac{1}{4}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}[\ln(\lambda^2/|p_0|^2)+\tfrac{1}{2}]+\tfrac{1}{2}\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\gamma_0\boldsymbol{\gamma}+O(\ln p_0/p_0)\}, \quad (A11)$$
$$\lim_{p_0'\to i\infty}\lim_{\lambda\to\infty}\Lambda(\gamma_{(1)};p,p')=(g_r^2/16\pi^2)\{-\tfrac{1}{4}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma^\tau\boldsymbol{\gamma}[\ln(\lambda^2/|p_0'|^2)+\tfrac{1}{2}]+\tfrac{1}{2}\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\gamma_0\boldsymbol{\gamma}+O(\ln p_0'/p_0')\}.$$

The box diagram is convergent even without regularization. In the regulator theory, $B_{(1)(2)}$ is the difference of two terms calculated with meson masses μ and λ, respectively, but the term with mass λ does not contribute in the limit $\lambda\to\infty$. [The situation is similar to the second term on the right-hand side of Eq. (A8).] A little care must be exercised in computing the Bjorken limit of $B_{(1)(2)}$. The reason is that, because of infrared singularities, the limit $q_0\to i\infty$ cannot be naively taken under the integrals over the Feynman parameters.[17] A detailed study yields

$$\lim_{q_0\to i\infty}\lim_{\lambda\to\infty}B_{(1)(2)}(p,p',q)=q_0^{-1}\Lambda(C;p,p')$$
$$+(g_r^2/16\pi^2)q_0^{-1}\{\tfrac{1}{4}\boldsymbol{\gamma}\gamma_\tau\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma^\tau\boldsymbol{\gamma}\ln(\lambda^2/|q_0|^2)$$
$$+\tfrac{1}{8}\boldsymbol{\gamma}[\gamma_\tau\gamma_{(1)}\gamma^\tau\gamma_{(2)}\gamma_0+\gamma_0\gamma_{(1)}\gamma_\tau\gamma_{(2)}\gamma^\tau]\boldsymbol{\gamma}$$
$$-\tfrac{1}{4}\boldsymbol{\gamma}\gamma_0\gamma_{(1)}\gamma_0\gamma_{(2)}\gamma_0\boldsymbol{\gamma}-[(1)\leftrightarrow(2)]\}+O(\ln q_0/q_0^2). \quad (A12)$$

Note that, according to Eqs. (3) and (A11), the $\ln\lambda^2$ dependence of $\Lambda(C;p,p')$ *precisely cancels* the $\ln\lambda^2$ in the curly bracket in Eq. (A12), as required by the absence of $\ln\lambda^2$ dependence on the left-hand side. Substituting Eqs. (A11) and (A12) into Eq. (A6), we obtain, finally,

$$\lim_{q_0\to i\infty}\tilde{T}_{(1)(2)}{}^{*\mathrm{Compt}}(p,p',q)=(1/q_0)$$
$$\times[\tilde{\Gamma}(C;p,p')+\Delta^{\mathrm{Compt}}]+O(q_0^{-2}\ln q_0), \quad (A13)$$

with Δ^{Compt} as given by Eq. (7c) of the text.

To conclude this appendix we remark that if, starting from the regulated quantities of Eqs. (A7) and (A8) and the regulated box-diagram part $B_{(1)(2)}$, one took the Bjorken limit $q_0\to i\infty$ *before* letting the regulator mass λ go to infinity, one would obtain

$$\lim_{p_0\to i\infty}\Sigma(p)=O(1/p_0),$$
$$\lim_{p_0\to i\infty}\Lambda(\gamma_{(1)};p,p')=O(1/p_0),$$
$$\lim_{p_0\to i\infty}\Lambda(\gamma_{(1)};p,p')=O(1/p_0'), \quad (A14)$$
$$\lim_{q_0\to i\infty}B_{(1)(2)}(p,p',q)=(1/q_0)\Lambda(C;p,p')+O(1/q_0^2).$$

[17] The problem which one encounters here can be illustrated by a simple example. Consider the integral $\int_0^1 dx\ (Ax^2q^2+B)/(xq^2+C)^2$, with A, B, and C constants. In the limit $q^2\to-\infty$, both terms in the numerator behave as $(q^2)^{-1}$, although at first glance one might expect the second term to behave like $(q^2)^{-2}$ and to be negligible compared to the first.

TABLE III. Regions of phase space where denominators in Eq. (2.19) vanish as $\mu^2\to 0$. n^s, g_1^s, $\cdots$ denote the spatial components ($s=1,2,3$) of n, g_1, $\cdots$.

	Phase-space region	Denominators which vanish
(1)	$n^s=0$	$(n+g_2)^2$, $(n+g_1)^2$
(2)	$g_1^s=0$	$(n+g_1)^2$, $(p-g_1)^2$
(3)	$g_2^s=0$	$(n+g_2)^2$, $(p-g_2)^2$
(4)	$g_1^s \parallel p^s$	$(p-g_1)^2$
(5)	$g_2^s \parallel p^s$	$(p-g_2)^2$
(6)	$g_1^s \parallel p^s$ and $g_2^s \parallel p^s$	$(p-g_1)^2$, $(p-g_2)^2$, $(p-g_1-g_2)^2$
(7)	$g_1^s \parallel n^s$	$(n+g_1)^2$
(8)	$g_2^s \parallel n^s$	$(n+g_2)^2$

As a consequence, one finds

$$\lim_{q_0\to i\infty} T_{(1)(2)}{}^{*\text{Compt}} = (1/q_0)\Gamma(C;p,p') + O(1/q_0^2)\,; \quad \text{(A15)}$$

that is, the Bjorken limit in the case of finite regulator mass agrees with the naive commutator. This result is expected for the regulator theory since the anomalous term Δ^{Compt} is independent of the gluon mass and is canceled by exactly the same term (with opposite sign) which must be present when the regulator mass is kept finite.

