Electroproduction Structure Functions, Integral Representations, and Light-Cone Commutators*

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The electroproduction inelastic structure functions $W_i(\kappa,\nu)$ ($i=1,2$; $\kappa=q^2=$ momentum transfer squared; $\nu=q\cdot p=$ energy transfer; $p^2=1$) are studied in the Bjorken (A) limit ($\nu\to\infty$, $\rho\equiv-\nu/\kappa$ fixed) and in the Regge (R) limit ($\nu\to\infty$, κ fixed). Finite A limits [$F_2(\rho)=\lim\nu W_2(\kappa,\nu)$, $F_1(\rho)=\lim W_1(\kappa,\nu)$] and Pomeranchuk-dominated R limits [$w^2(\kappa)\nu^{-1}=\lim W_2$, $w_1(\kappa)\nu=\lim W_1$] are assumed. These two limits are first related by use of Deser-Gilbert-Sudarshan (DGS) representations for causal functions $V_i(\kappa,\nu)$ related to the W_i by $W_2=\kappa V_2$ and $W_1=\kappa V_1-\nu^2V_2$. The above A- and R-limit assumptions, together with a smoothness assumption on the DGS spectral functions motivated by the existence of some equal-time commutators, are shown to imply that $F_2(\infty)=w_2(-\infty)\equiv w_2=$const and $\lim_{\rho\to\infty}F_1(\rho)/\rho=-\lim_{\kappa\to-\infty}w_2(\kappa)\kappa\equiv w_1=$const. These results agree with experiment if $w_i\neq0$. The properties of the spectral functions that are obtained are used to discuss some equal-time commutators and properties of the photon structure function and the cross section $\sigma_\gamma(\nu)$. The empirical value of $\sigma_\gamma(\infty)$ is used to roughly calculate w_2. It is stressed, however, that the above assumptions do not preclude the possibility that $w_2=0$. The Fourier transforms $\hat{W}_i(x^2,p\cdot x)$ of the $W_i(\kappa,\nu)$ are next studied and used to again relate the A and R limits. Restrictions on the W_i imposed by the requirements of finite A limits and correct R limits are determined and results equivalent to the above ones are obtained. This analysis determines the configuration-space behavior corresponding to the A and R limits for large ρ and κ, respectively. These limits are shown to determine the behavior of the $\hat{W}_i$ near the light cone $x^2=0$, and the results are that $\pi\hat{W}_2\sim\delta(x^2)\frac{1}{2}\epsilon(x_0)\hat{f}_2(x_0)$ and $\pi W_1\sim\delta'(x^2)2\,|x_0|\,\hat{f}_1(x_0)$ for $x^2\sim0$, where $w_2=\frac{1}{2}i\int dx_0\,\hat{f}_2(x_0)$ and $w_1=2i[\hat{f}_1(\infty)-\hat{f}_1(-\infty)]$. Corresponding properties of the V_i are derived, and the equivalence of this approach with the one using integral representations is established. The light-cone behavior of each component of $\langle p|[J_\mu(x),J_\nu(0)]|p\rangle$ can be determined, and it is shown in particular that $\langle p|[J_0-J_3,J_0-J_3]|p\rangle\sim\delta(x^2)\epsilon(x_0)\hat{f}_2(x_0)$, apart from total derivatives with respect to x_0-x_3. The light-cone behavior equivalent to the Fubini–Dashen–Gell-Mann sum rule is then put in a form which can accommodate this result so well that we are led to propose a highly symmetric universal structure for the light-cone commutator of two $SU(3)$ currents. The universality is made precise by use of the $[SU(3)\otimes SU(3)]_\beta$ equal-time commutation relations. The equal-time implications of the proposal are considered and are shown to be consistent with, and suggested by, the gluon mode. In the context of this model, w_2 is numerically estimated and found to agree with the experimental value.

I. INTRODUCTION

THE remarkable nontrivial scaling property experimentally[1,2] exhibited by the forward electroproduction structure function in the deep inelastic region has led to a number of theoretical speculations about the nature of inelastic electron-hadron scattering. Suggestions for behavior of the observed type have been based on "almost equal-time" commutators,[3] relation to Pomeranchuk exchange,[4–6] vector-meson dominance,[7] constituent "parton" models of the proton,[8] and relation to current-algebra sum rules.[9]

In this paper we shall investigate and relate two model-independent approaches to the study of the deep inelastic region. The first approach employs an integral representation for the scattering amplitude[10] and the second involves the behavior of the Fourier transform of the scattering amplitude near the light cone. In both cases, the observed behavior is deduced by relating the deep inelastic limit to the Pomeranchuk-dominated ordinary Regge limit. Although the integral-representation analysis is mathematically the simplest, the light-cone approach is more transparent physically in that it produces a simple configuration-space description of the experimental results. In fact, a simple algebraic generalization[11] of the configuration-space behavior implied by current-algebraic sum rules is seen to give an elegant and numerically accurate account of the empirical situation.

We recall that the total electron-proton cross section in order α^2 can be written[12,13]

$$d\sigma/d\Omega dE'=[\alpha^2/4E^2\sin^2(\tfrac{1}{2}\theta)]\times[W_2(\kappa,\nu)\cos^2(\tfrac{1}{2}\theta)+2W_1(\kappa,\nu)\sin^2(\tfrac{1}{2}\theta)],\quad(1.1)$$

where E, E', and θ are, respectively, the electron initial

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2 W. Albrecht *et al.* (unpublished).
3 J. D. Bjorken, Phys. Rev. **179**, 1547 (1969).
4 H. D. I. Abarbanel *et al.*, Phys. Rev. Letters **22**, 500 (1969).
5 R. A. Brandt, Phys. Rev. Letters **22**, 1149 (1969).
6 H. Harari, Phys. Rev. Letters **22**, 1078 (1969).
7 J. J. Sakurai, Phys. Rev. Letters **22**, 981 (1969).
8 R. P. Feynman, Phys. Rev. Letters **23**, 1415 (1969); and unpublished; S. D. Drell, D. Levy, and T. M. Yan, *ibid.* **22**, 744 (1969); Phys. Rev. **187**, 2159 (1970); D. J. Bjorken and E. A. Paschos, *ibid.* **185**, 1975 (1969).
9 R. A. Brandt, Phys. Rev. Letters **23**, 1260 (1969).
10 A short account of this work is given in Ref. 5.
11 A short account of this work is given in Ref. 9.
12 S. D. Drell and J. D. Walecka, Ann. Phys. (N.Y.) **28**, 18 (1968).
13 For an extensive review of the kinematics, see L. S. Brown, in Boulder Lectures in Theoretical Physics (unpublished).

energy, final energy, and scattering angle,

$$\kappa=q^2=-4EE'\sin^2(\tfrac{1}{2}\theta) \qquad (1.2)$$

is the square of the momentum transferred to electron, and

$$\nu=q\cdot p=E-E'. \qquad (1.3)$$

Throughout this paper we take the initial proton to be at rest and to have mass 1, so that the four-momentum p is

$$p^\mu=(1,0,0,0). \qquad (1.4)$$

The structure functions W_i can be defined from the forward spin-averaged current-proton scattering amplitude

$$T_{\mu\nu}=i\int d^4x\exp(iq\cdot x)\theta(x_0)\langle p\,|\,[J_\mu(x),J_\nu(0)]\,|\,p\rangle+\text{poly.}$$
$$=\left(p_\mu-\frac{\nu q_\mu}{\kappa}\right)\left(p_\nu-\frac{\nu q_\nu}{\kappa}\right)T_2(\kappa,\nu)$$
$$-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{\kappa}\right)T_1(\kappa,\nu) \qquad (1.5)$$

by

$$W_{\mu\nu}=(1/\pi)\,\text{Im}T_{\mu\nu}=(1/2\pi)\int d^4x\exp(iq\cdot x)$$
$$\times\langle p\,|\,[J_\mu(x),J_\nu(0)]\,|\,p\rangle$$
$$=\left(p_\mu-\frac{\nu q_\mu}{\kappa}\right)\left(p_\nu-\frac{\nu q_\nu}{\kappa}\right)W_2(\kappa,\nu)$$
$$-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{\kappa}\right)W_1(\kappa,\nu), \qquad (1.6)$$

so that

$$W_i(\kappa,\nu)=(1/\pi)\,\text{Im}T_i(\kappa,\nu). \qquad (1.7)$$

The positivity condition

$$W_i\geq 0 \qquad (1.8)$$

follows from (1.6). The transverse and longitudinal cross sections σ_T and σ_L are related to the W_i by

$$W_1=(\sigma_T/4\pi^2\alpha)(\nu-\tfrac{1}{2}|\kappa|),$$
$$W_2=\frac{(\sigma_T+\sigma_L)(\nu-\frac{1}{2}\kappa)}{4\pi^2\alpha(1+\nu^2/|\kappa|)}. \qquad (1.9)$$

Equation (1.5) becomes the physical Compton amplitude for $\kappa\to 0$, and we have

$$W_1\underset{\kappa\to 0}{\longrightarrow}(\nu\sigma_\gamma/4\pi^2\alpha),\qquad W_2\underset{\kappa\to 0}{\longrightarrow}(|\kappa|\sigma_\gamma/4\pi^2\alpha\nu), \qquad (1.10)$$

where σ_γ is the total photon-proton cross section. Experimentally,[14] one has

$$\sigma_\gamma(\nu)\underset{\nu\to\infty}{\longrightarrow}\text{const.} \qquad (1.11)$$

A consequence of (1.10) is that

$$W_2(0,\nu)=0. \qquad (1.12)$$

We refer to the Bjorken[3] deep inelastic limit as the A limit, defined by

A limit: $\nu\to\infty$, $\kappa\to-\infty$, $\rho\equiv-\nu/\kappa$ fixed,

and to the Regge limit as the R limit, defined by

R limit: $\nu\to\infty$, $\kappa<0$ fixed.

The limits for large values of the fixed parameter are

A' limit: $\nu\to\infty$, $\kappa\to-\infty$, $\rho\gg 1$,

R' limit: $\nu\to\infty$, $\kappa\ll -1$.

The amplitudes νW_2 and W_1 are dimensionless, and so one defines

$$\lim_A \nu W_2(\kappa,\nu)=F_2(\rho), \qquad (1.13)$$

$$\lim_A W_1(\kappa,\nu)=F_1(\rho). \qquad (1.14)$$

Bjorken[3] has argued that the F_i are expected to be finite so that, with (1.8),

$$0\leq F_i(\rho)<\infty. \qquad (1.15)$$

We shall assume this behavior in this paper. Nontrivial scaling means $F_i>0$ and this is empirically observed[1] for F_2, with

$$F_2(\rho)\underset{\rho\to\infty}{\longrightarrow}\text{const}\neq 0. \qquad (1.16)$$

In terms of σ_T and σ_L, one has

$$F_T\equiv F_1=(1/4\pi^2\alpha)(1-1/2\rho)\lim_A \nu\sigma_T\geq 0 \qquad (1.17)$$

and

$$F_L\equiv\rho F_2-F_1=(1/4\pi^2\alpha)(1-1/2\rho)\lim_A \nu\sigma_L\geq 0. \qquad (1.18)$$

We assume the usual Regge asymptotic behavior[15–18]

$$W_2\underset{R}{\longrightarrow}w_2(\kappa)\nu^{\alpha-2},$$
$$W_1\underset{R}{\longrightarrow}w_1(\kappa)\nu^{\alpha}, \qquad (1.19)$$

where α is the $t=0$ intercept of the leading appropriate Regge trajectory. We assume that the Pomeranchuk trajectory with $\alpha=1$ dominates, so that

$$W_2\underset{R}{\longrightarrow}w_2(\kappa)\nu^{-1}, \qquad (1.20)$$

$$W_1\underset{R}{\longrightarrow}w_1(\kappa)\nu. \qquad (1.21)$$

Although a naive analysis suggests that the Pomeran-

[14] Experimental limits: DESY Bubble Chamber Group [Phys. Letters **27B**, 474 (1968)] give $\sigma_\gamma=116\pm 17$ μb at 3.5–5.4 GeV and J. Ballam *et al.* (SLAC) [Phys. Rev. Letters **21**, 1544 (1968)] give $\sigma_\gamma=126\pm 17$ μb at 7.5 GeV.

