Because of causality, the limit in (A17) is the light-cone limit $x^2=4\sigma\lambda-\mathbf{x}^2\to 0$. It indeed follows from (A14) and (A17) that

$$\pi\hat{W}(z,\lambda)\sim\delta(z)\tfrac{1}{2}\epsilon(\lambda)\hat{f}(\lambda) \quad \text{(A23)}$$

for $z\sim 0$. We therefore have determined the leading singularity of $\hat{W}$ on the light cone and, by (A20), the large-λ behavior of its coefficient. The behavior of W near the light cone can also be determined from the configuration-space form

$$\pi\hat{W}(x^2,x_0)=-\tfrac{1}{2}i\int dadb\,\sigma(a,b)\,\exp(-ibx_0)\Delta(x;a+b^2) \quad \text{(A24)}$$

of (A1) and the behavior (3.58) of Δ. These give

$$\pi\hat{W}(z,\lambda)\sim\delta(z)L(\lambda) \quad \text{(A25)}$$

for $z\sim 0$, where

$$L(\lambda)=-[i\epsilon(\lambda)/4\pi]\int dadb\,\sigma(a,b)\,\exp(-ib\lambda). \quad \text{(A26)}$$

Equations (A6) and (A9) give

$$L(\lambda)\underset{\lambda\to\infty}{\longrightarrow}-2^{-\alpha-1}igw\epsilon(\lambda)\,|\lambda|^{\alpha}. \quad \text{(A27)}$$

In view of (A22), Eqs. (A25) and (A27) are in perfect agreement with (A23) and (A20).

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Three-Dimensional Formulation of the Relativistic Two-Body Problem and Infinite-Component Wave Equations

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A relativistic quasipotential equation is derived from the conventional Hamiltonian formalism and old-fashioned "noncovariant" off-energy-shell perturbation theory in a similar way to that by which the four-dimensional Bethe-Salpeter equation is obtained from the off-mass-shell Feynman rules. The three-dimensional equation for the (off-energy-shell) scattering amplitude appears as a straightforward generalization of the nonrelativistic Lippmann-Schwinger equation. The corresponding homogeneous equation for the bound-state wave function and the normalization condition for its solutions are derived from the equation for the complete four-point Green's function. In order to obtain a solvable model, we consider a simplified version of the quasipotential equation which still reproduces correctly the on-shell scattering amplitude and is consistent with the elastic unitarity condition. It involves a "local" approximation to the potential $V(p-q)$ which defines the kernel of our integral equation (the integration being carried over a two-sheeted hyperboloid in the energy-momentum space). It is shown that for the scalar Coulomb potential $V(p-q)=\alpha/(p-q)^2$, our model equation is equivalent to a simple infinite-component wave equation of the type considered by Nambu, Barut, and Fronsdal. The energy eigenvalues for the bound-state problem are calculated explicitly in this case and are found to be $O(4)$ degenerate (just as in the nonrelativistic Coulomb problem and in Wick and Cutkosky's treatment of the Bethe-Salpeter equation in the same approximation).

I. INTRODUCTION

THE purpose of this paper is to show the relationship between a modification of the quasipotential approach to the relativistic two-body problem developed in Refs. 1–3 and the infinite-component wave equations for the "relativistic hydrogen atom" of the type considered in Refs. 4–6.

A three-dimensional relativistic quasipotential equation for the two-particle scattering amplitude and for the bound-state wave function was first proposed by Logunov and Tavkhelidze.[7] It was derived in the framework of the Bethe-Salpeter equation using the nonuniqueness of the off-shell extrapolation of the scattering amplitude.

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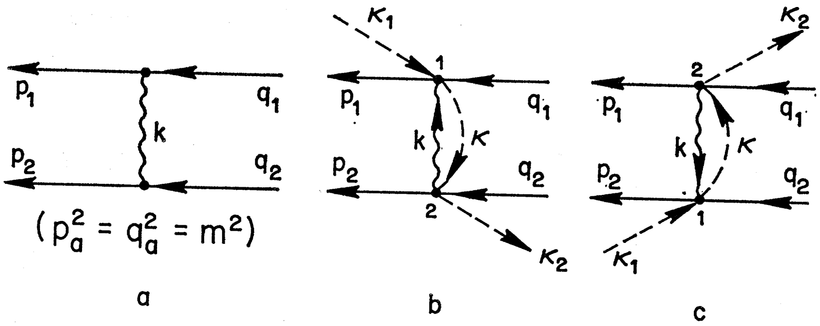

FIG. 1. (a) Second-order Feynman diagram; (b) and (c) the two corresponding diagrams of old-fashioned perturbation theory.

We consider here a different type of quasipotential equation closely related to the Hamiltonian formalism (cf. Ref. 1). The paper consists of two logically distinct parts.

In the first part we review the derivation of the quasipotential equation from a given local Hamiltonian (Sec. II). The kernel of the equation (i.e., the "quasipotential") is defined as a series of irreducible graphs, analogous to the series for the kernel of the Bethe-Salpeter equation. The equation obtained has a number of attractive features (including a straightforward nonrelativistic limit). However, it looks quite complicated and does not allow an exact solution even in the lowest approximation for the potential and for zero-mass exchange.

Therefore, in the second part, following the general idea of Refs. 7 and 2, we use the nonuniqueness of the off-energy-shell extrapolation of the scattering amplitude in order to obtain (Sec. III) a simpler equation which can be solved exactly at least in a special case. In doing that we modify both the free Green's function and the second-order potential. We only treat the equal-mass case. (We do not attempt to fit into the present scheme the approach to the unequal-mass case proposed in Ref. 8, since it is affecting the on-shell behavior of the amplitude.) In accordance with Ref. 7, our simplified equation still reproduces the correct on-shell scattering amplitude in perturbation theory. The second-order approximation in the model equation is compared with the exact Feynman amplitude for the box diagram. It is important to note that our modified equation involves integration over the two-sheeted hyperboloid $k^2=m^2$, thus exhibiting the symmetry between positive and negative frequencies. This property allows us to transform it, in the case of zero-mass exchange, into a form similar to the nonrelativistic Schrödinger equation for the Coulomb problem, and to find the energy eigenvalues. In Sec. IV we "algebraize" the equation for the scalar Coulomb problem, reducing it to a infinite-component wave equation of the type considered in Refs. 4–6. In this way, we find the same energy eigenvalues by just using the properties of the generators of a familiar representation of the conformal group $SO(4,2)$.