APPENDIX B: FOURTH-ORDER CALCULATION

In this appendix we consider an extension of our previous results to order g_r^4. Unfortunately, repeating the general calculation of Appendix A in the next order of perturbation theory would require a prohibitive amount of work, and therefore will not be attempted. Rather, we will content ourselves with the calculation of one special case, which is made tractable by a combination of tricks. The special case is the SU_3-antisymmetric piece of the vector-vector commutator, in the scalar- and pseudoscalar-gluon models. There are two further restrictions. We consider *only* the leading logarithmic behavior in the Bjorken limit, and we limit ourselves to the part of the commutator which, like Δ^{Compt}, is *independent* of the three-momenta $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{p}$, and $\mathbf{p}'$ and of the fermion mass m. This second restriction means that we can set $\mathbf{q}=\mathbf{q}'=\mathbf{p}=\mathbf{p}'=\mathbf{0}$ at the outset, so that we are dealing with the forward Compton scattering amplitude, and that we can take the limit $m\to 0$ wherever $\ln m$ divergences do not appear. (We will verify that there are no $\ln m$ factors in the leading $\ln|q_0|^2$ term.) The restrictions allow us to employ the following two tricks, which make the calculation tractable: (i) We exploit a connection, provided by unitarity, between the Bjorken limit of the forward Compton amplitude and current-fermion cross sections. This connection becomes especially simple in the $m\to 0$ limit. (ii) For dimensional reasons, $\ln|q_0|^2$ terms in the current-fermion cross section (at $m=0$) must be accompanied by $-\ln\mu^2$ terms, where μ is the gluon mass, so we can study the large-$|q_0|^2$ behavior by studying the small-μ^2 singularities. The latter arise from readily identifiable regions of phase space, and are much more easily evaluated than the complete current-fermion cross section itself.

We begin by reviewing the unitarity connection[6,9] between the current-fermion cross sections and the forward Compton amplitude. Since we are only interested in the commutator of two vector currents, we set $\gamma_{(1)}=\gamma_\mu\lambda^a$, $\gamma_{(2)}=\gamma_\nu\lambda^b$, $J_{(1)}=J_\mu{}^a$, and $J_{(2)}=J_\nu{}^b$. It will further be convenient to restrict a and b to lie in the isospin SU_2 subspace of SU_3 (a, $b=1$, 2); this has no effect on the part of the commutator antisymmetric in a and b, and has the virtue of making the charge structure of our problem identical to the familiar case of pion-nucleon scattering. Denoting $\omega=-q^2/p\cdot q$, we may write for the spin-averaged, forward-scattering current-"proton" amplitude,[18]

$$\begin{aligned}\tfrac{1}{4}\mathrm{Tr}\left[\left(\frac{\gamma\cdot p+m}{2m}\right)\tilde{T}_{(1)(2)}{}^*(p,p,q)\right] &= \text{polynomial}+\left(\frac{-i}{4}\right)\sum_{\text{spin}(p)}\int d^4x\, e^{iq\cdot x}\langle p\,|\,T(J_\mu{}^a(x)J_\nu{}^b(0))\,|\,p\rangle \\ &= \tfrac{1}{4}\mathrm{Tr}\left[\left(\frac{\gamma\cdot p+m}{2m}\right)\left(\gamma_\mu\lambda^a\frac{1}{\gamma\cdot p+\gamma\cdot q-m}\gamma_\nu\lambda^b+\gamma_\nu\lambda^b\frac{1}{\gamma\cdot p-\gamma\cdot q-m}\gamma_\mu\lambda^a\right)\right] \\ &\quad +T_1{}^{ab}(q^2,\omega)\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+T_2{}^{ab}(q^2,\omega)\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right). \quad \text{(B1)}\end{aligned}$$

On the third and fourth lines, we have explicitly separated off the Born approximation and made use of the vector Ward identities for $\tilde{T}_{(1)(2)}{}^*(p,p,q)$, which imply that the non-Born part is divergenceless. The isospin structure of the non-Born amplitudes may be written in the form

$$T_{1,2}{}^{ab}(q^2,\omega)=T_{1,2}{}^{(+)}(q^2,\omega)\tfrac{1}{2}\{\lambda^a,\lambda^b\}+T_{1,2}{}^{(-)}(q^2,\omega)\tfrac{1}{2}[\lambda^a,\lambda^b]. \quad \text{(B2)}$$

The standard forward dispersion relations analysis for pion-nucleon scattering[19] may now be taken over to show

[18] Here "proton" means the p-type quark, and similarly "neutron" means the n-type quark. The actual matrix element is obtained by sandwiching Eq. (B1) between "proton" isospinors.