[15] H. Harari, Phys. Rev. Letters **17**, 1303 (1966).

[16] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters **18**, 32 (1966); Phys. Rev. **157**, 1448 (1967).

[17] V. Singh, Phys. Rev. Letters **18**, 36 (1967); **18**, 300 (E) (1967).

[18] V. De Alfaro *et al.*, Ann. Phys. (N.Y.) **44**, 165 (1967).

chukon does not couple to W_2 at the relevant point $t=0$, it is believed[19,20] that a mechanism is operative which reinstates this coupling, quite consistent with (1.11). The specific mechanism, be it expressed in the language of singular residues, fixed poles, kinematical singularities, or whatever, does not concern us here. Nor does the J-plane nature of the Pomeranchuk singularity. We assume only the diffractive behavior (1.20) and (1.21).

Having stated the preliminary definitions and assumptions, we proceed to summarize the contents of this paper. In Sec. II A, we define functions V_1 and V_2 which are causally related to $W_{\mu\nu}$ and which satisfy $W_2=\kappa V_2$ and $W_1=\kappa V_1-\nu^2 V_2$. The V_i satisfy causal integral representations [with spectral functions $\sigma_i(a,b)$], and so we write representations for the W_i. We use some equal-time (ET) commutators in Sec. II B to motivate our smoothness assumption that the σ_i decrease fast for large a. This assumption, together with the A-limit and R-limit assumptions (1.13)–(1.15) and (1.20) and (1.21) are shown in Sec. II C to imply that

$$F_2(\infty)=w_2(-\infty)=\tfrac{1}{2}\int da\,\sigma(a)a\equiv w_2$$

and

$$\lim_{\rho\to\infty} F_1(\rho)/\rho=-\lim_{-\kappa\to\infty} w_2(\kappa)\kappa=w_2-2\int da\,\tilde{\sigma}(a)\equiv w_2-v_1,$$

where

$$-(\partial/\partial b)\sigma_2(a,b)\sim\sigma(a)b^{-1}$$

and

$$\sigma_1(a,b)\sim\tilde{\sigma}(a)b^{-2}$$

for $b\sim 0$. We then use some properties of the $\sigma_i(a,b)$ that have been obtained to discuss some ET commutators and also the photon amplitude. The expression for the photon amplitude, together with a saturation assumption, is used to roughly calculate w_2 in Sec. II D. In Sec. II E we discuss the crucial point that the above assumptions do *not* exclude the possibility that $w_2=0$.

In Sec. III A we again study the A limit by relating it to the Pomeranchuk-dominated Regge limit. This time we work with the Fourier transforms $\hat{W}_i(x^2,p\cdot x)$ of the $W_i(\kappa,\nu)$. We determine restrictions on the W_i imposed by the requirements that W_1 and νW_2 have finite A limits and correct R limits and then use these restrictions to find the A' limits and R' limits. Results equivalent to the above ones are obtained. This analysis is shown in Sec. III B to have determined the configuration-space behavior corresponding to the A' and R' limits in momentum space. The behaviors of the W_i near the light cone $x^2=0$ are shown to determine the A' and R' limits and the results are that $\pi\hat{W}_2\sim\delta(x^2)\tfrac{1}{2}\epsilon(x_0)\hat{f}_2(x_0)$ and $\pi\hat{W}_1\sim\delta'(x^2)2\,|\,x_0\,|\,\hat{f}_1(x_0)$ for $x^2\sim 0$, where

$$w_2=\tfrac{1}{2}i\int dx_0\,\hat{f}_2(x_0)\text{ and }w_1=2i[\hat{f}_1(+\infty)-\hat{f}_1(-\infty)].$$

Similar results in terms of the V_i are derived in Sec. III C and shown to be equivalent to these. In Sec. III D the equivalence of the above approach and the previous integral-representation approach is established.

The above results are sufficient to determine the light-cone (LC) behavior of each component of $\langle p\,|\,[J_\mu(x),J_\nu(0)]\,|\,p\rangle$. It is shown in Sec. III E in particular that

$$\langle p\,|\,[J_-(x),J_-(0)]\,|\,p\rangle\sim\delta(x^2)\epsilon(x_0)\hat{f}_2(x_0)$$

apart from total derivatives with respect to x_0-x_3, where $J_-\equiv J_0-J_3$. In Sec. III F the LC behavior equivalent to the Fubini–Dashen–Gell-Mann current-algebra sum rule is put in a form which can accommodate this result so well that we are led to propose a highly symmetric universal structure for the LC commutator of two $SU(3)$ currents. The universality is made precise by use of the $[SU(3)\otimes SU(3)]_\beta$ ET commutation relations. Although our proposal suggests an unusual algebraic structure in the R limit, the previous relations between the R' and A' limits are maintained. The ET implications of our proposal are considered in Sec. III G, and it is shown that the proposal is consistent with, and suggested by, the gluon model for ET commutators. In the context of this model, we numerically estimate w_2 and find excellent agreement with experiment. This supports our proposal and suggests why $w_2\neq 0$.

Our conclusions are summarized in Sec. IV and in an appendix we show how our methods can be applied when an arbitrary Regge trajectory is relevant.

II. INTEGRAL REPRESENTATIONS

A. Derivation

We shall assume the validity of the so called Deser-Gilbert-Sudarshan[21] (DGS) representation for forward matrix elements of commutators of relatively local field operators. Although this representation is not a strict consequence of the axioms of quantum field theory, it is known to be correct in every order of perturbation theory.[22] We use this representation for convenience only and could equally well employ the rigorous Jost–Lehmann–Dyson[23] representation.[24]

Although $W_{\mu\nu}$, as defined by (1.6), is the Fourier transform of a causal commentator, the W_i will not, in general, be causal because of the $1/\kappa$ factors in (1.6). It is therefore convenient to introduce additional

[19] A. H. Mueller and T. L. Trueman, Phys. Rev. **160**, 1296 (1967); **160**, 1306 (1967).

[20] H. D. I. Abarbanel *et al.*, Phys. Rev. **160**, 1329 (1967).

[21] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959); M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960).

[22] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **26**, 337 (1961).

[23] R. Jost and H. Lehmann, Nuovo Cimento **5**, 1598 (1957); F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

[24] Our calculation has been repeated using this representation in Ref. 13.

functions V_i defined by

$$W_{\mu\nu}=[\kappa p_\mu p_\nu-\nu(p_\mu q_\nu+q_\mu p_\nu)+g_{\mu\nu}\nu^2]V_2(\kappa,\nu)-(\kappa g_{\mu\nu}-q_\mu q_\nu)V_1(\kappa,\nu). \quad (2.1)$$

Comparison with (1.6) gives

$$W_2=\kappa V_2 \quad (2.2)$$

and

$$W_1=\kappa V_1-\nu^2 V_2. \quad (2.3)$$

The Fourier transform of (2.1) is

$$\hat{W}_{\mu\nu}\equiv(1/2\pi)\langle p\,|\,[J_\mu(x),J_\nu(0)]\,|\,p\rangle =[-\Box p_\mu p_\nu+(p\cdot\partial)(p_\mu\partial_\nu+\partial_\mu p_\nu)-g_{\mu\nu}(p\cdot\partial)^2] \times\hat{V}_2(x^2,p\cdot x)-(-\Box g_{\mu\nu}+\partial_\mu\partial_\nu)\hat{V}_1(x^2,p\cdot x), \quad (2.4)$$

where

$$V_i(\kappa,\nu)=\int d^4x\,\exp(iq\cdot x)\hat{V}_i(x^2,p\cdot x). \quad (2.5)$$

Note that

$$\hat{V}_i(x^2,-p\cdot x)=-\hat{V}_i(x^2,p\cdot x). \quad (2.6)$$

The $\hat{V}_i$ are locally related to $\hat{W}_{\mu\nu}$, and it can be shown that the $\hat{V}_i$ are themselves causal.[25] Thus we have the DGS representations

$$V_i(\kappa,\nu)=\int_0^\infty da\int_{-1}^1 db\,\sigma_i(a,b)\delta(\kappa+2b\nu-a)\epsilon(\nu+b). \quad (2.7)$$

In configuration space these become

$$\hat{V}_i(x^2,p\cdot x)=-\frac{i}{2\pi}\int_0^\infty da\int_{-1}^1 db\,\sigma_i(a,b) \times\exp(-ibp\cdot x)\Delta(x;a+b^2), \quad (2.8)$$

where

$$i\Delta(x;a)=[1/(2\pi)^3]\int d^4p\,\exp(-ip\cdot x)\delta(p^2-a)\epsilon(p^0) \quad (2.9)$$

is the usual mass-$a^{1/2}$ free-field commutator function. We shall always take $\nu>+1$ so that $\epsilon(\nu+b)$ can be replaced by $+1$ in (2.7). We can now use (2.2) and (2.3) to write representations for the W_i:

$$W_2=\kappa\int_0^\infty da\int_{-1}^1 db\,\sigma_2(a,b)\delta(\kappa+2b\nu-a), \quad (2.10)$$

$$W_1=\int_0^\infty da\int_{-1}^1 db\,[\kappa\sigma_1(a,b)-\nu^2\sigma_2(a,b)]\delta(\kappa+2b\nu-a). \quad (2.11)$$

We emphasize here that the above representations incorporate, in addition to causality, the constraints arising from the spectrum of the allowed intermediate states. These spectral conditions restrict the supports of the spectral functions $\sigma_i(a,b)$, as embodied in the integration limits in (2.7). These constraints, specifically the boundedness of the b-integration range, will be crucial to our analysis.

B. Smoothness Assumption

As they stand, the integral representations (2.7) are useless for our purposes because of their generality—they are valid in any decent causal theory. In order to make effective use of (2.7), we must restrict the class of spectral functions to be considered. We thus will assume that the $\sigma_i(a,b)$ are rapidly decreasing functions of a so that at least the moments $\int da\,\sigma_1(a,b)$ and $\int da(\partial/\partial b)\sigma_2(a,b)a$ exist. It is the purpose of this subsection to provide some justification for this assumption with reference to some ET commutation relations.

The connection between the representations (2.7) and ET current commutation relation has been established by Cornwall and Norton.[26] Their results, for the electromagnetic current, show that the existence of the time-space commutator implies the existence of $\int dadb\,\sigma_i(a,b)$, the existence of the space-space commutator implies the existence of $\int dadb\,\sigma_i(a,b)b$, and the existence of the (time derivative of space)-space commutator implies the existence of $\int dadb\,\sigma_i(a,b)b^2$ and $\int dadb\,\sigma_i(a,b)a$. Thus our smoothness assumption should be satisfied in any reasonable theory with finite ET commutators $[J_i,J_\mu]$ and $[\dot{J}_i,J_j]$.