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II. DERIVATION OF RELATIVISTIC QUASIPOTENTIAL EQUATION

A. Review of Noncovariant Perturbation Theory

Following Ref. 1, we start with the equation for the operator-valued function $R(\kappa_1,\kappa_2)$:

$$R(\kappa_1,\kappa_2)+\tilde{H}(\kappa_1-\kappa_2)+\frac{1}{2\pi}\int\tilde{H}(\kappa_1-\kappa)\times\frac{1}{\kappa-i0}R(\kappa,\kappa_2)d\kappa=0,\quad(2.1)$$

where $\tilde{H}(\kappa)$ is the Fourier transform of the interaction Hamiltonian $H(\tau)$

$$H(\tau)=\int_{x_0=\tau}H(x)d^3x,\qquad \tilde{H}(\kappa)=\int H(\tau)\exp(-i\kappa\tau)d\tau.\quad(2.2)$$

The direction of the time axis will be specified later. The operator R is related to the scattering operator $S=S(\infty,-\infty)$ by

$$S(\tau,-\infty)=1+(1/2\pi)\int R(\kappa,0)[\exp(i\kappa\tau)/(\kappa-i0)]d\kappa\quad(2.3)$$

or

$$S=S(\infty,-\infty)=1+iR(0,0).\quad(2.4)$$

{We have used the identity

$$(1/2\pi i)\lim_{\tau\to\pm\infty}[\exp(i\kappa\tau)/(\kappa-i0)]=\delta(\kappa)\quad\text{for }\tau\to+\infty$$
$$=0\quad\text{for }\tau\to-\infty.\}$$

A diagram technique was developed in Refs. 9 and 1 for the calculation of the matrix elements of R. In the case of a theory of spinless particles, it can be summarized in the following way. To any ordinary Feynman diagram we let correspond a set of new graphs with all possible numeration of the vertices $1,\ldots,N$. Every internal line is oriented toward the vertex with smaller number. Furthermore we let a spurion (dashed) line enter the vertex 1, connect 1 with 2, 2 with 3, and so on

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(always oriented toward the vertex with larger number), and finally go out of the vertex N. [For instance, to the second-order Feynman graph of Fig. 1(a) correspond the two diagrams of Figs. 1(b) and 1(c).] The conservation law in each vertex of the new diagrams takes into account the energies of the dashed lines. For instance, to vertex 1 of the graph shown in Fig. 1(b) corresponds the factor

$$-[g/(2\pi)^{1/2}]\delta(q_1+k-p_1+(\kappa_1-\kappa)n),$$

where n is a four-dimensional unit vector in the direction of the time axis. To a solid line with mass μ and momentum k, we let correspond the "on-mass-shell propagator"

$$\delta_\mu^+(k)=\theta(k_0)\delta(k^2-\mu^2). \tag{2.5}$$

To an internal dashed line with "energy" κ we let correspond the propagator

$$(1/2\pi)[1/(\kappa-i0)]. \tag{2.6}$$

These rules give rise to the old-fashioned ("noncovariant") perturbation expansion for the scattering amplitude. If we start with a local interaction Hamiltonian $H(x)$, the on-shell amplitude (for $\kappa_1=\kappa_2=0$) does not depend on the choice of the direction of the time axis. For instance, the contribution from the diagrams on Figs. 1(b) and 1(c) is

$$\langle p_1p_2 \mid R^{(2)}(\kappa_1,\kappa_2)\mid q_1q_2\rangle=[1/(2\pi)^2]\times\delta(p_1+p_2-q_1-q_2+(\kappa_2-\kappa_1)n)T^{(2)},$$

where

$$T^{(2)}=\tfrac{1}{2}g^2\left(\frac{1}{\omega_{p_1-q_1}}\frac{1}{\kappa_1+q_1^0-p_1^0+\omega_{p_1-q_1}-i0}+\frac{1}{\omega_{p_2-q_2}}\frac{1}{\kappa_1+q_2^0-p_2^0+\omega_{p_2-q_2}-i0}\right), \tag{2.7}$$

$$\omega_k=(\mu^2+\mathbf{k}^2)^{1/2}. \tag{2.8}$$

For $\kappa_1=\kappa_2=0$, $q_1-p_1=p_2-q_2$, Eq. (2.7) reduces to the covariant Feynman rule for the on-shell amplitude $\mathbf{T}$:

$$\begin{aligned}\mathbf{T}^{(2)}&=g^2/[\omega_{p_1-q_1}{}^2-(p_1^0-q_1^0)^2-i0]\\&=g^2/[\mu^2-(p_1-q_1)^2-i0].\end{aligned} \tag{2.9}$$

All ultraviolate divergences in higher-order diagrams can be reduced to divergences in the integration over the variables κ, and renormalization can be carried out in a way similar to the subtraction procedure in dispersion integrals (for more details see Ref. 9).

B. Two-Particle Quasipotential Equation

We consider the model of interaction of two charged scalar fields ψ_1 and ψ_2 of mass m with a real scalar field φ of mass μ with interaction Hamiltonian

$$H(x)=-L(x)=-g(:\psi_1^*(x)\psi_1(x)\varphi(x):+:\psi_2^*(x)\psi_2(x)\varphi(x):) \tag{2.10}$$

(the symbol : : stands, as usual, for the Wick normal product). Our aim will be to write an equation for the (off-shell) $\psi_1\psi_2$-elastic scattering amplitude. [It is known that, in contrast with the electromagnetic interaction via a vector field A_μ, the scalar interaction (2.10) of two equally charged particles is attractive.]

First of all, following Ref. 1, we remark that Eq. (2.1) may be written in a more compact symbolic form

$$R+\tilde{H}+\tilde{H}G_0R=0. \tag{2.11}$$

If we introduce the quasiparticle states $|\kappa)$ normalized by

$$(\kappa_1\mid\kappa_2)=\delta(\kappa_1-\kappa_2), \tag{2.12}$$

and put

$$\begin{aligned}(\kappa_1\mid R\mid\kappa_2)&=R(\kappa_1,\kappa_2),\\(\kappa_1\mid\tilde{H}\mid\kappa_2)&=\tilde{H}(\kappa_1-\kappa_2),\\(\kappa_1\mid G_0\mid\kappa_2)&=\delta(\kappa_1-\kappa_2)/2\pi(\kappa_1-i0),\end{aligned} \tag{2.13}$$

and define the "matrix" multiplication as an integral over κ, then Eq. (2.1) is "obtained" from (2.11) by taking the matrix element between κ_1 and κ_2. Iterating once Eq. (2.11), we get

$$R=-\tilde{H}+\tilde{H}G_0\tilde{H}+\tilde{H}G_0\tilde{H}G_0R. \tag{2.14}$$

We take the matrix element of both sides of (2.14) between two $\psi_1\psi_2$-particle states $\langle p_1p_2|$ and $|q_1q_2\rangle$. [We use the covariant normalization,