[19] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

that the amplitudes $T_{1,2}^{(\pm)}$ satisfy the following dispersion relations:

$$T_1^{(+)}(q^2,\omega)=T_1^{(+)}(q^2,\infty)-\int_0^2 d\omega'[W_1^-(q^2,\omega')+W_1^+(q^2,\omega')][(\omega'-\omega)^{-1}+(\omega'+\omega)^{-1}],$$

$$T_1^{(-)}(q^2,\omega)=-\int_0^2 d\omega'[W_1^-(q^2,\omega')-W_1^+(q^2,\omega')][(\omega'-\omega)^{-1}-(\omega'+\omega)^{-1}],$$

$$T_2^{(+)}(q^2,\omega)=-\omega\int_0^2 \frac{d\omega'}{\omega'}[W_2^-(q^2,\omega')+W_2^+(q^2,\omega')][(\omega'-\omega)^{-1}-(\omega'+\omega)^{-1}], \tag{B3}$$

$$T_2^{(-)}(q^2,\omega)=-\omega\int_0^2 \frac{d\omega'}{\omega'}[W_2^-(q^2,\omega')-W_2^+(q^2,\omega')][(\omega'-\omega)^{-1}+(\omega'+\omega)^{-1}],$$

with absorptive parts given by

$$-(2\pi)^3\tfrac{1}{4}\sum_{\mathrm{spin}(p)}\sum_N\langle p\,|\,2^{-1/2}(J_\mu{}^1\mp iJ_\mu{}^2)\,|\,N\rangle\langle N\,|\,2^{-1/2}(J_\nu{}^1\pm iJ_\nu{}^2)\,|\,p\rangle\delta^4(p+q-N)$$

$$=2W_1^\pm(q^2,\omega)\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+2W_2^\pm(q^2,\omega)\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right). \tag{B4}$$

In writing Eq. (B3), we have assumed one subtraction each for $T_1^{(\pm)}$ [the subtraction constant $T_1^{(-)}(q^2,\infty)$ vanishes by crossing symmetry] and no subtraction for $T_2^{(\pm)}$. To second order in g_r^2 we have explicitly checked the validity of these assumptions. Since the asymptotic behavior as $\omega\to 0$ of higher orders of perturbation theory will differ from second order only by powers of $\ln\omega$, and not by powers of ω, we expect these assumptions to be true to arbitrary order, and in particular, to order g_r^4.

Let us now set $\mathbf{q}=\mathbf{p}=\mathbf{0}$, $\mu=\nu=1$ in Eq. (B1) and take the limit $q_0\to i\infty$. Using Eqs. (B2) and (B3), we find that the right-hand side of Eq. (B1) becomes

$$q_0^{-1}\tfrac{1}{2}[\lambda^a,\lambda^b]\times\left\{1-\lim_{q^2\to-\infty}2m\int_0^2 d\omega'[W_1^-(q^2,\omega')-W_1^+(q^2,\omega')]\right\}+(\text{term symmetric in } a,b)+O(q_0^{-2}\ln q_0). \tag{B5}$$

We know that the Bjorken limit of $\tilde{T}_{(1)(2)}{}^*(p,p,q)$ must have the general form

$$\lim_{q_0\to i\infty;\mathbf{q}=\mathbf{p}=0}\tilde{T}_{(1)(2)}{}^*=q_0^{-1}\tfrac{1}{2}[\lambda^a,\lambda^b]\times[\gamma_\mu\gamma_0\gamma_\nu+\gamma_\nu\gamma_0\gamma_\mu+2(g_{\mu\nu}-g_{\mu0}g_{\nu0})\gamma_0 f(q^2/\mu^2,m^2/\mu^2)]+(\text{term symmetric in } a,b)+O(q_0^{-2}\ln q_0), \tag{B6}$$

with f the difference between the Bjorken limit and the naive commutator. Setting $\mu=\nu=1$ in Eq. (B6), substituting for the left-hand side of Eq. (B1), and comparing with Eq. (B5), we get a sum rule for f,

$$f(q^2/\mu^2,m^2/\mu^2)=\lim_{q^2\to-\infty}2m\int_0^2 d\omega'\times[W_1^-(q^2,\omega')-W_1^+(q^2,\omega')]. \tag{B7}$$

Equation (B7) can be rewritten in a more useful form by recalling that the usual fixed-q^2 sum rule, following from the Gell-Mann time-component algebra, is

$$0=\int_0^2\frac{d\omega'}{\omega'^2}[W_2^-(q^2,\omega')-W_2^+(q^2,\omega')], \tag{B8}$$

and is valid to all orders in perturbation theory in our models. Multiplying Eq. (B8) by $2mq^2$ and adding to Eq. (B7), we get the modified sum rule

$$f(q^2/\mu^2,m^2/\mu^2)=\lim_{q^2\to-\infty}2\int_0^2 d\omega'\times[L^-(q^2,\omega')-L^+(q^2,\omega')], \tag{B9}$$

with

$$L^\mp(q^2,\omega)=2m[W_1^\mp(q^2,\omega)+(q^2/\omega^2)W_2^\mp(q^2,\omega)], \tag{B10}$$

the total longitudinal cross section for current-fermion scattering. The great virtue of Eq. (B9) is that, in the limit $m\to 0$, the longitudinal cross sections are given by the simple formula[20]

$$L^\mp=-(m\omega^2/q^2)(2\pi)^3\times\tfrac{1}{4}\sum_{\mathrm{spin}(p)}\sum_N|\langle p\,|\,p^\tau\tfrac{1}{2}(J_\tau{}^1\pm iJ_\tau{}^2)\,|\,N\rangle|^2\delta^4(p+q-N), \tag{B11}$$

as may be readily verified by comparison of Eqs. (B11) and (B4). We will see that the factor p^τ in Eq. (B11) enormously simplifies the subsequent calculation.

We are now ready to proceed with the calculation of f to order g_r^4. Before doing this, however, let us illustrate the procedure and check the arithmetic done so far by using Eqs. (B9) and (B11) to recalculate the order g_r^2 result contained in Eq. (11) of the text. To second order, the intermediate states which may contribute are the single "neutron,"[18] $N=n$, and the "neutron" plus gluon, $N=n+g$ (Fig. 5). Neither of these contributes to L^+, and the single "neutron" contribution to L^- vanishes to order g_r^2, because the zeroth-order

[20] We wish to thank D. J. Gross for pointing this out to us.