The ultimate justification of our assumption rests, of course, with the reasonable and experimentally correct nature of our results. We feel that these results do, in fact, suggest the rapid decrease of the $\sigma_i(a,b)$ for $a\to\infty$. It might have been that more singular spectral functions were required by the data, but this seems not to be the case here.

C. Limits

R′ and A limits

We consider first (2.10) and proceed as in Ref. 5. For reference, we list our three basic assumptions discussed above as (i) finite A limit, (ii) Pomeranchukon dominance in R limit, and (iii) rapid convergence of spectral functions.

In the A limit, we find

$$\nu W_2 \underset{A}{\longrightarrow} \tfrac{1}{2}\kappa\int da\,\sigma_2(a,1/2\rho).$$

Thus, in order that the limit be finite, we must have

$$\int da\,\sigma_2(a,1/2\rho)=0. \quad (2.12)$$

Then (2.10) gives

$$\nu W_2 \underset{A}{\longrightarrow} (1/4\rho)\int da\,\bar{\sigma}_2(a,1/2\rho)a\equiv F_2(\rho), \quad (2.13)$$

[25] J. W. Meyer and H. Suura, Phys. Rev. **160**, 1366 (1967). Note that this result depends on the Abelian nature of the electromagnetic current.

[26] J. M. Cornwall and R. E. Norton, Phys. Rev. **173**, 1637 (1968).

where

$$\bar{\sigma}_2(a,b) = -(\partial/\partial b)\sigma_2(a,b).$$

If $\bar{\sigma}_2$ were finite at $b=0$, we would have

$$F_2(\rho) \overset{?}{\underset{\rho\to\infty}{\longrightarrow}} \frac{1}{4\rho}\int da\, \bar{\sigma}_2(a,0)a.$$

To see if this is possible, we calculate the R limit of (2.10):

$$\nu W_2 \underset{R}{\longrightarrow} (\kappa/2\nu)\int da\, \sigma_2[a, (a-\kappa)/2\nu]. \qquad (2.14)$$

We now take $\nu \gg -\kappa \gg 1$ so that, since σ_2 is assumed to vanish rapidly for large a, we can assume that $-a/\kappa \ll 1$ inside the integral. Then, using (2.12), (2.14) becomes

$$M_2 \underset{R'}{\longrightarrow} -(\kappa/4\nu^2)\int da\, \bar{\sigma}_2(a, -\kappa/2\nu)a, \qquad \nu \gg -\kappa \gg 1. \qquad (2.15)$$

Thus, ignoring possible difficulties at $b=0$, we find

$$W_2 \overset{?}{\underset{R'}{\longrightarrow}} -\frac{\kappa}{4\nu^2}\int da\, \bar{\sigma}_2(a,0)a. \qquad (2.16)$$

Since this violates (1.19), we must conclude that $\bar{\sigma}_2$ is singular at $b=0$. It must, in fact, diverge linearly to account for the extra power of ν^{-1} in (2.16). Therefore we can write

$$\bar{\sigma}_2(a,b) = \sigma(a)b^{-1} + \tau_2(a,b), \qquad (2.17)$$

where $\tau_2(a,b)$ is less singular at $b\sim 0$ than b^{-1}. It then follows from (2.12) [which implies that $\int da\, \bar{\sigma}_2(a,b) = 0$] that

$$\int da\, \sigma(a) = 0 \qquad (2.18)$$

and $\int da\, \tau_2(a,b) = 0$. Insertion of (2.17) into (2.15) gives

$$W_2 \underset{R'}{\longrightarrow} (1/2\nu)\int da\, \sigma(a)a \equiv w_2\nu^{-1}, \qquad \nu \gg -\kappa \gg 1. \qquad (2.19)$$

Thus we are now consistent with (1.19) and find that $w_2(\kappa) \sim w_2$ is independent of κ for large κ.

We now return to (2.13) and use (2.17) to find

$$F_2(\rho) \underset{\rho\to\infty}{\longrightarrow} \tfrac{1}{2}\int da\, \sigma(a)a = w_2, \qquad (2.20)$$

which is the desired result (1.16). Although it is conceivable that $w_2=0$, we shall indicate below that this is not the case. Thus (1.16) will be satisfied in any theory obeying Bjorken behavior (1.13), Regge behavior (1.20), and the representation (2.10) with a rapidly decreasing spectral function.

In the same way, from (2.11), we find

$$W_1 \underset{A}{\longrightarrow} -(1/2\rho)\int da\, \sigma_1(a, 1/2\rho) + \tfrac{1}{4}\int da\, \bar{\sigma}_2(a, 1/2\rho)a \equiv F_1(\rho), \qquad (2.21)$$

and consideration of the R' limit leads to the requirement that

$$\sigma_1(a,b) = \tilde{\sigma}(a)b^{-2} + \tau_1(a,b), \qquad (2.22)$$

for some (possibly vanishing) $\tilde{\sigma}(a)$ and some $\tau_1(a,b)$ less singular at $b\sim 0$ than b^{-2}. Thus

$$W_1 \underset{R'}{\longrightarrow} -w_1\kappa^{-1}\nu, \qquad (2.23)$$

where

$$w_1 = w_2 - v_1, \qquad (2.24)$$

with

$$v_1 = 2\int da\, \tilde{\sigma}(a), \qquad (2.25)$$

and so

$$F_1(\rho) \underset{\rho\to\infty}{\longrightarrow} w_1\rho. \qquad (2.26)$$

This prediction is consistent with the present rough experimental information[2] on F_1.

It is convenient to consider explicitly the combinations (1.17) and (1.18). F_L is determined by σ_1 according to

$$F_L(\rho) = (1/2\rho)\int da\, \sigma_1(a, 1/2\rho) \underset{\rho\to\infty}{\longrightarrow} v_1\rho. \qquad (2.27)$$

If $v_1=0$, then

$$F_T(\rho) \underset{\rho\to\infty}{\longrightarrow} w_2\rho \qquad (2.28)$$

and

$$F_L(\rho) \underset{\rho\to\infty}{\longrightarrow} 0\rho, \qquad (2.29)$$

so that

$$\sigma_L/\sigma_T \underset{A}{\longrightarrow} F_L/F_T \underset{\rho\to\infty}{\longrightarrow} 0. \qquad (2.30)$$

This behavior is also roughly indicated experimentally.[2]

Connection with Equal-Time Commutators

By comparing our results with the Cornwall-Norton[26] expressions for ET commutators discussed in Sec. II B, we can find some restrictions imposed by the assumption of a finite scaling limit. We have seen that $F_2(\rho) < \infty$ implies that $\int da\, \sigma_2(a,b) = 0$ and hence that

$$\int dadb\, \sigma_2(a,b) = \int dadb\, \sigma_2(a,b)b = 0. \qquad (2.31)$$

Imposing this requirement on the Cornwall-Norton Eqs. (2.20) gives

$$\langle p \mid [J_i(0,\mathbf{x}), J_j(0)] \mid p\rangle = 0 \qquad (2.32)$$

and

$$\langle p \mid [J_0(0,\mathbf{x}), J_i(0)] \mid p\rangle \propto \partial_j\delta(\mathbf{x}). \qquad (2.33)$$

Conversely, (2.32) or (2.33) imply (2.31) and hence finite scaling for νW_2. In a similar way one can obtain the Callan-Gross[27] results from comparison of the Cornwall-Norton representation for $\langle p \mid [\dot{J}_i, J_j] \mid p\rangle$ and our expressions (2.13) and (2.21).

R Limit

We have determined the large-κ behavior of the Regge residue functions defined in (1.20) and (1.21)

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[27] C. Callan and D. J. Gross, Phys. Rev. Letters **22**, 156 (1969).

to be

$$w_2(\kappa) \underset{\kappa\to\infty}{\longrightarrow} w_2, \tag{2.34}$$

$$w_1(\kappa) \underset{\kappa\to\infty}{\longrightarrow} -w_1\kappa^{-1}. \tag{2.35}$$

We can obtain an expression for $w_2(\kappa)$ for general κ by integrating (2.17) to obtain

$$\sigma_2(a, b) = -\sigma(a) \ln b + \eta(a, b), \tag{2.36}$$

where η is less singular at $b\sim 0$ than $\ln b$. Substitution in (2.14), using (2.18), gives

$$w_2(\kappa) = -\tfrac{1}{2}\kappa \int da\, \sigma(a) \ln(a-\kappa). \tag{2.37}$$

This reduces the (2.34) for large κ. Similarly, we find

$$w_1(\kappa) = \int da \left[\tfrac{1}{2}\sigma(a) \ln(a-\kappa) - 2\tilde{\sigma}(a)\kappa(a-\kappa)^{-1}\right], \tag{2.38}$$

which satisfies (2.35).

We define here the photon amplitudes

$$Y_2(\nu) = -\lim_{\kappa\to 0} \kappa^{-1} W_2(\kappa, \nu), \tag{2.39}$$

$$Y_1(\nu) = \lim_{\kappa\to 0} W_1(\kappa, \nu), \tag{2.40}$$

and find from (2.37) and (2.38) that

$$Y_2(\nu) \underset{\nu\to\infty}{\longrightarrow} (1/2\nu)\int da\, \sigma(a) \ln a \equiv y\nu^{-1}, \tag{2.41}$$

$$Y_1(\nu) \underset{\nu\to\infty}{\longrightarrow} \tfrac{1}{2}\nu \int da\, \sigma(a) \ln a = y\nu. \tag{2.42}$$

Using (1.10), we obtain

$$\sigma_\gamma(\infty) = 4\pi^2\alpha y. \tag{2.43}$$

This agrees with the experimental result (1.11) if $y\neq 0$.

D. Saturation Assumption

Any of the constants w_1, w_2, and y introduced above can, in principle, vanish. In this subsection we shall assume that $w_2\neq 0$ and obtain a rough estimate of it from the experimental (nonvanishing) value of y. The possibility that $w_2=0$ will be considered in Sec. II E.

To relate w_2 and y, we shall assume that the a integration in (2.41) is approximately saturated near some effective squared mass a_0. It is important to emphasize that we are not making a saturation approximation for the scattering amplitude (2.10) for any energy. The point is that $\sigma(a)$ is like a two-point spectral function in that, according to (2.17), it describes the intermediate-state spectrum for $b=0$ at which point, according to (2.10), $W_2(\kappa, \nu)$ loses its ν dependence and becomes a function of κ alone. The $b=0$ contribution to (2.10) indeed has the form of a two-point spectral representation in the variable κ. This single-variable character of $\sigma(a)$ is also clear from (2.37). For these reasons, we expect the saturation approximation to be useful in (2.37) although not in the general representation (2.10). Furthermore, since the $b=0$ part of (2.10) or, more clearly, (2.37) describes the κ dependence of W_2 so that $\sigma(a)$ corresponds to the singularity structure of W_2 in κ, we expect the ρ-meson intermediate state to be the dominant one for small κ. Our assumption thus amounts to "ρ dominance" of (2.37) for small κ. Since $\sigma(a)$ has no simple expression in terms of intermediate states, however, this assumption is not on the same footing as the usual vector-meson dominance of electromagnetic processes. Furthermore, since, in view of (2.18), $\sigma(a)$ is not positive definite, our assumption in the zero-width approximation corresponds to taking something like $\delta'(a-a_0)$ for $\sigma(a)$.