$$\langle p\mid q\rangle=2E_p\delta(\mathbf{p}-\mathbf{q}),\qquad E_p=(m^2+\mathbf{p}^2)^{1/2}, \tag{2.15}$$

for one-particle states of momenta p and q and mass m.] Observing that for the interaction Hamiltonian (2.10)

$$\langle p_1p_2\mid\tilde{H}\mid q_1q_2\rangle=0, \tag{2.16}$$

and separating the contribution of the intermediate $\psi_1\psi_2$-particle state, we obtain

$$\begin{aligned}\langle p_1p_2\mid R\mid q_1q_2\rangle=&\langle p_1p_2\mid\tilde{H}G_0\tilde{H}\mid q_1q_2\rangle\\&+\int\langle p_1p_2\mid\tilde{H}G_0\tilde{H}\mid k_1k_2\rangle G_0\langle k_1k_2\mid R\mid q_1q_2\rangle(dk_1)(dk_2)\\&+\sum_{n>2}\int d\sigma_n(k_1,\ldots,k_n)\langle p_1p_2\mid\tilde{H}G_0\tilde{H}\mid k_1,\ldots,k_n\rangle\\&\times G_0\langle k_1,\ldots,k_n\mid R\mid q_1q_2\rangle,\end{aligned} \tag{2.17}$$

where

$$(dk)=\delta_m^+(k)d^4k$$

and $d\sigma_n(k_1,\ldots,k_n)$ is the corresponding invariant measure for the n-particle intermediate state. (We mention that the matrix elements of the type $\langle p_1p_2\mid R\mid q_1q_2\rangle$ are still considered as operators in the space of the spurion energies κ.) Our aim is to define a kernel K which incorporates the contribution from the n-particle states ($n>2$) in order to obtain a linear equation for the two-particle amplitude. Let Π_2 be the projection operator on the subspace of two-particle states containing one ψ_1 and one ψ_2 particle. We define the kernel K by

$$\begin{aligned}K&=\tilde{H}G_0\tilde{H}\{1/[1-G_0(1-\Pi_2)\tilde{H}G_0\tilde{H}]\}\\&=\tilde{H}G_0\tilde{H}+\tilde{H}G_0\tilde{H}G_0(1-\Pi_2)\tilde{H}G_0\tilde{H}+\cdots.\end{aligned} \tag{2.18}$$

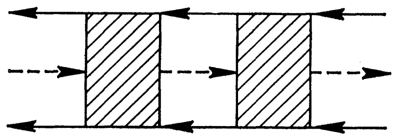

FIG. 2. Reducible diagram of nonconvariant perturbation theory.

Then Eq. (2.17) can be rewritten in the form

$$\langle p_1p_2 | R | q_1q_2\rangle = \langle p_1p_2 | K | q_1q_2\rangle + \int \langle p_1p_2 | K | k_1k_2\rangle \times G_0\langle k_1k_2 | R | q_1q_2\rangle (dk_1)(dk_2) \quad (2.19)$$

[we have taken into account that Π_2 commutes with G_0 so that $G_0(1-\Pi_2)=(1-\Pi_2)G_0(1-\Pi_2)$].

Let us define the connected scattering amplitude R^c as the sum of all "solid-line-connected" graphs (i.e., graphs which remain connected when the dashed lines carrying momenta κn are removed). Then an equation identical to (2.19) holds true for R^c with a modified Green's function (to be defined later) instead of G_0 and with $\langle p_1p_2 | K | k_1k_2\rangle$ replaced by the sum $\langle p_1p_2 | K^c | k_1k_2\rangle$ of all (two-particle) irreducible diagrams, which are defined in the following way: A (connected) diagram of the $\psi_1\psi_2$ scattering amplitude is called irreducible if it cannot be split into two graphs of the same process which are linked by a ψ_1 and a ψ_2 line oriented in the direction of the external lines (say, from right to left) and one dashed line oriented in the opposite direction (see Fig. 2).

In order to take the energy-momentum conservation explicitly into account, we put

$$\langle p_1p_2 | (\kappa_1 | R^c | \kappa_2) | q_1q_2\rangle = \delta(p_1+p_2-q_1-q_2-(\kappa_1-\kappa_2)n)T_{\kappa_1\kappa_2}(p_1p_2; q_1q_2),$$

$$\langle p_1p_2 | (\kappa_1 | K^c | \kappa_2) | q_1q_2\rangle = -\delta(p_1+p_2-q_1-q_2-(\kappa_1-\kappa_2)n)V_{\kappa_1\kappa_2}(p_1p_2; q_1q_2). \quad (2.20)$$

This allows us to rewrite Eq. (2.19) in the form

$$T_{\kappa_1\kappa_2}(p_1p_2; q_1q_2)+V_{\kappa_1\kappa_2}(p_1p_2; q_1q_2) + \int\cdots\int\delta(p_1+p_2-k_1-k_2-(\kappa_1-\kappa)n)V_{\kappa_1\kappa}(p_1p_2; k_1k_2) \times G_\kappa(k_1, k_2)T_{\kappa\kappa_2}(k_1k_2; q_1q_2)(dk_1)(dk_2)d\kappa = 0. \quad (2.21)$$

Here $G_\kappa(k_1, k_2)$ is defined through the free two-particle Green's function $(\kappa | G_0(k_1k_2; k_1'k_2') | \kappa')$ (equal to the sum of all solid-line disconnected diagrams of the complete $\psi_1\psi_2$ four-point Green's function) by

$$(\kappa | G_0(k_1k_2; k_1'k_2') | \kappa') = 4E_{k_1}E_{k_2}\delta(\mathbf{k}_1-\mathbf{k}_1') \times\delta(\mathbf{k}_2-\mathbf{k}_2')\delta(\kappa-\kappa')G_\kappa(k_1, k_2). \quad (2.22)$$