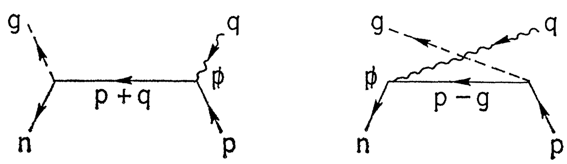

FIG. 5. Diagrams of order g_r contributing to the "neutron" plus gluon intermediate state.

part of $\langle p \mid p^\tau(J_\tau^1+iJ_\tau^2) \mid n\rangle$ is proportional to $\bar{u}(p)\gamma\cdot p u(n)=0$. So we have

$$L^+=0,$$

$$L^-=\frac{-m\omega^2}{q^2}(2\pi)^3\tfrac{1}{4}\sum_{\text{spin}(p)}\sum_{\text{spin}(n)}\int\frac{d^3n}{(2\pi)^3}\times\frac{m}{n^0}\int\frac{d^3g}{(2\pi)^3}\frac{1}{2g^0}\delta^4(p+q-n-g)\,|\mathfrak{M}|^2, \quad \text{(B12)}$$

$$\mathfrak{M}=g_r\bar{u}(n)\left(\frac{1}{\gamma\cdot p+\gamma\cdot q}\gamma\cdot p+\gamma\cdot p\frac{1}{\gamma\cdot p-\gamma\cdot g}\right)u(p),$$

with the factors $\gamma\cdot p$ in $\mathfrak{M}$ a result of the factor p^τ multiplying the current in Eq. (B11).[21] The factor $(\gamma\cdot p+\gamma\cdot q)^{-1}=(\gamma\cdot p+\gamma\cdot q)/[p\cdot q(2-\omega)]$ in the first term in $\mathfrak{M}$ would, if it survived, lead to a divergence in Eq. (B9) at the end point $\omega=2$, but it vanishes on account of the $\gamma\cdot p$ in the numerator. The second term in $\mathfrak{M}$ is also simplified by the presence of $\gamma\cdot p$, since it can be written as

$$g_r\bar{u}(n)[-2p\cdot g/(\mu^2-2p\cdot g)]u(p),$$

which approaches the finite quantity $g_r\bar{u}(n)u(p)$ in the limit of vanishing gluon mass μ^2. As a result, L^- remains finite in the limit as $\mu^2\to 0$ and, by the dimensional argument stated above, we expect L^- to be finite in the limit $q^2\to-\infty$. This reasoning can be confirmed by direct evaluation of Eq. (B12), which gives

$$\lim_{q^2\to-\infty} L^-(q^2,\omega)=(g_r^2/64\pi^2)\omega; \quad \text{(B13)}$$

substituting into Eq. (B9) then gives

$$\lim_{q^2\to-\infty;\,m\to 0} f=g_r^2/16\pi^2, \quad \text{(B14)}$$

in agreement with the $[\lambda^a,\lambda^b]$ term in Eq. (11).

To order g_r^4, we will not try to calculate the finite part of f, but only the part which diverges logarithmically as $q^2\to-\infty$. By our dimensional argument, this can be accomplished by isolating the part of f which diverges like $\ln\mu^2$ as $\mu^2\to 0$. There are four intermediate states which contribute in fourth order: (i) single "neutron", $N=n$ (Fig. 6); (ii) "neutron" plus one gluon, $N=n+g$ (Fig. 7); (iii) "neutron" plus two gluons, $N=n+g_1+g_2$ (Fig. 8); (iv) trident, $N=n+p+\bar{p}$, $n+n+\bar{n}$, or $p+p+\bar{n}$ (Fig. 9). The first three contribute only to L^-, while the trident intermediate state contributes to both L^+ and L^-. We consider the cases in turn.

(i) Single "neutron." The second-order part of $\langle n \mid p^\tau(J_\tau^1-iJ_\tau^2) \mid p\rangle$ is proportional to $\bar{u}(n)\tilde{\Lambda}(\gamma\cdot p;$

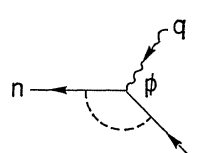

FIG. 6. Diagram of order g_r^2 contributing to the one "neutron" intermediate state.

[21] As we noted, the fermion mass m is zero. The factor m^2 in front of Eq. (B12) and subsequent equations just cancels a corresponding factor m^{-2} coming from our choice of spinor normalization.

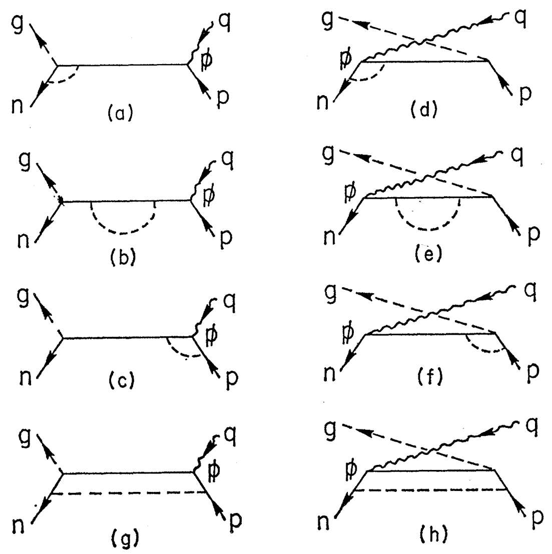

FIG. 7. Diagrams of order g_r^3 contributing to the "neutron" plus gluon intermediate state.