With this motivation, we proceed to approximate the integral in (2.41) by

$$\int da\, \sigma(a) \ln a \simeq \int da\, \sigma(a)\left[\ln a_0 + (a-a_0)/a_0 + \cdots\right]$$
$$\simeq a_0^{-1} \int da\, \sigma(a)\, a,$$

where we have used (2.18). Thus we have

$$y \simeq w_2 a_0^{-1},$$

so that, by (2.43),

$$\sigma_\gamma(\infty) \simeq 4\pi^2\alpha w_2 a_0^{-1}.$$

Taking $a_0^{1/2}$ to be the ρ mass and using $\sigma_\gamma(\infty)\sim 120\ \mu b$,[14] this gives $w_2\sim 0.6$, in rough agreement with the experimental value ~ 0.3 in view of the uncertainties in our choice of a_0 and $\sigma_\gamma(\infty)$.

E. Discussion

Our derived result (2.20) is in agreement with experiment only if $w_2\neq 0$. Although, as we saw in Sec. II D, if $w_2\neq 0$, then a rough estimate of it is in reasonable agreement with experiment, it is nevertheless theoretically possible for w_2 to vanish.[28] We shall therefore discuss here consequences of $w_2=0$ and what further assumption is necessary to guarantee that $w_2\neq 0$.

All of equations above remain valid for $w_2=0$. In fact, our general result can be written as

$$\lim_{\rho\to\infty} F_2(\rho) = \lim_{-\kappa\to\infty} w_2(\kappa) = w_2, \tag{2.44}$$

and $w_2=0$ simply means that both limits vanish. Note that the vanishing of $w_2=w_2(\infty)$ does not contradict the coupling of the Pomeranchukon (for finite κ). Our assumption of Pomeranchukon dominance does preclude the vanishing of $w_2(\kappa)$ for (almost all) finite κ and, in particular, for $\kappa=0$ (so that $y\neq 0$). According to (2.37), we therefore must have $\sigma(a)\neq 0$.

Let us now determine the asymptotic behavior of our functions when $w_2=0$. More generally, we suppose that

$$\int da\, \sigma(a)\, a^i = 0 \quad \text{for} \quad i=0, 1, 2, \cdots, n-1$$

and

$$\int da\, \sigma(a)\, a^n \neq 0.$$

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[28] I thank Henry Abarbanel for emphasizing this to me.

Then from (2.14) and (2.36),

$$W_2 \underset{R'}{\longrightarrow} (1/2n\nu\kappa^{n-1})\int da\, \sigma(a) a^n$$

and

$$F_2(\rho) \underset{\rho\to\infty}{\longrightarrow} 0,$$

so that

$$w_2(\kappa) \underset{\kappa\to\infty}{\longrightarrow} \mathrm{const}\,\kappa^{1-n}.$$

Although we cannot have $\sigma(a)\equiv 0$, it is still possible for $F_2(\rho)$ to vanish identically. This happens when $\int da\, \bar{\sigma}_2(a,b)a\equiv 0$. It follows from (2.24) and (2.27) that $w_1=0\Rightarrow w_2=0$. The converse is not true, however, since $w_2=v_1\Rightarrow w_1=0$. In fact, in a ladder-diagram model,[29] it is explicitly found that $w_2\neq 0$ but $w_1=0$.

Since it is consistent with our previous assumptions for w_2 to vanish, another assumption is necessary in order to agree with experiment. Any one of the following will suffice:

(i) $\lim_{\rho\to\infty} F_2(\rho)\neq 0$.

(ii) $\lim_{\kappa\to\infty} w_2(\kappa)\neq 0$.

(iii) $\sigma(a)$ oscillates as little as possible.

(iv) $w_2(\kappa)$ vanishes as slowly as possible for $\kappa\to\infty$.

Although each of these is *a priori* weaker than the blatant assumption that $w_2\neq 0$, there is little theoretical basis for any of them. The situation will be clarified in Sec. III, where theoretical support for the nonvanishing of w_2 will be adduced.

III. LIGHT-CONE COMMUTATORS

A. W_i Limits

Configuration-space representation for the W_i can be obtained from (2.5) by using (2.2) and (2.3). Thus

$$W_i(\kappa,\nu)=\int d^4x\, \exp(iq\cdot x)\hat{W}_i(x^2, p\cdot x), \qquad (3.1)$$

where

$$\hat{W}_2=-\Box\hat{V}_2 \qquad (3.2)$$

and

$$\hat{W}_1=-\Box\hat{V}_1+(p\cdot\partial)^2\hat{V}_2. \qquad (3.3)$$

We see that the $\hat{W}_i$, like the $\hat{V}_i$, are antisymmetric under $x\to -x$. The representations (3.1) are the most convenient ones for studying and relating the A and R limits and so we shall use them first. We shall afterwards consider the V_i.

It is convenient to introduce new configuration-space variables according to the identification

$$x^\mu=(\sigma+\lambda, \mathbf{x}, \sigma-\lambda). \qquad (3.4)$$

Thus

$$2\sigma\equiv x_-\equiv x_0-x_3, \qquad 2\lambda\equiv x_+\equiv x_0+x_3, \qquad (3.5)$$

and

$$\mathbf{x}=(x^1,x^2). \qquad (3.6)$$

We shall also write

$$z=x^2=4\sigma\lambda-\mathbf{x}^2 \qquad (3.7)$$

and

$$d\mathbf{x}=dx^1dx^2. \qquad (3.8)$$

We fix for now the momentum transfer vector to be

$$q^\mu=(\nu,0,0,-(\nu^2-\kappa)^{1/2}). \qquad (3.9)$$

Thus

$$q\cdot x \underset{A}{\longrightarrow} 2\sigma\nu+(\sigma-\lambda)/2\rho \qquad (3.10)$$

and

$$q\cdot x \underset{R}{\longrightarrow} 2\sigma\nu-(\sigma-\lambda)\kappa/2\nu. \qquad (3.11)$$

With the choice (3.9), the exponential in (3.1) is independent of $\mathbf{x}$ and so we define

$$\begin{aligned} f_i(\sigma,\lambda) &= \mathcal{W}_i(\sigma\lambda, \sigma+\lambda) \\ &= \int d\mathbf{x}\hat{W}_i(4\sigma\lambda-\mathbf{x}^2, \sigma+\lambda) \\ &= \pi\int_0^{4\sigma\lambda} dz\, \hat{W}_i(z,\sigma+\lambda). \end{aligned} \qquad (3.12)$$

In obtaining the last equality, we have used the causality property of W_i:

$$\hat{W}_i(z,x_0)=0 \quad \text{for} \quad z<0. \qquad (3.13)$$

Equation (3.13) follows from (3.2) and (3.3) and the causality of the $\hat{V}_i$. This causality further implies that (3.12) has the form

$$f_i(\sigma,\lambda)=\theta(\sigma\lambda)\mathfrak{F}_i(\sigma\lambda,\sigma+\lambda). \qquad (3.14)$$

Using (3.12), the A and R limits of (3.1) become

$$W_i \underset{A}{\longrightarrow} 2\int d\sigma d\lambda\, \exp[2i\sigma\nu+i(\sigma-\lambda)/2\rho] f_i(\sigma,\lambda) \qquad (3.15)$$

and

$$W_i \underset{R}{\longrightarrow} 2\int d\sigma d\lambda\, \exp[2i\sigma\nu-i(\sigma-\lambda)\kappa/2\nu] f_i(\sigma,\lambda). \qquad (3.16)$$

Let us first consider the A limit of νW_2. It follows from the form of (3.15) and the non-negativity of $F_2(\rho)$ that the behavior of W_2 in the A limit is controlled by the behavior of $f_2(\sigma,\lambda)$ for $\sigma\to 0$. Furthermore, because of the oscillatory factor $\exp(-i\lambda/2\rho)$, only bounded values of λ are relevant in the A limit. Thus the behavior of $f_2(\sigma,\lambda)$ for $\sigma\to 0$ and λ bounded, i.e., for $\sigma\to 0$ and $\sigma\lambda\to 0$, determines the A limit of W_2. It therefore follows from the finiteness of $F_2(\rho)$ that

$$2f_2(\sigma,\lambda) \underset{\sigma\to 0;\sigma\lambda\to 0}{\longrightarrow} \hat{f}_2(\lambda)\theta(\sigma)+\text{L.S.}, \qquad (3.17)$$

i.e., that the leading possible singularity of f_2 for $\sigma\sim 0$ and bounded λ is proportional to $\theta(\sigma)$. Here L.S.

[29] G. Altarelli and H. R. Rubinstein, Phys. Rev. **187**, 2111 (1969).

stands for less singular terms in the specified limit. We emphasize that (3.17) exhibits the lending *possible* singularity so that perhaps $f_2(\lambda)\equiv 0$. Substitution of (3.17) into (3.15) gives (1.13) with

$$F_2(\rho)=\tfrac{1}{2}i\int d\lambda\, \exp(-i\lambda/2\rho)\hat{f}_2(\lambda). \qquad (3.18)$$

To obtain information about $\hat{f}_2(\lambda)$, we next consider the R limit (3.16) of W_2. The R limit for general κ is again related to the $\sigma\to 0$ behavior of $f_2(\sigma,\lambda)$. It is clear from (3.16), however, that in order to determine the R limit of W_2, we need to know $f_2(\sigma,\lambda)$ for large λ as well as for small σ. In fact, for arbitrary f_2, (3.16) is an essentially arbitrary function $F_2(\nu,\kappa/\nu)$ of ν and κ/ν and therefore of ν and κ.

We can nevertheless proceed because the exponentials in (3.16) limit the integration regions which contribute significantly in the R limit. Thus we expect the values of σ and λ such that $\sigma<\nu^{-1}$ and $\lambda<\nu/\kappa$ to dominate. *If* we take $-\kappa\gg 1$, i.e., consider the R' limit, then the relevant region is $\sigma\to 0$ and $\lambda\ll\sigma^{-1}$, i.e., $\sigma\to 0$ and $\sigma\lambda\to 0$. We can therefore relate the R' limit to the A limit. In view of (3.17), we obtain

$$W_2 \underset{R'}{\to} (i/2\nu)\int d\lambda\, \exp(i\lambda\kappa/2\nu)\hat{f}_2(\lambda). \qquad (3.19)$$

It now follows from the assumed behavior (1.20) that $\hat{f}_2(\lambda)$ is integrable and so we again get (2.34), where now

$$w_2=\tfrac{1}{2}i\int d\lambda\, \hat{f}_2(\lambda). \qquad (3.20)$$

It is again possible that $w_2=0$, even if $\hat{f}_2(\lambda)\not\equiv 0$. This just means that $w_2(\infty)=0$, in which case, to obtain a more specific large-κ behavior, further terms must be kept in (3.17). The point is simply that weaker σ singularities, with stronger large-λ behaviors, can also give the required Regge term ν^{-1}, but with a coefficient function $w_2(\kappa)$ which approaches zero suitably fast for $\kappa\to\infty$. In any case, we see from (3.18) that

$$F_2(\rho) \underset{\rho\to\infty}{\to} w_2, \qquad (3.21)$$

the same result as (2.20). The (experimentally observed) nonvanishing of w_2 is now seen to be the statement that the large-λ behavior of $\hat{W}_2(x^2,x_0)$ is as nice as possible.