[We have made use of the one-particle normalization condition (2.15).] To see that the free Green's function has indeed the diagonal form (2.22), we observe that a typical contribution to it comes from a graph of the type shown on Fig. 3. As a consequence of the conservation law at each vertex we have

$$k_2-k_2' = (\kappa_1-\kappa_2+\cdots+\kappa_{2r-1}-\kappa_{2r})n.$$

Since both k_2 and k_2' lie on the mass shell [the expression for $(\kappa | G_0 | \kappa')$ being always accompanied by the factor $\prod_{a=1}^2 \delta_m^+(k_a)\delta_m^+(k_a')$], the vector k_2-k_2' is spacelike. It could be proportional to the timelike vector n only if each side of the equality vanishes separately. Thus, we have $k_2=k_2'$, $\kappa_1-\kappa_2+\cdots+\kappa_{2r-1}-\kappa_{2r}=0$. The over-all energy-momentum conservation then gives

$$k_1-k_1' = (\kappa-\kappa')n.$$

Repeating the above argument once more, we see that $k_1=k_1'$, $\kappa=\kappa'$, which justifies Eq. (2.22). The first two terms in the perturbation expansion for G_κ have the form

$$G_\kappa^{(0)}+G_\kappa^{(2)}(k_1, k_2) = \frac{1}{2\pi}\left[\frac{1}{\kappa-i0}+\left(\frac{g}{2\pi}\right)^2 \times\left(\frac{F(\kappa n, k_1)}{2k_{20}}+\frac{F(\kappa n, k_2)}{2k_{10}}\right)\right], \quad (2.23a)$$

where

$$F(\kappa n, k) = \int_{x_0}^\infty dx\, f(x^2-\mathbf{k}^2)\, \frac{(x+\kappa)(k_0^2+x^2)+2xk_0^2}{(x^2-k_0^2)^2[(x+\kappa-i0)^2-k_0^2]},$$

$$x_0 = x_0(k) = [(m+\mu)^2+\mathbf{k}^2]^{1/2}, \quad (2.23b)$$

$$k_0 = kn, \qquad k^2 = k_0^2-\mathbf{k}^2 = m^2,$$

and f is defined by the phase-space integral

$$(1/\pi)\int\delta_m^+(k-q)\delta_\mu^+(q)d^4q = \theta(k_0)\theta(k^2-(m+\mu)^2)f(k^2),$$

$$f(z) = (1/8z)[z^2-2(m^2+\mu^2)z+(m^2-\mu^2)^2]^{1/2}. \quad (2.23c)$$

The term $G_\kappa(k_1, k_2)$ in Eq. (2.21) corresponds to the product $\Delta_F'(k_1)\Delta_F'(k_2)$ of complete Feynman propagators in the Bethe-Salpeter equation (see, e.g., Ref. 10 and references cited there). It can be shown that in the nonrelativistic limit $V_{\kappa_1\kappa_2}$ [see Eq. (2.20)] goes into the nonrelativistic potential. We shall see this later in the special case of the scalar Coulomb interaction.

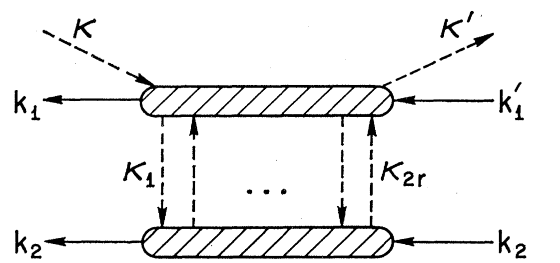

FIG. 3. Typical diagram of the two-particle propagator.

[10] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) Suppl. **43**, 1 (1969).

C. Center-of-Mass Variables and Equation for Wave Function

In what follows we shall treat Eq. (2.21) in the center-of-mass frame and will assume that the unit vector n, which defines the time axis, is collinear to p_1+p_2 (and hence to q_1+q_2). [If we were interested in the t-channel behavior of the scattering amplitude (for timelike p_1-q_1), it would have been advantageous to choose n along p_1-q_1.] In this frame we put

$$\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p},\quad \mathbf{q}_1=-\mathbf{q}_2=\mathbf{q},\quad \mathbf{n}=0$$

$$p_1{}^0=p_2{}^0=p^0,\quad q_1{}^0=q_2{}^0=q^0, \tag{2.24}$$

$$p_0=E_p=(m^2+\mathbf{p}^2)^{1/2},\quad p_0-\tfrac{1}{2}\kappa_1=q_0-\tfrac{1}{2}\kappa_2\equiv E;$$

$$T_{\kappa_1\kappa_2}(p_1,p_2;q_1q_2)=T_E(p,q),$$

$$V_{\kappa_1\kappa_2}(p_1,p_2;q_1,q_2)=V_E(p,q), \tag{2.25}$$

$$G_\kappa(k_1,k_2)=G_{2(k_0-E)}(k_0,\mathbf{k},k_0,-\mathbf{k})\equiv 2k_0G_E(k).$$

In these variables, Eq. (2.21) can be written in the form

$$T_E(p,q)+V_E(p,q)+\int V_E(p,k)G_E(k)T_E(k,q)(dk)=0. \tag{2.26}$$

The corresponding equation for the complete two-particle Green's function

$$\mathcal{G}_E(p;q)=G_E(p)(p_0+q_0)\delta(\mathbf{p}-\mathbf{q})+G_E(p)T_E(p,q)G_E(q) \tag{2.27}$$

is

$$\mathcal{G}_E(p,q)+G_E(p)\int V_E(p,k)\mathcal{G}_E(k,q)(dk)=(p_0+q_0)\delta(\mathbf{p}-\mathbf{q})G_E(p). \tag{2.28}$$

Let there exist an r-fold degenerate ($r\geq 1$) bound state of mass $2B$ in the $\psi_1\psi_2$ system. Assume that in analogy with the Bethe-Salpeter equation (cf. Ref. 10) and with the nonrelativistic Lippmann-Schwinger equation that the Green's function $\mathcal{G}_E(p,q)$ has a simple pole for $E=B$ and in the neighborhood of this pole can be written in the form[11]

$$\mathcal{G}_E(p,q)=N_B\sum_{a=1}^{r}[\phi_{Ba}(p)\bar{\phi}_{Ba}(q)/(B-E-i0)]+(\text{regular terms for } E\to B), \tag{2.29}$$

where $\phi_{Ba}(p)$ will be interpreted as the wave function of the bound state of mass $2B$ and other quantum numbers specified by a, and N_B is a normalization factor. Inserting (2.29) in Eq. (2.28) and comparing the residues at the pole $E=B$, we obtain

$$\sum_{a=1}^{r}[\phi_{Ba}(p)+G_B(p)\int V_B(p,k)\phi_{Ba}(k)(dk)]\bar{\phi}_{Ba}(q)=0.$$

[11] The form of the singular term in (2.29) is consistent with the noncovariant perturbation rules described in Sec. II A if we assume the existence of r particles of mass $2B$ coupled to ψ_1 and ψ_2.