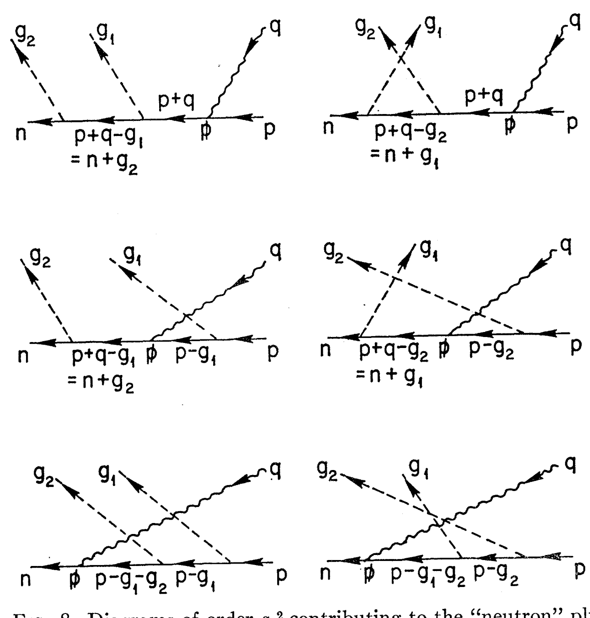

FIG. 8. Diagrams of order g_r^2 contributing to the "neutron" plus two gluon intermediate state.

TABLE IV. Phase-space regions and pieces of $|\mathfrak{M}|^2$ which actually make divergent contributions to Eq. (2.18). $n^s, g_1^s, \cdots$ denote the spatial components ($s=1, 2, 3$) of $n, g_1, \cdots$.

	Phase-space region	Piece of $\vert\mathfrak{M}\vert^2$
(4)	$g_1^s \parallel p^s$	$\vert g_r^2\bar{u}(n)\{\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2)][1/(\gamma\cdot p-\gamma\cdot g_1)]\}u(p)\vert^2$
(5)	$g_2^s \parallel p^s$	$\vert g_r^2\bar{u}(n)\{\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2)][1/(\gamma\cdot p-\gamma\cdot g_2)]\}u(p)\vert^2$
(7)	$g_1^s \parallel n^s$	$\vert g_r^2\bar{u}(n)\{[1/(\gamma\cdot n+\gamma\cdot g_1)]\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_2)]\}u(p)\vert^2$
(8)	$g_2^s \parallel n^s$	$\vert g_r^2\bar{u}(n)\{[1/(\gamma\cdot n+\gamma\cdot g_2)]\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_1)]\}u(p)\vert^2$

$n, p)u(p)$, where Λ is the renormalized vertex part. Using the fact that $p^2=0$, one sees from Eq. (A8) that $\tilde{\Lambda}(\gamma\cdot p; n, p)$ contains only a piece proportional to $\gamma\cdot p$ and a piece proportional to $(\gamma\cdot n)(\gamma\cdot p)(\gamma\cdot n)$, both of which vanish when sandwiched between the spinors. So the single- "neutron" contribution is zero.

(ii) "Neutron" plus one gluon. The "neutron"-plus- one- gluon contribution, in fourth order, arises from the interference of the first-order diagrams in Fig. 5 with the third-order diagrams in Fig. 7. The third-order diagrams are clearly of the same structure as the diagrams in Fig. 1, which we have already evaluated in our general treatment of the order $-g_r^2$ case. We note first that, because of the factor $\gamma\cdot p$, the contributions of Figs. 7(a) and 7(b) vanish. Thus, just as in the case of the first-order matrix element, the terms containing $(\gamma\cdot p+\gamma\cdot q)^{-1}\propto(2-\omega)^{-1}$ vanish, and

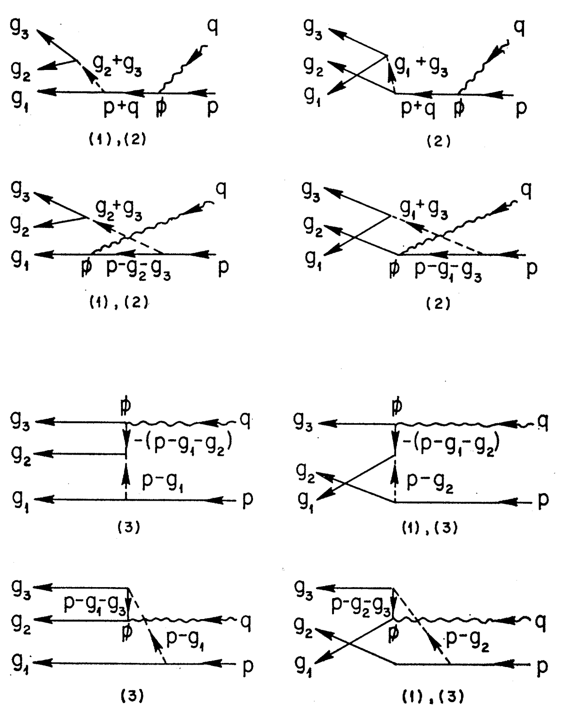

FIG. 9. Diagrams of order g_r^2 contributing to the trident intermediate states.

as a result the integral over ω' in Eq. (B9) converges, even for vanishing gluon mass μ^2. This means that any $\ln\mu^2$ singularities in f must result from $\ln\mu^2$ singularities in L^- itself.