We emphasize that the results (2.34) and (3.21) depend on the relevance of the specific behavior (3.17) to the R' limit. In general we can write

$$2f_2(\sigma,\lambda)=\hat{f}_2(\lambda)\theta(\sigma)+g_2(\sigma,\lambda), \qquad (3.22)$$

where $g_2(\sigma,\lambda)$ is less singular than $\theta(\sigma)$ for $\sigma\to 0$ *and* $\sigma\lambda\to 0$. In the limit $\sigma\to 0$ with λ unspecified, however, g_2 can become important. For the case in hand, this can, in fact, be explicitly seen to happen. The point is that, as a consequence of the constraint (1.12), we have

$$0=\nu W_2(0,\nu)=2\nu\int d\sigma d\lambda\, \exp(2i\sigma\nu)f_2(\sigma,\lambda). \qquad (3.23)$$

The behavior of f_2 for $\sigma\to 0$ controls the $\nu\to\infty$ behavior of (3.23), but, if $w_2\neq 0$, the contribution of the first term in (3.22) is $w_2\neq 0$. The second term must therefore satisfy

$$\int d\lambda\, g_2(\sigma,\lambda) \underset{\sigma\to 0}{\to} 2iw_2\theta(\sigma). \qquad (3.24)$$

This is not inconsistent with (3.17) and (3.22), because, owing to the absence of an oscillatory cutoff as in (3.16), large λ's can be important in (3.23) and (3.24) so that the limit (3.17) is not relevant. Thus the $\sigma\to 0$ limit cannot be taken inside of the integral in (3.23) or (3.24). One can construct quite elementary functions $g_2(\sigma,\lambda)$ with the above properties.

The same type of analysis can be used to study $W_1(\kappa,\nu)$. Starting again from (3.15), it follows from the finiteness of $F_1(\rho)$ in (1.14) that

$$2f_1(\sigma,\lambda) \underset{\sigma\to 0;\sigma\lambda\to 0}{\to} \hat{f}_1(\lambda)\delta(\sigma)+\text{L.S.} \qquad (3.25)$$

for some (possibly vanishing) function $\hat{f}_1(\lambda)$. Then (1.14) is satisfied with

$$F_1(\rho)=\int d\lambda\, \exp(-i\lambda/2\rho)\hat{f}_1(\lambda), \qquad (3.26)$$

and (3.16) gives

$$W_1 \underset{R'}{\to} \int d\lambda\, \exp(i\lambda\kappa/2\nu)\hat{f}_1(\lambda). \qquad (3.27)$$

In order to satisfy (1.21), we must have

$$\hat{f}_1(\lambda)=f_1\epsilon(\lambda)+\text{L.S.} \qquad (3.28)$$

for some (possibly vanishing) constant f_1. We mean by (3.28) that for large λ, $f_1(\lambda)$ behaves like the sum of $f_1\epsilon(\lambda)$ and a function $g_1(\lambda)$ whose integral

$$\int_{-\Lambda}^{\Lambda} d\lambda\, g(\lambda)$$

diverges less strongly for $\Lambda\to\infty$ than const$\times\Lambda$.[30] We now again obtain the result (2.35) with

$$w_1=4if_1, \qquad (3.29)$$

and the result (2.26).

B. Light-Cone Analysis

We saw in Sec. III A that the limit $\sigma\to 0$, $\sigma\lambda\to 0$ of the $W_i(x^2,x_0)$ determined the A and R' behavior of the $W_i(\kappa,\nu)$. Let us study this limit in more detail. Since $x^2=4\sigma\lambda-\mathbf{x}^2$, and since the W_i vanish for $x^2<0$ by causality, the above limit is precisely the limit $x^2\to 0$. Thus it is the behavior of the W_i on the light cone $x^2=0$ which determines the A and R' limits of the W_i.[31] We shall therefore be able to determine from the results of Sec. III A what is the nature of the $\hat{W}_i$ near the light cone (LC).

We consider first W_2 and return to Eq. (3.17). The θ function in f_2, which is the strongest singularity allowed by the finiteness of F_2, arises naturally from

[30] This is a consequence of generalized Fourier-transform theory. Another way of stating (3.28) is that $\hat{f}_1(+\infty)-\hat{f}_1(-\infty)=2f_1$.

[31] A relation between the A limit and the light cone has also been noted by B. L. Joffe, Phys. Letters **30B**, 123 (1969).

the causality-dictated θ function exhibited in (3.14). Using the fact that

$$\theta(\sigma\lambda)=\theta(\sigma)\epsilon(\lambda)+\text{L.S.}, \qquad (3.30)$$

(3.17) is seen to simply specify the limit of $\mathfrak{F}_2(\alpha,\beta)$ for $\alpha\to 0$. We easily find

$$\mathfrak{F}_2(0,\lambda)=\tfrac{1}{2}\epsilon(\lambda)\hat{f}_2(\lambda). \qquad (3.31)$$

It therefore follows from (3.12) that

$$\pi\hat{W}_2(z,\lambda)\sim\delta(z)\mathfrak{F}_2(0,\lambda) \qquad (3.32)$$

for $z\sim 0$. We mean by (3.32) that the leading singularity of $\pi\hat{W}_2(z,\lambda)$ for $z\to 0$ and λ fixed in as indicated. We do not specify lesser singularities such as $z^{-1+\epsilon}$ or $\theta(z)$. Lesser singularities would not contribute to $\mathfrak{F}_2(0,\lambda)$ as defined by (3.12) and (3.14).

We explicitly see from (3.32) that it is the leading singularity of $\hat{W}_2$ on the LC which was determined in Sec. III A and which controls the A and R' limits of W_2. We also learned in Sec. III A that the coefficient of this leading singularity, $\mathfrak{F}_2(0,\lambda)$, is an integrable function of λ, consequences of which were the derived behaviors (2.20) and (2.34). In the LC language, the finiteness of F_2 implies that $\pi\hat{W}_2\sim\delta(z)L_2(\lambda)$ for $z\sim 0$, and then the desired Pomeranchuk behavior requires that L_2 be integrable so that consequently (2.20) and (2.34) hold.

In general we can write

$$\pi W_2(x^2,p\cdot x)=\delta(x^2)L_2(p\cdot x)+R_2(x^2,p\cdot x), \qquad (3.33)$$

where $R_2(z,\lambda)$ is less singular than $\delta(z)$ for $z\to 0$ with fixed λ. The corresponding decomposition of W_2 in (3.12) is

$$\mathfrak{W}_2(\sigma\lambda,\sigma+\lambda)=\theta(\sigma\lambda)[\mathfrak{F}_2(\sigma\lambda,\sigma+\lambda)+\mathfrak{G}_2(\sigma\lambda,\sigma+\lambda)], \qquad (3.34)$$

where, for fixed λ,

$$\mathfrak{F}_2(\alpha,\lambda)\underset{\alpha\to 0}{\longrightarrow}\tfrac{1}{2}\epsilon(\lambda)\hat{f}_2(\lambda) \qquad (3.35)$$

and

$$\mathfrak{G}_2(\alpha,\lambda)\underset{\alpha\to 0}{\longrightarrow}0. \qquad (3.36)$$

The behavior (3.36) cannot be correct for unbounded λ if $w_2\neq 0$, however, because of the constraint (1.12), which requires that

$$0=\nu W_2(0,\nu)\underset{R}{\longrightarrow}2\nu\int d\sigma d\lambda\,\exp(2i\sigma\nu)\theta(\sigma\lambda)[\mathfrak{F}_2+\mathfrak{G}_2]. \qquad (3.37)$$

We can determine the behavior of $\hat{W}_1$ on the LC in the same way. It follows from (3.25) and the first equality of (3.12) that

$$\mathfrak{W}_1(\sigma\lambda,\sigma+\lambda)\underset{\sigma\to 0;\sigma\lambda\to 0}{\longrightarrow}\tfrac{1}{2}\delta(\sigma\lambda)\,|\lambda|\,\hat{f}_1(\lambda). \qquad (3.38)$$

The final equality of (3.12) then gives

$$\pi\hat{W}_1(z,\lambda)\sim\delta'(z)2\,|\lambda|\,\hat{f}_1(\lambda) \qquad (3.39)$$

for $z\sim 0$. The leading LC singularity of $\hat{W}_1$ is thus specified to be $\delta'(z)$. The large-λ behavior of the coefficient of this singularity was determined in Sec. III A to be given by (3.28).

C. V_i Behavior

From the above results we can determine the light-cone behavior of the $\hat{V}_i(x^2,p\cdot x)$ by inverting Eqs. (3.2) and (3.3). It is most simple, however, to derive this behavior directly from (2.5). The assumed momentum-space behavior follows from (1.13)–(1.15) and (2.2) and (2.3). We use the same variables as in Sec. III B and therefore define

$$e_i(\sigma,\lambda)\equiv\int d\mathbf{x}\,\hat{V}_i(4\sigma\lambda-\mathbf{x}^2,\sigma+\lambda)$$
$$=\pi\int_0^{4\sigma\lambda}dz\,\hat{V}_i(z,\sigma+\lambda), \qquad (3.40)$$

where we have used the causal properties of the $\hat{V}_i$. Causality further requires the e_i to have the form

$$e_i(\sigma,\lambda)=\theta(\sigma\lambda)\mathcal{E}_i(\sigma\lambda,\sigma+\lambda). \qquad (3.41)$$

The existence of the A limits requires the leading possible singularities of the e_i to have the forms

$$2e_2(\sigma,\lambda)\underset{\sigma\to 0;\sigma\lambda\to 0}{\longrightarrow}\sigma\theta(\sigma)\hat{e}_2(\lambda)+\text{L.S.} \qquad (3.42)$$

and

$$2e_1(\sigma,\lambda)\underset{\sigma\to 0;\sigma\lambda\to 0}{\longrightarrow}\theta(\sigma)\hat{e}_1(\lambda)+\text{L.S.} \qquad (3.43)$$

for some (possibly vanishing) functions $\hat{e}_i(\lambda)$. This behavior then gives the representations

$$F_2(\rho)=(1/4\rho)\int d\lambda\,\exp(-i\lambda/2\rho)\hat{e}_2(\lambda) \qquad (3.44)$$

and

$$F_1(\rho)=(-i/4\rho)\int d\lambda\,\exp(-i\lambda/2\rho)\hat{e}_1(\lambda)+\rho F_2(\rho). \qquad (3.45)$$

The assumed behavior in the R limit specifies the large-λ behavior of the $\hat{e}_i(\lambda)$ to be

$$\hat{e}_2(\lambda)=e_2\theta(\lambda)+\text{L.S.}, \qquad (3.46)$$

$$\hat{e}_1(\lambda)=e_1\,|\lambda|+\text{L.S.} \qquad (3.47)$$

for some (possibly vanishing) constants e_i.[32] These imply again the behavior (2.34) and (2.35) with

$$w_2=-\tfrac{1}{2}ie_2 \qquad (3.48)$$

and

$$w_1=ie_1+w_2. \qquad (3.49)$$

Equations (3.46) and (3.47) determine the behavior of the $\mathcal{E}_i(\alpha,\beta)$ in (3.41) for $\alpha\to 0$. We obtain

$$\mathcal{E}_2(\alpha,\lambda)\underset{\alpha\to 0}{\longrightarrow}\tfrac{1}{2}\alpha\epsilon(\lambda)\lambda^{-1}\hat{e}_2(\lambda) \qquad (3.50)$$

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[32] Note that (3.42) cannot hold for unbounded λ since otherwise, according to (3.46), $Y_2(\nu)$, defined by (2.39), would not exist.

and

$$\mathcal{E}_1(\alpha,\lambda) \underset{\alpha\to 0}{\longrightarrow} \tfrac{1}{2}\epsilon(\lambda)\hat{e}_1(\lambda). \tag{3.51}$$

The LC behaviors of the $\hat{V}_i$ then follow from (3.40) to be

$$\pi\hat{V}_2(z,\lambda) \sim \tfrac{1}{8}\theta(z)\epsilon(\lambda)\lambda^{-1}\hat{e}_2(\lambda) \tag{3.52}$$

and

$$\pi\hat{V}_1(z,\lambda) \sim \tfrac{1}{2}\delta(z)\epsilon(\lambda)\hat{e}_1(\lambda) \tag{3.53}$$

for $z\sim 0$. Thus the leading singularities of the $\hat{V}_i$ on the LC have been determined.