Since $\bar{\phi}_{Ba}(q)$ are linearly independent, this implies the following homogeneous equation for each of the wave functions $\phi_B(p)$:

$$G_B{}^{-1}(p)\phi_B(p)+\int V_B(p,k)\phi_B(k)(dk)=0. \tag{2.30}$$

In order to obtain the normalization condition for the wave function, we apply to both sides of Eq. (2.28) the integral operator

$$\int\mathcal{G}_E(p,p')G_E{}^{-1}(p')\cdot(dp').$$

This leads to the following nonlinear equation for $\mathcal{G}_E$:

$$\int\mathcal{G}_E(p,k)G_E{}^{-1}(k)\mathcal{G}_E(k,q)(dk)+\int\int\mathcal{G}_E(p,k_1)V_E(k_1,k_2)\times\mathcal{G}_E(k_2,q)(dk_1)(dk_2)=\mathcal{G}_E(p,q). \tag{2.31}$$

Inserting (2.29) in (2.31) and comparing the residues at the pole $E=B$ on both sides, we obtain the following orthonormalization condition (cf. Refs. 12 and 13):

$$N_B\int\int\bar{\phi}_{Ba}(k_1)\{-(\partial/\partial B)[G_B{}^{-1}(k_1)2E_{k_1}\delta(\mathbf{k}_1-\mathbf{k}_2)+V_B(k_1,k_2)]\}\phi_{Bb}(k_2)(dk_1)(dk_2)=\delta_{ab}. \tag{2.32}$$

Consider the special case when $V_E(p,q)$ does not depend on E and G_κ is replaced by the first term in the expansion (2.23), so that, according to (2.25),

$$G_E{}^{-1}(k)\approx 8\pi E_k(E_k-E). \tag{2.33}$$

Choosing the normalization factor $N_B=1/4\pi$, we reduce Eq. (2.32) in this case to the normalization condition for the nonrelativistic Schrödinger wave function

$$\int\bar{\phi}_{Ba}(\mathbf{k})\phi_{Bb}(\mathbf{k})d^3k=\delta_{ab}. \tag{2.34}$$

We stress that Eq. (2.30) does not have the well-known defects[14] of the four-dimensional Bethe-Salpeter equation related to the presence of the nonphysical variable of relative energy (or relative time). (The most serious defect of the Bethe-Salpeter equation is the existence of extra nonphysical solutions.) In contrast to the Bethe-Salpeter equation, the three-dimensional equation (2.30) admits an unambiguous nonrelativistic limit. However, we pay a certain price for the nice features of the quasipotential equation. If we replace the potential V_E by its second-order approximation, then the known analytic properties of the scattering amplitude will be distorted by the iterative solution of Eq. (2.26) (which is not the case for the corresponding Bethe-Salpeter equation).

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[14] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

III. SIMPLIFIED VERSION OF QUASIPOTENTIAL EQUATION CONSISTENT WITH ELASTIC UNITARITY

A. Nonuniqueness of Off-Shell Extrapolation of Scattering Amplitude and of Corresponding Quasipotential Equation

In spite of the attractive general properties of the quasipotential equation (2.26) [or (2.30)] discussed at the end of Sec. II C, it has one defect: It is too complicated to provide exactly soluble problems in any reasonable approximation. Indeed, already in lowest order in perturbation theory the potential $V_E^{(2)}$ evaluated from the second-order graphs in Fig. 1 has the "nonlocal"[15] form

$$V_E^{(2)}(p,q)=[g^2/\omega_{p-q}(2E-p_0-q_0-\omega_{p-q}+i0)], \quad (3.1)$$

where $\omega_{p-q}=[\mu^2+(\mathbf{p}-\mathbf{q})^2]^{1/2}$ and the corresponding quasipotential equation cannot be solved exactly even in the limit of zero-mass exchange ($\mu=0$) (the Bethe-Salpeter equation has been treated exactly in this particular case in Refs. 14 and 16).

However, it is known that one can write different three-dimensional (quasipotential) equations that give rise to the same perturbation expansion for the on-shell amplitude. For instance, the original quasipotential equation of Logunov and Tavkhelidze[7]

$$T_{E_q}(\mathbf{p},\mathbf{q})=V_{E_q}(\mathbf{p},\mathbf{q})+\frac{1}{4\pi}\int V_{E_q}(\mathbf{p},\mathbf{k}) \times\frac{T_{E_q}(\mathbf{k},\mathbf{q})}{E_k^2-(E_q+i0)^2}\frac{d^3k}{2E_k} \quad (3.2)$$

differs from our Eq. (2.26) both in the Green's function (i.e., the energy denominator) and the potential,[17] their choice of the second-order off-shell amplitude and potential being

$$T_{E_q}^{(2)}(\mathbf{p},\mathbf{q})=V_{E_q}^{(2)}(\mathbf{p},\mathbf{q})=g^2/[\mu^2+(\mathbf{p}-\mathbf{q})^2].$$

Both Eqs. (2.26) and (3.2) belong to a large family of linear equations of the type

$$T+V+VGT=0, \quad (3.3)$$

which have the following property in common: For real V in the physical region they lead automatically (at least formally) to the elastic unitarity condition.

To describe the whole class of equations of the type (3.3) with this property, we write the solution of (3.3) as

$$T=-[1/(1+VG)]V=-V[1/(1+GV)]. \quad (3.4)$$

[15] In analogy with the nonrelativistic Schrödinger equation we call a potential $V(p,q)$ "local" if it depends only on the difference $p-q$.

[16] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

[17] The difference in the sign convention is not important. Our choice fits the nonrelativistic limit of the potential.

If the potential is Hermitian, $V=V^*$, then the discontinuity of T in the s channel is given by

$$T-T^*=[-1/(1+VG)]V+[1/(1+VG^*)]V =T(G-G^*)T^*. \quad (3.5)$$

In order to make Eq. (3.5) identical with the elastic unitarity condition

$$\mathbf{T}(p,q)-\mathbf{T}^*(p,q)=(i/4E)\int\mathbf{T}(p,k) \times\mathbf{T}^*(k,q)\delta(E_k^2-E^2)d^3k \quad (3.6)$$

[where, for the on-shell amplitude $\mathbf{T}(p,q)$, $p_0=q_0=E$], we have to specify accordingly the discontinuity of the Green's function. It is readily verified that for both Green's functions

$$G_E(k_0)=[1/8\pi E_k(k_0-E-i0)]$$

and

$$G_E'(E_k)=\{1/4\pi[E_k^2-(E+i0)^2]\}$$

[corresponding to Eqs. (2.26) and (3.2), respectively], the discontinuity is the same:

$$G_E-G_E^*=G_E'-G_E'^*=(i/4E)\delta(E_k-E), \quad (3.7)$$

and it leads to (3.6) (we use in both cases the invariant volume element $d^3k/2E_k$ on the upper hyperboloid).