To evaluate the contribution of Figs. 7(c)–7(f), we calculate the *renormalized* self-energy and vertex parts $\tilde{\Sigma}$ and $\tilde{\Lambda}$, by performing the usual mass and wave-function renormalizations on the unrenormalized quantities of Eqs. (A10). Note that the renormalized quantities contain no dependence on the cutoff λ, guaranteeing the validity of our dimensional arguments. In the treatment of the gluon vertex correction in Fig. 7(*f*), a subtlety arises. Instead of subtracting the vertex part at $g^2=\mu^2$, as required by the Watson-Lepore[22] convention, we subtract at $g^2=0$. The difference between the two methods of subtraction makes a contribution to L^- which is proportional to $\ln(m^2/\mu^2)$, but which, for fixed ω, is independent of q^2 and therefore can be dropped. This is the only place in the entire calculation where we encounter $\ln m^2$ terms and where the presence of a $\ln\mu^2$ term does not indicate the presence of a term $-\ln q^2$. When the gluon vertex part is subtracted at $g^2=0$, the $m\to 0$ limit is finite, and our usual dimensional argument applies. On substituting the expressions for $\tilde{\Sigma}$ and $\tilde{\Lambda}$ into the third-order matrix element, we find that the integration over the intermediate state $(n+g)$ variables is always convergent, so that $\ln\mu^2$ terms in L^- can *only arise* from the explicit $\ln\mu^2$ dependence of $\tilde{\Sigma}$ and $\tilde{\Lambda}$. We then find for the contributions of the various diagrams to L^-,

$$\begin{aligned}
&\lim_{\mu^2\to 0} L_{7(c)}^- = \text{finite},\\
&\lim_{\mu^2\to 0} L_{7(d)}^- = (g_r^2/4\pi)^2(\omega/64\pi^2)\ \ln\mu^2+\text{finite},\\
&\lim_{\mu^2\to 0} L_{7(e)}^- = -(g_r^2/4\pi)^2(\omega/64\pi^2)\ \ln\mu^2+\text{finite},\\
&\lim_{\mu^2\to 0} L_{7(f)}^- = -2(g_r^2/4\pi)^2(\omega/64\pi^2)\ \ln\mu^2+\text{finite}.
\end{aligned} \tag{B15}$$

Next, we must examine the contribution of the box diagrams of Figs. 7(g) and 7(h). We deal with these diagrams by writing them in Feynman parametrized form and substituting into the expression for L^-. For example, the contribution of Fig. 7(g) to L^- is pro-

[22] K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949).

portional to

$$\int_0^{\bar{\nu}_{\max}} d\bar{\nu} \int_0^1 x^2 dx \int_0^1 z\, dz \int_0^1 dy$$
$$\times \{ (1/D_a^2) 2x[(1-z+yz)\nu - yz\bar{\nu}][(1-x)\mu^2 + 2xz(1-y)\nu] + (2/D_a) [\nu[1-2x(1-z+yz)] + 2xyz\bar{\nu}]\}, \quad \text{(B16)}$$

$$D_a = \mu^2[x-1+x^2yz(1-z)] + x(1-x)(1-z)(p+q)^2 - x^2(1-y)z[2(1-z+yz)\nu - 2yz\bar{\nu} - (1-z)(p+q)^2],$$
$$\nu = p\cdot q, \qquad \bar{\nu} = p\cdot g.$$

For general values of $q\cdot p$ and q^2, a singularity of Eq. (B16) at $\mu^2=0$ can only arise from the integration end points $\bar{\nu}=0$, $x=0$, $x=1$, $\cdots$, $y=1$.[23] A careful analysis of the behavior of Eq. (B16) at these end points in all possible combinations shows that there is no $\ln\mu^2$ term as $\mu^2\to 0$. A similar analysis yields the same result for Fig. 7(h), so we get, finally,

$$\lim_{\mu^2\to 0} L_{7(g)}^- = \lim_{\mu^2\to 0} L_{7(h)}^- = \text{finite}. \quad \text{(B17)}$$

This completes our analysis of the "neutron" plus one gluon intermediate state.

(iii) "Neutron" plus two gluons. The "neutron" plus two gluon contribution arises from the square of the second-order matrix element corresponding to the diagrams in Fig. 8. We have

$$L_{2\ \text{gluon}}^- = \frac{-m\omega^2}{q^2}(2\pi)^3 \tfrac{1}{4} \sum_{\text{spin}(p)} \sum_{\text{spin}(n)} \int \frac{d^3n}{(2\pi)^3}\frac{m}{n^0}\frac{1}{2} \int \frac{d^3g_1}{(2\pi)^3}\frac{1}{2g_1^0} \int \frac{d^3g_2}{(2\pi)^3}\frac{1}{2g_2^0}\delta^4(p+q-n-g_1-g_2)\,|\mathfrak{M}|^2, \quad \text{(B18)}$$

with

$$\mathfrak{M} = g_r^2\bar{u}(n)\bigg(\frac{1}{\gamma\cdot n+\gamma\cdot g_2}\gamma\cdot p\frac{1}{\gamma\cdot p-\gamma\cdot g_1} + \frac{1}{\gamma\cdot n+\gamma\cdot g_1}\gamma\cdot p\frac{1}{\gamma\cdot p-\gamma\cdot g_2} + \gamma\cdot p\frac{1}{\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2}\frac{1}{\gamma\cdot p-\gamma\cdot g_1}$$
$$+\gamma\cdot p\frac{1}{\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2}\frac{1}{\gamma\cdot p-\gamma\cdot g_2}\bigg)u(p). \quad \text{(B19)}$$