The consistency of the above results with those of Secs. III A and III B is easily established. Consider first the relations (3.48) and (3.49). These are consistent with the representations (3.20) and (3.29) provided we make the identifications

$$e_2 = -\int d\lambda\, \hat{f}_2(\lambda) \tag{3.54}$$

and

$$e_1 = f_1 + \tfrac{1}{4} i w_2. \tag{3.55}$$

The equivalence of (3.44) and (3.45) with (3.18) and (3.26) requires the local identifications

$$\hat{f}_2(\lambda) = -\hat{e}_2'(\lambda) \tag{3.56}$$

and

$$\hat{f}_1(\lambda) = -\hat{e}_1'(\lambda) + \tfrac{1}{4}\hat{e}_2(\lambda). \tag{3.57}$$

These are consistent with (3.54) and (3.55) and the behaviors (3.46) and (3.47). Finally, the relations (3.2) and (3.3) can be explicitly seen to be consistent with the leading LC singularities (3.32) and (3.39) of W_2 and W_1 and those (3.52) and (3.53) of $\hat{V}_2$ and V_1 and also with the large-λ behavior of the coefficient of these leading singularities.

D. Connection with Integral Representations

In this subsection we establish the equivalence between the results of Sec. II and the preceding results of Sec. III. We thus assume now the validity of the DGS representations (2.7) and the properties of the spectral functions determined in Sec. II. We shall show that these imply a LC behavior for the $\hat{V}_i$ in agreement with that given in Sec. III C. We shall work directly with the configuration-space representations (2.8) and use the fact that

$$\Delta(x;a+b^2) \to (1/2\pi)\epsilon(x_0)[\delta(x^2) - \tfrac{1}{4}(a+b^2)\theta(x^2)] \tag{3.58}$$

for $x^2\to 0$.

Substituting (3.58) in (2.8) with $i=2$, and using (2.12), we see that

$$\pi\hat{V}_2(x^2,\lambda) \to \theta(x^2)K_2(\lambda) \tag{3.59}$$

for $x^2\to 0$, where

$$K_2(\lambda) = [i\epsilon(\lambda)/16\pi]\int da\,db\, \sigma_2(a,b)\,a\exp(-ib\lambda). \tag{3.60}$$

Using (2.17) and (2.19), we find further that

$$K_2(\lambda) \underset{\lambda\to\infty}{\longrightarrow} iw_2/8\lambda. \tag{3.61}$$

These results are in complete agreement with (3.52), (3.46), and (3.48).

Similarly, from (3.58) in (2.8) with $i=1$, we find

$$\pi\hat{V}_1(x^2,\lambda) \to \delta(x^2)K_1(\lambda) \tag{3.62}$$

for $x^2\to 0$, where

$$K_1(\lambda) = [-i\epsilon(\lambda)/4\pi]\int da\,db\, \sigma_1(a,b)\exp(-ib\lambda). \tag{3.63}$$

In view of (2.22), (2.24), and (2.25), Eq. (3.63) satisfies

$$K_1(\lambda) \underset{\lambda\to\infty}{\longrightarrow} \tfrac{1}{4} i v_1 \lambda = \tfrac{1}{8} i (w_2 - w_1)\lambda. \tag{3.64}$$

We again obtain perfect agreement with (3.53), (3.47), and (3.49).

Thus the representations (2.8) with the spectral functions satisfying (2.12), (2.17), and (2.22) imply the LC behaviors (3.52) and (3.53). It is easily seen that, conversely, the behaviors (3.52) and (3.53), with (3.46) and (3.47), imply that the spectral functions do satisfy (2.12), (2.17), and (2.18). Given the existence of the DGS representations with nice spectral functions, the equivalence of the DGS and LC methods is therefore established. We feel that the LC approach is preferable, however, both because it requires fewer assumptions and, as we shall see below, because it suggests an appealing explanation of the nonvanishing of w_2.

All of the considerations of Secs. III A–III D can be carried out for an arbitrary Regge trajectory intercept $\alpha=\alpha(0)$. This is illustrated in the Appendix for a scalar amplitude.

E. Light-Cone Commutation Relations

By using the results (3.59) and (3.62) in (2.4), we can determine the behavior of each component of $\langle p|[J_\mu(x),J_\nu(0)]|p\rangle$ in the neighborhood of the LC. It is convenient for this purpose to work with the components $A_\pm$, $\mathbf{A}$ of a general four-vector A_μ defined by

$$A_\pm = A_0 \pm A_3, \qquad \mathbf{A} = (A_1, A_2). \tag{3.65}$$

Indices are then raised by the metric tensor $g^{\mu\nu}$ with nonvanishing components $g^{+-}=g^{-+}=-\tfrac{1}{2}g^{11}=-\tfrac{1}{2}g^{22}=\tfrac{1}{2}$. Thus $x^2=x_+x_- - \mathbf{x}^2 = x_\mu x_\nu g^{\mu\nu} = x_\mu x^\mu$.

As an important example, we consider the commutator $[J_-(x),J_-(0)]$. We find from (2.4) that

$$\hat{W}_{--}(x,p) = (-\Box + 2\partial_0\partial_-)\hat{V}_2 - \partial_-^2\hat{V}_1. \tag{3.66}$$

This can be written as

$$\hat{W}_{--}(x,p) = Y_0(x^2,x_0) + \partial_- Y_1(x^2,x_0) + \partial_-^2 Y_2(x^2,x_0), \tag{3.67}$$

where

$$Y_0 = -\Box\hat{V}_2 = \hat{W}_2, \tag{3.68}$$

$$Y_1 = 2\partial_0\hat{V}_2, \tag{3.69}$$

and

$$Y_2 = -\hat{V}_1. \tag{3.70}$$

We define the LC limits by

$$Y_i(x^2, x_0) \sim \tilde{Y}_i(x^2, x_0), \qquad i=0, 1, 2 \tag{3.71}$$

for $x^2 \sim 0$, and we see from (3.33), (3.59), and (3.62) that

$$\pi \tilde{Y}_0 = \delta(x^2) L_2(x_0), \tag{3.72}$$

$$\pi \tilde{Y}_1 = 4x_0 \delta(x^2) K_2(x_0) + 2\theta(x^2) K'_2(x_0), \tag{3.73}$$

$$\pi \tilde{Y}_2 = -\delta(x^2) K_1(x_0). \tag{3.74}$$

Thus we can write

$$\pi \hat{W}_{--}(x, p) \sim \delta(x^2) L_2(x_0) + \partial_- M(x^2, x_0) \tag{3.75}$$

for $x^2 \sim 0$. For general functions A and B we let $A \leftrightarrow B$ mean that $A \sim B$ for $x^2 \sim 0$ apart from terms of the form $\partial_- C$. Then (3.75) gives

$$\pi \hat{W}_{--}(x, p) \leftrightarrow \delta(x^2) L_2(x_0). \tag{3.76}$$

The Fourier transform

$$W_{--}(q, p) = \int d^4x \exp(iq\cdot x) \hat{W}_{--}(x, p) \tag{3.77}$$

s, according to (2.1), given by

$$W_{--}(q, p) = (\kappa - 2\nu q_-) V_2 + q_-^2 V_1. \tag{3.78}$$

We learn from (3.77) and (3.67)–(3.74) that

$$W_{--} \underset{A}{\rightarrow} (\kappa^2/4\nu^2) V_1. \tag{3.79}$$

This also follows from (3.78).

Similarly, we can easily compute the other components of $\hat{W}_{\mu\nu}$. For the given q [Eq. (3.9)], we find in the A limit that W_{00}, W_{03}, and W_{33} are proportional to W_{--}, $W_{12} = W_{13} = W_{01} = 0$, and $W_{11} = -W_1$.

It is convenient at this point to consider, in addition to (3.9), the vector

$$k^\mu = (\nu, \sqrt{(-\kappa)}, 0, -\nu). \tag{3.80}$$

k^μ is obtained from q^μ by a spatial rotation and satisfies

$$k_- \equiv 0, \tag{3.81}$$

$k^2 = \kappa$, and $k\cdot p = \nu$. Replacing q by k in (1.6), and using the fact that $k_- = g_{--} = 0$, we obtain

$$W_2(\kappa, \nu) = \int d^4x \exp(ik\cdot x) Y_0(x^2, x_0), \tag{3.82}$$

where we have used (3.67) in the integrand. We can now use rotational invariance to replace k by q in (3.82):

$$W_2(\kappa, \nu) = \int d^4x \exp(iq\cdot x) Y_0(x^2, x_0). \tag{3.83}$$

In view of (3.68), this is simply Eq. (3.1) for $i=2$. We can take the A limit as before with (3.83) to get

$$W_2(\kappa, \nu) \underset{A}{\rightarrow} \int d^4x \exp(iq\cdot x) \tilde{Y}_0(x^2, x_0), \tag{3.84}$$

and then use rotational invariance again to obtain

$$W_2(\kappa, \nu) \underset{A}{\rightarrow} \int d^4x \exp(ik\cdot x) \tilde{Y}_0(x^2, x_0). \tag{3.85}$$

We thus learn that the A limit corresponds to the LC behavior of the integrand in (3.82) as well as in (3.83). Using (3.72), the integral in (3.85) can easily be done and, of course, gives (1.13).

F. Connection with Current Algebra

Until now our procedure has been to derive consequences of the assumed existence of the A limit and nature of the R limit. We have obtained the desired result $F_2(\infty) = w_2 =$ const except for the possibility that $w_2 = 0$. In the language of LC commutators, this possibility means that the function $L_2(\lambda) = \mathfrak{F}_2(0, \lambda) = \frac{1}{2}\epsilon(\lambda) f_2(\lambda)$, given by (3.76) or (3.32), satisfies $w_2 = \frac{1}{2} i \int d\lambda\, f_2(\lambda) = 0$. Our purpose now will be to provide an argument which may explain why $w_2 \neq 0$. We shall depart from our past procedure, forget what we have learned, and try to guess what the LC commutator might be. Our guess will be based on a generalization of the LC implication of a widely accepted current-algebra sum rule suggested by an analogy with the weak-interaction universality principle. The guess will be seen to imply a LC behavior of the form (3.76), with an $L_2(\lambda)$ giving $w_2 \neq 0$ and, furthermore, giving a numerical value for w_2 in good agreement with experiment.