We will exploit the freedom in the off-shell extrapolation of the scattering amplitude in order to write a simpler equation consistent with (3.7) (i.e., with the elastic unitarity condition). The potential in any such modified quasipotential equation is calculated from a definite off-shell extrapolation of the perturbation expansion of the amplitude T (see Ref. 7). We require, for instance, that in the lowest order $V^{(2)}=-T^{(2)}$, where $T^{(2)}$ coincides on shell with (2.9). (An example of a quasipotential equation in which this latter requirement is not fulfilled is considered in Refs. 3 and 18.)

B. Model Quasipotential Equation

We will consider the model equation of the type (3.3)

$$T_E(p,q)+V_E(p,q)+(1/8\pi E) \times\int V_E(p,q)[\epsilon(k_0)/(k_0-E-i0)] \times T_E(k,q)\delta(k^2-m^2)d^4k=0, \quad (3.8)$$

where

$$\epsilon(k_0)=\operatorname{sgn}k_0=\theta(k_0)-\theta(-k_0),$$

and the corresponding homogeneous equation

$$(E-p_0)\phi_E(p)=(1/8\pi E)\int V_E(p,k) \times\phi_E(k)\epsilon(k_0)\delta(k^2-m^2)d^4k=0. \quad (3.9)$$

It is readily checked that the Green's function

$$G_E(k)=[\epsilon(k_0)/8\pi E(k_0-E-i0)]$$

[18] C. Itzykson and I. T. Todorov, in *Proceedings of the First Coral Gables Conference on Fundamental Interactions at High Energy, University of Miami, 1969*, edited by T. Gudehus *et al.* (Gordon and Breach, New York, 1969).

corresponding to these equations fulfills the elastic unitarity condition (3.7). This choice of G_E is among the simplest possibilities [consistent with (3.7)], since the operator on the left-hand side of (3.9) is a first-degree polynomial in p_0. Besides we will restrict ourselves to the second-order approximation in the potential, choosing it as the "local" energy-independent extrapolation

$$V_E(p,q)=g^2/[(p-q)^2-\mu^2+i0] \qquad (3.10)$$

of $-T^{(2)}$ [Eq. (2.9)]. An important feature of Eqs. (3.8) and (3.9) is that they involve integration over the two-sheeted hyperboloid $k^2=m^2$. [We mention that $T_E(k,q)$ could be interpreted for $k_0<0$ as the amplitude of a process with four incoming particles, which is possible off the energy shell.] We will see in Sec. IV that in the case of scalar Coulomb potential (i.e., for $\mu=0$), the presence of the lower sheet of the hyperboloid $k^2=m^2$ in the domain of integration in (3.9) is essential in order to ensure the $O(4)$ symmetry of the bound-state problem.

A more complicated model, with Green's function

$$G_E(k)=[1/8\pi E_k(k_0-E-i0)] \qquad (3.11)$$

(and also involving integration over a two-sheeted hyperboloid), was considered in Refs. 18 and 19. It leads to the same $O(4)$ degeneracy of the energy levels.

As some justification of Eq. (3.8), we observe that the exact expression for the fourth-order box diagram (Fig. 4) after integration over the internal energy k_0 in the center-of-mass frame can be written in the form

$$T_{\text{box}}(p,q)=\frac{1}{8\pi E}\int V_E(p,k)\bigg[\frac{1}{k_0-E-i0} + \epsilon(k_0)\,\frac{\omega_{p-k}{}^2+\omega_{q-k}{}^2+\omega_{p-k}\omega_{q-k}-(E-k_0)^2}{\omega_{p-k}\omega_{q-k}(\omega_{p-k}+\omega_{q-k})}\bigg] \times V_E(k,q)\,\delta(k^2-m^2)\,d^4k, \qquad (3.12)$$

where V_E is given by (3.10). We see that this expression contains the second iteration of Eq. (3.8) plus a term which is regular in the physical region and, hence, does not contribute to the imaginary part of T.

The comparison between (3.12) and the second iteration of (3.8), i.e., the integral

$$T_2=\frac{g^4}{8\pi E}\int\frac{1}{(p-k)^2-\mu^2}\,\frac{\epsilon(k_0)}{k_0-E-i0}\,\frac{1}{(k-q)^2-\mu^2} \times\delta(k^2-m^2)\,d^4k \qquad (3.13)$$

(for $p_0=q_0=E$), may give us a feeling of the discrepancy of the fourth order calculated from the Bethe-Salpeter equation in the ladder approximation and the quasipotential equation (3.8) (there is no *a priori* reason to trust any of these approximations more than the other one). It is possible to evaluate explicitly both (3.12) and

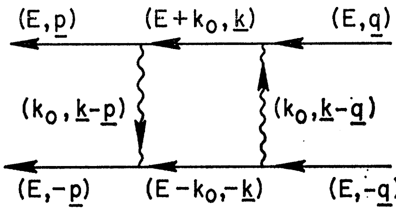

FIG. 4. Box diagram in center-of-mass variables.

(3.13) for the case of forward scattering. The result for the box diagram is

$$T_{\text{box}}(p,p)=(g^4/4m^2\mu^2)F(E), \qquad (3.14)$$

where

$$F(E)=\frac{\frac{1}{2}i\pi\tanh\chi+\theta_0\cot\theta_0-\chi\tanh\chi}{\cosh2\chi+\cos2\theta_0} \quad \text{for } E\geq m \qquad (3.15)$$

(with $\cos\theta_0=\mu/2m$, $\cosh\chi=E/m$, $\chi>0$) and

$$F(E)=\frac{\theta\cot\theta-\theta_0\cot\theta_0}{\cos2\theta-\cos2\theta_0} \quad \text{for } E^2=m^2\sin^2\theta<m^2. \qquad (3.16)$$

Equation (3.15) may be considered as an analytic continuation of (3.16) for complex θ $(\theta=\frac{1}{2}\pi+i\chi)$. We mention that the forward contribution of the crossed box diagram is obtained from (3.15) by the substitution $\chi\to\frac{1}{2}i\pi+\chi$:

$$F_{\text{cr}}=\frac{\chi\coth\chi-\theta_0\coth\theta_0}{\cosh2\chi-\cos2\theta_0} \qquad (E=m\cosh\chi). \qquad (3.17)$$

As is well known, the dominant high-energy terms in F and F_{cr} cancel each other and we get

$$T_{\text{box}}+T_{\text{cr}}\approx\frac{g^4}{16\mu^2E^2}\left[i\pi+\frac{m^2}{E^2}\ln\frac{E^2}{m^2}+O\left(\frac{1}{E^2}\right)\right] \quad \text{for } E\to\infty. \qquad (3.18)$$

With the same notation, the forward contribution from (3.13) for $E\geq m$ is

$$T_2=\frac{g^4}{4m^2\mu^2}\,\tfrac{1}{2}\pi\,\frac{i\tanh\chi+\cot\theta_0}{\cosh2\chi+\cos2\theta_0}. \qquad (3.19)$$

The imaginary parts of (3.15) and (3.19) coincide as they should. On the other hand, we saw that the term $-\chi\tanh\chi$ in the numerator of the right-hand side of (3.15) [which is absent in (3.19)] is canceled at high energy (i.e., for large χ) by the contribution of the crossed box. It is remarkable that the high-energy behavior (3.18) of the complete fourth-order term is exactly reproduced by the second iteration of the quasi-potential equation (2.26) with G_E and V_E given by (2.33) and (3.1). In other words, the old-fashioned perturbation rules of Sec. II A seem to be well suited for calculating the high-energy behavior in a given order in g.