Only four terms appear in $\mathfrak{M}$ because the contributions of the two diagrams on the first line of Fig. 8 are proportional to $(\gamma\cdot p+\gamma\cdot q)^{-1}\gamma\cdot p u(p)$, and therefore vanish. Just as before, this means that the integral over ω' in Eq. (B9) converges, and any $\ln\mu^2$ behavior in $f_{2\ \text{gluon}}$ must originate in $L_{2\ \text{gluon}}^-$ itself. Possible divergences in $L_{2\ \text{gluon}}^-$ as $\mu^2\to 0$ arise from the eight regions of three-particle phase space listed in Table III, where denominators in the matrix element of Eq. (B19) vanish. To extract the divergent part, we make a careful study of the behavior of the integral of Eq. (B18) in each of the eight regions. In this connection, the following simple inequality is very useful: Let p be a null vector and let $Q(=g_1,\ g_2,\ g_1+g_2)$ be timelike with $p^0>0$ and $Q^0>0$. Then we may write

$$(\gamma\cdot p)(\gamma\cdot Q) = p\cdot Q + \tfrac{1}{2}\gamma_\alpha\gamma_\beta T^{\alpha\beta},$$
$$T^{\alpha\beta} = p^\alpha Q^\beta - p^\beta Q^\alpha, \quad \text{(B20)}$$

with the following simple bounds on $T^{\alpha\beta}$:

$$|T^{AB}| \le [4p^0Q^0 p\cdot Q]^{1/2}, \qquad A, B, = 1, 2, 3$$
$$|T^{A0}| \le [2(p\cdot Q)^2 + 4p^0Q^0 p\cdot Q]^{1/2}. \quad \text{(B21)}$$

In other words, for small $p\cdot Q$, the γ-matrix product $(\gamma\cdot p)(\gamma\cdot Q)$ is always bounded by $(p\cdot Q)^{1/2}$. Application of this inequality shows that many of the potentially divergent phase-space regions actually make a finite contribution to Eq. (B18), and that the only divergent contributions come from the phase-space regions and pieces of $|\mathfrak{M}|^2$ shown in Table IV. Evaluation of the spin sums and phase-space integrals show that regions (4) and (5) each make a contribution to L^- of

$$-\tfrac{1}{2}(g_r^2/4\pi)^2(\omega/64\pi^2)[\ln(\tfrac{1}{2}\omega) + (2/\omega) - 1]\ln\mu^2 + \text{finite}, \quad \text{(B22)}$$

while regions (7) and (8) each make a contribution of

$$-\tfrac{1}{4}(g_r^2/4\pi)^2(\omega/64\pi^2)\ln\mu^2, \quad \text{(B23)}$$

giving a total of

$$\lim_{\mu^2\to 0} L_{2\ \text{gluon}}^- = -(g_r^2/4\pi)^2(\omega/64\pi^2)[\ln(\tfrac{1}{2}\omega) + (2/\omega) - \tfrac{1}{2}] \times \ln\mu^2 + \text{finite}. \quad \text{(B24)}$$

(iv) Trident. The three trident contributions arise from the squares of the second-order matrix elements corresponding to the diagrams of Fig. 9. In Table V we list the momentum labeling for each of the three states and indicate to which L it contributes. The

TABLE V. Four-momentum labeling for trident production.

Trident state	Four-momentum label g_1	g_2	g_3	Contributes to
(1)	n	p	$\bar{p}$	L^-
(2)	n	n	$\bar{n}$	L^-
(3)	p	p	$\bar{n}$	L^+

[23] T. Kinoshita, J. Math. Phys. 3, 650 (1952).

TABLE VI. Phase-space regions and pieces of $|\mathfrak{M}^{(1,2)}|^2$ which actually make divergent contributions to Eq. (3.25). g_1^s, g_2^s, and g_3^s denote the spatial components ($s=1, 2, 3$) of g_1, g_2, and g_3.

Phase-space region	Piece of $\lvert\mathfrak{M}\rvert^2$	Occurs in
$g_2^s \parallel g_3^s$	$\lvert g_r^2\bar{u}(g_1)\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_2-\gamma\cdot g_3)]u(p)[1/((g_2+g_3)^2-\mu^2)]\bar{u}(g_2)v(g_3)\rvert^2$	$\lvert\mathfrak{M}^{(1)}\rvert^2$, $\lvert\mathfrak{M}^{(2)}\rvert^2$
$g_1^s \parallel g_3^s$	$\lvert g_r^2\bar{u}(g_2)\gamma\cdot p[1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_3)]u(p)[1/((g_1+g_3)^2-\mu^2)]\bar{u}(g_1)v(g_3)\rvert^2$	$\lvert\mathfrak{M}^{(2)}\rvert^2$

matrix element for state (j) $(j=1, 2, 3)$ receives contributions from only those diagrams in Fig. 9 which are labeled below with (j). We find [the factors of $\frac{1}{2}$ in Eq. (B26) are statistical]

$$L^{\mp}{}_{\text{trident}}=\frac{-m\omega^2}{q^2}(2\pi)^3\tfrac{1}{4}\sum_{\text{spin}(p)}\sum_{\text{spin}(g_1,g_2,g_3)}\int\frac{d^3g_1}{(2\pi)^3}\frac{m}{g_1^0}\int\frac{d^3g_2}{(2\pi)^3}\frac{m}{g_2^0}\int\frac{d^3g_3}{(2\pi)^3}\frac{m}{g_3^0}\delta^4(p+q-g_1-g_2-g_3)\,|\mathfrak{M}^{\mp}|^2, \tag{B25}$$