We begin by generalizing the definitions (1.5) and (1.6) by using the $SU(3)$ vector currents $J_\mu{}^a$, $a = 1, \ldots, 8$, or axial-vector currents $J_{5\mu}{}^a$. Then

$$\begin{aligned} T_{\mu\nu}{}^{ab} &= i\int dx \exp(iq\cdot x)\theta(x_0) \\ &\quad \times \langle p \,|\, [J_\mu{}^a(x), J_\nu{}^b(0)] \,|\, p\rangle + \text{poly.} \\ &= p_\mu p_\nu T_2{}^{ab}(\kappa, \nu) + \cdots \end{aligned} \tag{3.86}$$

and

$$\begin{aligned} W_{\mu\nu}{}^{ab} &= (1/\pi)\, \mathrm{Im} T_{\mu\nu}{}^{ab} \\ &= (1/2\pi) \int dx \exp(iq\cdot x) \\ &\qquad \times \langle p \,|\, [J_\mu{}^a(x), J_\nu{}^b(0)] \,|\, p\rangle \\ &= p_\mu p_\nu W_2{}^{ab}(\kappa, \nu) + \cdots. \end{aligned} \tag{3.87}$$

The Gell-Mann[33] equal-time $[SU(3) \otimes SU(3)]_{\gamma_5}$ commutation relations[34]

$$\begin{aligned} [J_0{}^a(x), J_0{}^b(0)]\delta(x_0) &= if^{abc} J_0{}^c \delta(x), \\ [J_0{}^a(x), J_{50}{}^b(0)]\delta(x_0) &= if^{abc} J_{50}{}^c \delta(x), \\ [J_{50}{}^a(x), J_{50}{}^b(0)]\delta(x_0) &= if^{abc} J_0{}^c \delta(x), \end{aligned} \tag{3.88}$$

together with an assumption about interchanging an infinite momentum limit with an integral or about the validity of some unsubtracted dispersion relations, yield the celebrated Fubini[35]–Dashen–Gell-Mann[36] sum

[33] M. Gell-Mann, Physics **1**, 63 (1964).

[34] We use the notation of Dashen and Gell-Mann (Ref. 36) and denote a subgroup of $U(12)$ by using as subscripts the Dirac matrices with which the generators commute.

[35] S. Fubini, Nuovo Cimento **43A**, 475 (1966).

[36] R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966*, edited by A. Perlmutter, J. Wojtaszek, E. Sudarshan, and B. Kursunoglu (Freeman, San Francisco, 1966).

rule

$$\int d\nu\, \tilde{W}_2^{ab}(\kappa,\nu) = -f^{abc}F^c, \quad (3.89)$$

where

$$iF^c \equiv \langle p \mid J_0^c \mid p \rangle \quad (3.90)$$

and $\tilde{W}^{ab} = \frac{1}{2}(W^{ab} - W^{ba})$.

Because of the extra assumption needed to obtain (3.89) from (3.88), Eq. (3.89) is essentially[37] equivalent[38] not to the equal-time commutators (3.88) but to the LC commutator[39]

$$\tfrac{1}{2}\int dx_+ [J_-^a(x), J_-^b(0)]\delta(x_-) = if^{abc}J_-^c\delta(\mathbf{x})\delta(x_-). \quad (3.91)$$

Thus, since the experimental support[40] for (3.89) is really support for (3.91), we shall accept (3.91) in the following. We write (3.91) in the more suggestive form

$$[J_-^a(x), J_-^b(0)] \sim 2\pi^{-1} i\delta(x^2)[f^{abc}J_-^c f(2x_0) + R^{ab}(x)] \quad (3.92)$$

for $x^2 \sim 0$, where

$$\tfrac{1}{2}\int dx_+ f(x_+) = 1 \quad (3.93)$$

and

$$\int dx_+ R^{ab}(x)\delta(x_-) = 0. \quad (3.94)$$

Equation (3.91) is clearly equivalent to (3.92)–(3.94). Consequences of (3.92) are the symmetry properties

$$f(-x_+) = f(x_+) \quad (3.95)$$

and

$$R^{ab}(-x) = -R^{ba}(x). \quad (3.96)$$

We now make a guess about the form of $R^{ab}(x)$. We shall explore some consequences of the assumption that the form of R^{ab} is such as to give (3.92) a universal structure maximally symmetric in f- and d-type octet couplings. Thus we propose that[41]

$$[J_-^a(x), J_-^b(0)] \leftrightarrow 2\pi^{-1} i\delta(x^2) f(2x_0) \times [f^{abc}J_-^c - d^{abc}S^c\epsilon(x_0)]. \quad (3.97)$$

Here S^c is the scalar current density given by $\frac{1}{2}\bar{\psi}\lambda^a\psi$ in the quark model. The quark model $[SU(3)\otimes SU(3)]_\beta$ commutation relations

$$[J_0^a(x), J_0^b]\delta(x_0) = if^{abc}J_0^c\delta(x),$$
$$[J_0^a(x), S^b]\delta(x_0) = if^{abc}S^c\delta(x), \quad (3.98)$$
$$[S^a(x), S^b]\delta(x_0) = if^{abc}J_0^c\delta(x)$$

establish the scale of S relative to J_0,[42] and make precise the universality implicit in (3.97) in exactly the same way[33] that the $[SU(3)\otimes SU(3)]_{\gamma_5}$ commutation relations (3.88) establish the scale of J_{50} relations to J_0 and make precise weak-interaction universality. We shall see that (3.97) gives results usually taken to imply a composite structure[8] for the nucleon. A more precise discussion of the operator nature of (3.97) will be given elsewhere.

In addition to the weak $V-A$ universality analogy, some motivation for (3.97) comes from the following formalisms which have been applied in other contexts: (a) The Cabibbo-Horwitz-Ne'eman[43] proposal applied to vector-meson–nucleon scattering gives an operator structure similar to that of (3.97) in the $[U(3)\otimes U(3)]_\beta$ symmetry limit if one takes all Regge trajectories to cross $j=1$ at $t=0$. (b) Okubo[44] has suggested that the pseudoscalar-meson source commutator for $x^2=0$ involves (unspecified) unitary singlets and octets. (c) Given the presence of the first term in (3.97), its relation to nonsense right-signature $j=1$ fixed poles,[16,17] and its analogy with odd-signature vector-meson Regge-pole exchange, the presence of the second term can be inferred from a (very) generalized interpretation of exchange degeneracy.[45]

(a)–(c) are concerned with the on-shell Regge limit, whereas, as we have seen, the LC should only describe the Regge limit far off the mass shell. They also all involve only Regge poles, whereas (3.97) involves (perhaps only) fixed poles. Exchange degeneracy, for example, essentially equates the odd-signature vector-meson Regge-pole residues and trajectories with those of the even-signature tensor-meson Regge poles. We assume the same relation between the right-signature fixed poles and the wrong-signature singularities generalized by $SU(3)$ from the combined singular residue[19,20] and Pomeranchuk singularity (or any other mechanisms) which give rise to the forward coupling of the Pomeranchukon to photons.

Now the relations (3.98) do not fix the scale of S^0, and so it is not *a priori* clear whether or not we should include S^0 in (3.97). In Ref. 9 we did not include S^0. Evidence from models to be discussed below, as well as positivity requirements,[46] suggests that S^0 should be included, however, and so, in this paper, we take as usual $d^{ab0} = (\sqrt{\tfrac{2}{3}})\delta^{ab}$.

In order to project out T_2 and W_2 from (3.86) and (3.87), we substitute k^μ for q^μ and use T_{--} and W_{--}. We obtain

$$T_2^{ab} = i\int dx\, \exp(ik\cdot x)\theta(x_0)\langle p \mid [J_-^a(x), J_-^b(0)] \mid p\rangle \quad (3.99)$$

---

[37] An additional assumption (e.g., Regge theory) concerning the high-energy behavior of absorptive parts is required.

[38] All operator relations in this paper are written in a form valid only for forward rest matrix elements.

[39] H. Leutwyler, Acta Phys. Austriaca Suppl. **V**, 320 (1968); J. Jersák and J. Stern, Nuovo Cimento **59A**, 315 (1969).

[40] This consists mainly of the successes of the Adler-Weisberger and Cabibbo-Radicati sum rules. Thus (3.89) is only verified for small κ.

[41] We recall that "↔" means equality of leading light-cone singularities apart from total derivatives with respect to x_-. Such total derivatives must satisfy (3.94). We can allow the J and S terms to have different functions $f(x_+)$, as long as the normalization condition (3.93) is satisfied by each.

[42] In view of Ref. 38, J_- in (3.97) is equivalent to J_0.

[43] N. Cabibbo, L. Horwitz, and Y. Ne'eman, Phys. Letters **22**, 336 (1966).

[44] S. Okubo, Physics **3**, 165 (1967).

[45] R. Arnold, Phys. Rev. Letters **14**, 657 (1965); A. Ahmadzadeh and C. H. Chan, Phys. Letters **22**, 692 (1966).

[46] If we assume (3.97) is valid with the same $f(2x_0)$ for all matrix elements and ignore $SU(3)$ symmetry breaking, then negative total cross sections can result if S^0 is not included. I thank A. H. Mueller for emphasizing this to me.

and

$$W_2^{ab} = (1/2\pi)\int dx \exp(ik\cdot x)\langle p \mid [J_-^a(x), J_-^b(0) \mid p\rangle. \tag{3.100}$$

Because $k_- = 0$, the A and R' limits of (3.99) and (3.100) are determined by (3.97). The proton expectation value of (3.97) is

$$\langle p \mid [J_-^a(x), J_-^b(0)] \mid p\rangle \leftrightarrow 2\pi^{-1} i\delta(x^2) f(2x_0) \times [if^{abc}F^c - d^{abc}D^c\epsilon(x_0)], \tag{3.101}$$

where we have defined

$$D^c = \langle p \mid S^c \mid p\rangle. \tag{3.102}$$

Using the methods of Secs. III A and III B, and the result (3.85), we find from (3.100) and (3.101) that in the A limit,

$$\nu W_2^{ab}(\kappa, \nu) \underset{A}{\rightarrow} F_2^{ab}(\rho), \tag{3.103}$$

where

$$F_2^{ab}(\rho) = (1/2\pi)\int dx_+ \exp(-ix_+/4\rho) f(x_+) \times [d^{abc}D^c - i\epsilon(x_+) f^{abc}F^c]. \tag{3.104}$$

Using (3.93), we find further that

$$F_2^{ab}(\rho) \underset{\rho\to\infty}{\rightarrow} \pi^{-1} d^{abc}D^c \equiv w_2^{ab}. \tag{3.105}$$

Thus our proposal (3.101) implies both the scaling property (3.103) *and* the constant asymptotic behavior (3.105). In view of (3.76), this is hardly surprising. The new point here is that part of our universality assumption implies the general nonvanishing of (3.105). Thus the relation (3.91) implies the normalization condition (3.93) on $f(x_+)$ and our universality proposal makes the same $f(x_+)$ relevant in (3.105). In the same way, the universal $\delta(x^2)$ in (3.97) was responsible for the existence of the scaling limit (3.103).