[19] I. T. Todorov, in *Proceedings of the Battelle-Seattle Rencontres in Mathematics and Physics, 1969* (Springer-Verlag, Berlin, to be published).

IV. SOLUTION OF SCALAR COULOMB PROBLEM AND RELATION TO INFINITE-COMPONENT WAVE EQUATIONS

A. Connection with Schrödinger Equation for Hydrogen Atom

In this section we solve the equation

$$(E-p_0)\phi_E(p)=\frac{g^2}{8\pi E}\int\frac{\epsilon(k_0)}{(p-k)^2}\phi_E(k)\delta(k^2-m^2)d^4k, \tag{4.1}$$

which is obtained by inserting the potential (3.10) with $\mu=0$ in Eq. (3.9).

Our first step will be to map the two-sheeted hyperboloid $p^2=m^2$ on the three-dimensional space $\mathbf{p}'$ using the formulas

$$\begin{aligned} p_0/m&=(4m^2+\mathbf{p}'^2)/(4m^2-p'^2),\\ \mathbf{p}&=\frac{\mathbf{p}'}{1-(\mathbf{p}'/2m)^2},\\ \mathbf{p}'&=[2m/(p_0+m)]\mathbf{p}\end{aligned} \tag{4.2}$$

(and analogously for $\mathbf{q}\to\mathbf{q}'$, $\mathbf{k}\to\mathbf{k}'$). Under this change of variables the upper hyperboloid $p_0=E_p$ goes into the inside of the sphere $\mathbf{p}'^2<4m^2$, while the lower hyperboloid $p_0=-E_p$ is mapped onto the outside of this sphere ($\mathbf{p}'^2>4m^2$); the points at infinity ($\mathbf{p}^2\to\infty$) are transformed on the finite sphere $p'^2=4m^2$. The invariant-volume element on the hyperboloid is transformed as follows:

$$\frac{d^3k}{E_k}=\frac{d^3k'}{m\,|\,1-\mathbf{k}'^2/4m^2\,|^3}. \tag{4.3}$$

The scalar Coulomb potential

$$V(p,k)=g^2/(p-k)^2$$

goes into

$$V'(\mathbf{p}',\mathbf{k}')=-\frac{g^2}{16m^4}\frac{(4m^2-\mathbf{p}'^2)(4m^2-\mathbf{k}'^2)}{(\mathbf{p}'-\mathbf{k}')^2}. \tag{4.4}$$

Defining further the relativistic binding energy as $\mathcal{E}=2(E-m)$, and putting

$$\begin{aligned} m_{\mathcal{E}}&=m^2/(E+m)=2m^2/(4m+\mathcal{E}),\\ \psi_{\mathcal{E}}(\mathbf{p}')&=(1-\mathbf{p}'^2/4m^2)^{-2}\phi_E(p),\end{aligned} \tag{4.5}$$

we transform Eq. (3.10) into

$$\left(\frac{\mathbf{p}'^2}{2m}-\mathcal{E}\right)\psi_{\mathcal{E}}(\mathbf{p}')=\frac{g^2}{2\pi(\mathcal{E}+2m)m}\int\frac{\psi_{\mathcal{E}}(\mathbf{k}')}{(\mathbf{p}'-\mathbf{k}')^2}d^3k'. \tag{4.6}$$

[We have used the fact that $\epsilon(k_0)=(4m^2-\mathbf{k}'^2)/|\,4m^2-\mathbf{k}'^2\,|$.] This equation is of the same form as the nonrelativistic Schrödinger equation for the Coulomb problem:

$$\left(\frac{\mathbf{p}^2}{2\mu}-\mathcal{E}\right)\psi_{\mathcal{E}}(\mathbf{p})=\frac{\alpha}{2\pi^2}\int\frac{\psi_{\mathcal{E}}(\mathbf{k})}{(\mathbf{p}-\mathbf{k})^2}d^3k. \tag{4.7}$$

This similarity can be used to find the energy eigenvalues for Eq. (4.6), using the known result that the nonrelativistic hydrogen levels are given by

$$-\mu\alpha^2/2n^2. \tag{4.8}$$

Thus, the energy eigenvalues for Eq. (4.6) are to be determined from the equation

$$\mathcal{E}_n=-m_{\mathcal{E}_n}\frac{2m^2\alpha^2}{(2m+\mathcal{E}_n)^2n^2}=-\frac{4m^4\alpha^2}{(4m+\mathcal{E}_n)(2m+\mathcal{E}_n)^2n^2}, \tag{4.9}$$

where we have put

$$\alpha=(\pi/2m^2)g^2 \tag{4.10}$$

(we mention that the coupling constant g has the dimension of mass). The solution of (4.9) is

$$M_n{}^2=4E_n{}^2=(2m+\mathcal{E}_n)^2=2m^2[1+(1-\alpha^2/n^2)^{1/2}]. \tag{4.11}$$

Let us note that (4.9) [or (4.11)] leads (in the lowest order in α^2) to the correct nonrelativistic formula for the Coulomb energy levels $\mathcal{E}_n$ of a particle of reduced mass $\frac{1}{2}m$. We see that in the relativistic case the $O(4)$ degeneracy of the energy levels is still preserved just as well as in the case of the Bethe-Salpeter equation (see Refs. 14 and 16 as well as the recent discussion in Ref. 20). This is the main qualitative distinction between the scalar "Coulomb" interaction and the real electromagnetic interaction (via a four-vector potential), which necessarily leads to a fine splitting of the relativistic energy levels with respect to the total angular momentum.