with

$$|\mathfrak{M}^-|^2=|\mathfrak{M}^{(1)}|^2+\tfrac{1}{2}|\mathfrak{M}^{(2)}|^2,\qquad |\mathfrak{M}^+|^2=\tfrac{1}{2}|\mathfrak{M}^{(3)}|^2, \tag{B26}$$

$$\begin{aligned}
\mathfrak{M}^{(1)}=g_r^2\{&\bar{u}(g_1)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_2-\gamma\cdot g_3)^{-1}u(p)[(g_2+g_3)^2-\mu^2]^{-1}\bar{u}(g_2)v(g_3)\\
&+\bar{u}(g_2)u(p)[(p-g_2)^2-\mu^2]^{-1}\bar{u}(g_1)[-1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2)]\gamma\cdot pv(g_3)\\
&+\bar{u}(g_2)u(p)[(p-g_2)^2-\mu^2]^{-1}\bar{u}(g_1)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_2-\gamma\cdot g_3)^{-1}v(g_3)\},\\
\mathfrak{M}^{(2)}=g_r^2\{&\bar{u}(g_1)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_2-\gamma\cdot g_3)^{-1}u(p)[(g_2+g_3)^2-\mu^2]^{-1}\bar{u}(g_2)v(g_3)\\
&+\bar{u}(g_2)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_3)^{-1}u(p)[(g_1+g_3)^2-\mu^2]^{-1}\bar{u}(g_1)v(g_3)\},
\end{aligned}\tag{B27}$$

$$\begin{aligned}
\mathfrak{M}^{(3)}=g_r^2\{&\bar{u}(g_1)u(p)[(p-g_1)^2-\mu^2]^{-1}\bar{u}(g_2)[-1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2)]\gamma\cdot pv(g_3)\\
&+\bar{u}(g_2)u(p)[(p-g_2)^2-\mu^2]^{-1}\bar{u}(g_1)[-1/(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_2)]\gamma\cdot pv(g_3)\\
&+\bar{u}(g_1)u(p)[(p-g_1)^2-\mu^2]^{-1}\bar{u}(g_2)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_1-\gamma\cdot g_3)^{-1}v(g_3)\\
&+\bar{u}(g_2)u(p)[(p-g_2)^2-\mu^2]^{-1}\bar{u}(g_1)\gamma\cdot p(\gamma\cdot p-\gamma\cdot g_2-\gamma\cdot g_3)^{-1}v(g_3)\}.
\end{aligned}$$

The two diagrams on the first line of Fig. 9 make no contribution to the matrix elements since they contain the factor $(\gamma\cdot p+\gamma\cdot q)^{-1}\gamma\cdot pu(p)=0$, and as before, this means that divergences in f_{trident} must originate in $L^{\mp}{}_{\text{trident}}$ themselves. Potential divergences in $L^{\mp}{}_{\text{trident}}$ are associated with special regions of three-body phase space where denominators in Eq. (B27) vanish. In studying the actual behavior of Eq. (B25) in these regions, we use the inequality of Eq. (B21) and the estimates

$$\begin{aligned}
|\bar{u}(g_{1,2})u(p)|&\propto(g_{1,2}\cdot p)^{1/2}\quad\text{as } g_{1,2}\cdot p\to 0,\\
|\bar{u}(g_{1,2})v(g_3)|&\propto(g_{1,2}\cdot g_3)^{1/2}\quad\text{as } g_{1,2}\cdot g_3\to 0.
\end{aligned}\tag{B28}$$

We find that most of the dangerous phase-space regions actually give finite results in the $\mu^2\to 0$ limit, with logarithmic divergences coming from the regions of phase space and pieces of $|\mathfrak{M}^{(1,2)}|^2$ shown in Table VI. Evaluation of the spin sums and phase-space integrals gives the result

$$\lim_{\mu^2\to 0}L^+{}_{\text{trident}}=\text{finite},$$

$$\lim_{\mu^2\to 0}L^-{}_{\text{trident}}=-4(g_r^2/4\pi)^2(\omega/64\pi^2)\ln\mu^2+\text{finite}, \tag{B29}$$

with $\frac{3}{4}$ of this result coming from the phase-space region $g_2^s \parallel g_3^s$ and $\frac{1}{4}$ from the region $g_1^s \parallel g_3^s$.

This completes our analysis of intermediate states which contribute in order g_r^4. Adding up the contributions from Eqs. (B15), (B24), and (B29), we find, for the total fourth-order contribution,

$$\lim_{\mu^2\to 0}L^+(q^2,\omega)=\text{finite},$$

$$\lim_{\mu^2\to 0}L^-(q^2,\omega)=-(g_r^2/4\pi)^2(\omega/64\pi^2)\times[\ln(\tfrac{1}{2}\omega)+(2/\omega)+\tfrac{11}{2}]\ln\mu^2+\text{finite}, \tag{B30}$$

which, by our dimensional argument, implies that

$$\lim_{q^2\to-\infty}L^+(q^2,\omega)=\text{finite},$$

$$\lim_{q^2\to-\infty}L^-(q^2,\omega)=(g_r^2/4\pi)^2(\omega/64\pi^2)\times[\ln(\tfrac{1}{2}\omega)+(2/\omega)+\tfrac{11}{2}]\ln(q^2/\mu^2)+\text{finite}. \tag{B31}$$

Substituting this result into Eqs. (B6) and (B9) yields the fourth-order Bjorken limit quoted in Eq. (12) of the text.[24]

[24] A fourth-order calculation of the longitudinal cross section in the inequivalent limit in which $|q^2|$ and ω^{-1} simultaneously become large has been given recently by H. Cheng and T. T. Wu, Phys. Rev. Letters 22, 1409 (1969).