B. Algebraization of Scalar Coulomb Problem

Now we will establish a one-to-one correspondence between the quasipotential equation

$$(E-p_0)\phi_E(p)=(g^2/8\pi E)\int[1/(p-k)^2]\times\phi_E(k)\epsilon(k_0)\delta(k^2-m^2)d^4k \tag{4.12}$$

and an infinite-component wave equation written in terms of the generators of the zero-helicity representation of the conformal group $SO(4,2)$. A similar algebraization has been carried out for the Bethe-Salpeter equation (for the same case of scalar Coulomb interaction) in Ref. 21.

We will make use of the well-known degenerate representation of $SO(4,2)$ which can be realized on the set of homogeneous functions on the upper light cone $u_0=(u_1{}^2+u_2{}^2+u_3{}^2+u_4{}^2)^{1/2}$ of degree of homogeneity -2 or -1 (see, e.g., Ref. 5). It is equivalent to the zero-helicity representation[22] of the conformal group [for an explicit demonstration of the equivalence of the two representations see Appendix of Ref. 18]. This representation can be realized equivalently on the space $\mathcal{H}_1$ of functions defined on the double-sheeted hyperboloid $p^2=1$, equipped with scalar product

$$(\phi,\psi)=(1/\pi^4)\int\int\bar{\phi}(p)[-1/(p-q)^2]\times\psi(q)\delta(p^2-1)\delta(q^2-1)d^4p\,d^4q. \tag{4.13}$$

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The (homogeneous) Lorentz group acts in $\mathcal{H}_1$ as a group of argument transformations:

$$[U(\Lambda)\psi](q)=\psi(\Lambda^{-1}q).$$

The generators Γ_μ and Γ_5 of $SO(4,2)$ (i.e., the representatives of the Dirac γ matrices $\frac{1}{2}\gamma_\mu$ and $\frac{1}{2}\gamma_5$ in this infinite-dimensional unitary representation) are defined by the following nonlocal operators:

$$[\Gamma_\mu\phi](p)=-(2/\pi^2)\int\{q_\mu/[(p-q)^2]^2\}\times\phi(q)\,\epsilon(q_0)\,\delta(q^2-1)\,d^4q, \quad (4.14)$$

$$[\Gamma_5\phi](p)=-(2/\pi^2)\int\{1/[(p-q)^2]^2\}\times\phi(q)\,\epsilon(q_0)\,\delta(q^2-1)\,d^4q. \quad (4.15)$$

Comparing (4.14) with (4.15) we see that

$$p_\mu\phi(p)=[(1/\Gamma_5)\Gamma_\mu\phi](p). \quad (4.16)$$

[It can be verified directly (see Ref. 18) that the operators $p_\mu=(1/\Gamma_5)\Gamma_\mu$ commute between themselves and that $p_\mu p^\mu=1$.] It is also not difficult to check that the inverse of the operator (4.15) is given by

$$[(1/\Gamma_5)\phi](p)=(1/2\pi^2)\int[-1/(p-q)^2]\times\phi(q)\,\epsilon(q_0)\,\delta(q^2-1)\,d^4q. \quad (4.17)$$

Changing p and q in (4.16) and (4.17) to p/m and q/m, and inserting in Eq. (4.12), we get the following "algebraic form" of the quasipotential equation:

$$((m/\Gamma_5)\Gamma_0-E)\phi_E=(\alpha m^2/2E\Gamma_5)\phi_E, \quad (4.18)$$

where α is given by (4.10). The discrete spectrum corresponding to Eq. (4.18) can be found by multiplying both sides by Γ_5 from the left and performing a rotation in the (0, 5) plane (cf. Ref. 4). The result is

$$[E(m^2-E^2)^{1/2}\Gamma_0-\tfrac{1}{2}\alpha m^2]\phi_E=0. \quad (4.19)$$

Finally we recall that the eigenvalues of Γ_0 in the given representation are all positive integers (see, e.g., Ref. 18) and find

$$E_n{}^2=\tfrac{1}{2}m^2[1+(1-\alpha^2/n^2)^{1/2}],$$

in agreement with (4.11). Equation (4.18) admits also a continuous spectrum corresponding to the two-particle scattering states.

Equation (4.11) for the total-energy eigenvalues in our model appears as a special case (for $m_1=m_2=m$) of the equation

$$M_n{}^2=m_1{}^2+m_2{}^2+2m_1m_2[1-(\alpha/n)^2]^{1/2} \quad (4.20)$$

obtained by Barut and Baiquni in Ref. 6 [see Eq. (10) of this reference]. It is interesting to note that the same bound-state eigenvalues can be obtained from the so-called eikonal approximation[23,24] [it is sufficient to replace the photon by a massless scalar particle in the scheme of Ref. 24 in order to obtain (4.20)]. This supports the relevance of our approximate quasipotential equation and indicates that Barut's algebraic model[6] is related to a more conventional dynamical model. We mention that unlike the Barut-Baiquni infinite-component wave equation, our Eq. (4.18) is symmetric with respect to the two initial particles.

In conclusion, we would like to make the following remarks:

(1) The preceding argument gives a simple prescription for the "algebraization" of the (free) four-momentum:

$$p_\mu=(m/\Gamma_5)\Gamma_\mu \quad (4.21)$$

[see (4.16)]. This prescription is independent of the interaction under consideration.

(2) The simple algebraization of the potential based on Eq. (4.17) is peculiar to the case of zero-mass exchange. The potential (3.10) with $\mu>0$ already leads to considerable complications (see Sec. III 2 of Ref. 18). The reason is that the kernel in the scalar product (4.13) in $\mathcal{H}_1$ is closely related to the Coulomb potential. If, however, we adapt the scalar product in our representation space to the potential for $\mu>0$, the simplicity of the free Hamiltonian will be destroyed.

(3) The potential on the right-hand side of (4.18) will coincide with the nonrelativistic attractive Coulomb potential in coordinate space $(-\alpha/r)$ if we identify r with $(1/m)\Gamma_5$. This observation is not accidental. It has been argued in Ref. 2 that, in general, for spin-0 particles the relativistic generalization of r is given by $r^2=m^{-2}(\mathbf{N}^2-\mathbf{L}^2)$, where $\mathbf{L}$ and $\mathbf{N}$ are the generators of the homogeneous Lorentz group. In our case, $\mathbf{N}^2-\mathbf{L}^2=\Gamma_5{}^2$.

(4) The simplest evaluation of the Lamb-shift corrections and of the hyperfine splitting of the hydrogen levels due to the nucleon form factor[25,26] is made on the basis of the Logunov-Tavkhelidze quasipotential equation.[7] It would be interesting to carry out this more realistic calculation on the basis of the quasipotential equation (3.9) by extending the algebraic technique developed here to spinor particles.

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