Using (3.101), the R' limits of (3.99) and (3.100) are seen to be

$$T_2^{ab} \underset{R'}{\rightarrow} \nu^{-1}(f^{abc}F^c + id^{abc}D^c) \tag{3.106}$$

and

$$W_2^{ab} \underset{R'}{\rightarrow} (\pi\nu)^{-1} d^{abc}D^c. \tag{3.107}$$

The first term in (3.106) corresponds to the fixed-pole asymptotic behavior well known to be equivalent to (3.89). The second term, however, or, equivalently, Eq. (3.107), is not consistent with the *usual* Regge picture which says that only the Pomeranchuk contribution (which here corresponds to D^0 and perhaps part of D^8) goes as $1/\nu$.[47] Assuming the usual Regge *trajectory* behavior, (3.107) requires, for example, additonal fixed nonsense wrong-signature (double) poles.[48] Alternatively, our proposal can be easily altered to be consistent with the usual Regge picture, but then universality, as we have formulated it, would be lost. In any case, according to (3.105) and (3.107), (3.97) is seen to incorporate the suggestion[4–6] that the leading Regge contribution continues to dominate in the A limit.

We have not yet exploited the full content of the universality in (3.97). As we mentioned, we expect the ET commutation relations (3.98) to fix the scale of S^a and thus provide a *numerical* value for (3.105). We shall accomplish this in Sec. III G within the context of a specific model.

G. Equal-Time Behavior

The relation (3.97) cannot be used to calculate the ET commutator $[J_-^a(x), J_-^b(0)]\delta(x_0)$ because the terms of the form $\partial_- M(x)$ are not specified. The function $f(2x_0)$ can, of course, be chosen so that the first term in (3.97) contributes to the quark-model ET commutation relation

$$[J_-^a(x), J_-^b(0)]\delta(x_0) = if^{abc}J_-^c\delta(x). \tag{3.108}$$

The second term in (3.97) does not contribute to this commutator.[49] Equation (3.97) gives even less information about $\delta^3(\mathbf{x})$ terms in the ET commutator $[\dot{J}_-^a(x), J_-^b(0)]\delta(x_0)$ because, in addition to the unspecified $\partial_- M(x)$ terms, the unspecified nonleading LC singularities can contribute. The second term in (3.97) does, however, contribute to this commutator. We easily find

$$\langle p \mid [\dot{J}_-^a(0, \mathbf{x}), J_-^b(0)] \mid p\rangle = 4i\delta(\mathbf{x}) f(0) d^{abc}D^c + H^{ab}, \tag{3.109}$$

where H^{ab} represents the unknown contributions of the unspecified terms.

A more precise connection between LC and ET commutators can be obtained by use of the old Bjorken[50] limit $q_0 \to \infty$ for fixed $\mathbf{q}$ and p. One can write, in general,

$$\int d^4x\, \delta(x_0)\langle p \mid [\dot{J}_i^a(x), J_j^b(0)] \mid p\rangle = iE_2^{ab}(p_0)p_ip_j + E_1^{ab}(p_0)\delta_{ij} + E_3^{ab}(p_0)\epsilon_{ijk}p^k. \tag{3.110}$$

Assuming scaling, Regge asymptotics, and the validity of the Bjorken limit, one obtains the generalized Callan-Gross-type[27] relation

$$2\int_0^2 d\omega\, F_2^{ab}(\omega^{-1}) = E_2^{ab}(\infty), \tag{3.111}$$

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[47] The Harari (Ref. 15) mass-shift analysis remains, however, essentially unchanged.

[48] Double poles may not be unreasonable since there exist two independent mechanisms [that of Refs. 19 and 20 and that of S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967)] for producing fixed single poles. If, in addition, the Pomeranchukon really is uncoupled, then (3.106) involves only fixed poles.

[49] We define ET limits by smearing with symmetric testing functions $k_n(x_0)$ converging to $\delta(x_0)$.

[50] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

where $\omega\equiv-\kappa/\nu=\rho^{-1}$. On the other hand, it follows from (3.104) that [assuming that (3.104) has the correct support]

$$2\int_0^2 d\omega\, F_2{}^{ab}(\omega^{-1})=4f(0)\,d^{abc}D^c. \qquad (3.112)$$

Thus we learn that

$$E_2{}^{ab}(\infty)=4f(0)\,d^{abc}D^c; \qquad (3.113)$$

i.e., the infinite momentum limit of the $[\dot{J}_i{}^a, J_j{}^b]$ ET commutator determines the $[J_-{}^a, J_-{}^b]$ LC commutator at $x_0=0$, and conversely. This result is obviously generally valid and does not depend on our specific assumption (3.101). Comparison with (3.109) gives the further result that the infinite momentum limit of the $[\dot{J}_i{}^a, J_j{}^b]$ ET commutator is that part of the $[\dot{J}_-{}^a, J_-{}^b]$ ET commutator coming from the $[J_-{}^a, J_-{}^b]$ LC commutator.

These results place strong restrictions on models which can accommodate (3.101). Thus (3.101) can only be valid in models for which the ET commutator $[\dot{J}_i{}^a, J_j{}^b]$ satisfies $E_2{}^{ab}(\infty)\propto d^{abc}D^c$. In the remainder of this section we shall discuss such a model. This will give support to our proposal (3.101) and will enable us to calculate the asymptotic constants $w_2{}^{ab}$ numerically.

The model we consider is the so-called gluon model in which a quark triplet field is coupled to a $SU(3)$ singlet vector-meson field and the only $[SU(3)\otimes SU(3)]_{\gamma_5}$ symmetry breaking is in the quark-mass term $\alpha_0\lambda^0+\alpha_8\lambda^8$. We assume that perturbation-theoretic difficulties are not relevant, so that the canonical commutation relations can be freely applied. This model has been shown to have desirable properties in problems connected with radiative corrections to weak interactions[51] and nonleptonic weak interactions.[52] It also predicts that for inelastic electroproduction $\sigma_L/\sigma_T\to 0$,[27] in good agreement with the recent experimental results. Finally, Brandt and Preparata[53] have shown that the model, supplemented with a Reggeized theory of symmetry breaking, gives a numerical value for the electroproduction integral (3.112) in excellent agreement with experiment.

In the gluon model, $E_2{}^{ab}(p_0)\equiv E_2{}^{ab}$ is independent of p_0. It furthermore has the form[53] $E_2{}^{ab}=d^{abc}E^c$, so that (3.113) is suggested. With Reggeized symmetry breaking, in fact we have $E^c\propto D^c$ in the model since both octets have the same D/F ratios.[53] Thus $f(0)$ can be chosen so that (3.113) is correct in the gluon model and so (3.101) is consistent with (and is suggested by) the gluon model. This fact supports the validity of both (3.101) and the model.

Equation (3.101) can, in fact, be partially derived in the gluon model. The existence of (3.111) implies scaling and hence that $\langle p\mid R^{ab}(x)\mid p\rangle$ in (3.92) has the form $\delta(x^2)\epsilon(x_0)\mathfrak{D}^{ab}(x_0)$. The ET commutator then tells us that $\mathfrak{D}^{ab}(0)\propto d^{abc}D^c$. If, further, one has $D^{ab}(x_0)=h(x_0)T^{ab}$, then in the model we must have $T^{ab}\propto d^{abc}D^c$, as in (3.101).

We conclude this section by estimating the constants (3.105) in the context of the gluon model. We assume $SU(3)$ symmetry for the vertex function D^c and thus we need to know the magnitudes of D, F, and D^0, where

$$\langle a\mid S^c\mid b\rangle\equiv Dd^{abc}+Ff^{abc}+[D^0-(\sqrt{\tfrac{2}{3}})D]\delta^{ab}\delta^{c0}. \qquad (3.114)$$

Mass-shift calculations[54] accurately give $D/F=-0.31\pm 0.02$. If the commutators (3.98) are saturated with low-lying states, then $F\sim 1.2$ and $D\sim -0.4$. These values we used in Ref. 9 without a D^0 term. In the present model we must retain the D^0 term but we do not assume saturation.[55] Instead, we use the fact that $D^a\propto E^a$. In Ref. 53, the magnitudes of 2α and 2β, the F and D for E^a, were determined and E^0 was found to be $\simeq\frac{1}{2}E^0{}_{\rm free}=\frac{1}{2}\sqrt{\frac{2}{3}}$, the subscript "free" denoting the value in the limit of vanishing interaction between the quarks. We therefore take $D^0\simeq\frac{1}{2}D^0{}_{\rm free}=\frac{1}{2}\sqrt{\frac{3}{2}}$. The proportionality $D^a\propto E^a$ then requires that $F=2\alpha D^0/E^0\simeq 3/7$ and $D\simeq -1/7$. With these values, all the constants (3.105) can be calculated.

For electroproduction from protons, we have

$$w_2\equiv w_2{}^{QQ}=\pi^{-1}d^{QQc}D^c=[6F+2D+(12\sqrt{\tfrac{2}{3}})D^0]/9\pi. \qquad (3.115)$$

The above values give $w_2\simeq 0.30$, in excellent agreement with the experimental value.[1] Values for inelastic electron, neutrino, and antineutrino scattering from any $\frac{1}{2}^+$ baryon target can be similarly calculated, and some corresponding experimental results are needed to really test (3.97). Needless to say, these precise numerical results should not be taken too seriously. The experimental nonvanishing of w_2, however, already implies the nontrivial nature of R^{ab} in (3.92). Given this, (3.97) appears to be algebraically the most appealing possibility.

IV. CONCLUSIONS

In Sec. II we used the DGS representation with an assumed rapidly decreasing spectral function $\sigma_2(a,b)$ to show that if $\nu W_2(\kappa,\nu)$ has a finite A limit $F_2(\rho)$ and the Pomeranchuk-dominated Regge behavior $w_2(\kappa)\nu^{-1}$, then $F_2(\infty)=w_2(\infty)\equiv\frac{1}{2}\int da\,\sigma(a)a\equiv w_2=$ const, where $-(\partial/\partial b)\sigma_2(a,b)\sim\sigma(a)b^{-1}$ for $b\sim 0$. In Sec. III we arrived at the same conclusion by relating the asymptotic limits to the behavior of $\hat{W}_2(x^2, p\cdot x)$, the Fourier

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transform of $W_2(\kappa,\nu)$, in the neighborhood of the LC $x^2=0$. There we found $w_2=\frac{1}{2}i\int dx_0\,\hat{f}_2(x_0)$, where $\pi\hat{W}_2(x^2,x_0)\sim\delta(x^2)\frac{1}{2}\epsilon(x_0)\,f_2(x_0)$ for $x^2\sim 0$. This result gives a configuration-space description of the A limit and of scaling behavior in momentum space. Both approaches were shown to be equivalent, but in neither case do we have a good reason why $w_2\neq 0$.

The above LC behavior can be expressed directly in terms of the currents as $\langle p\,|\,[J_-(x),J_-(0)]\,|\,p\rangle\sim\delta(x^2)\epsilon(x_0)\hat{f}_2(x_0)$, apart from total derivatives with respect to x_-. This behavior is so strikingly well accommodated by the LC behavior (3.92) equivalent to (3.89) that we were led to propose the highly symmetric universal form (3.97) for the $SU(3)$ LC commutator. This commutator is consistent with, and is suggested by, the gluon model for ET commutators. In the context of this model, the numerical value of w_2 was estimated and found to agree with the experimental value.

The relation (3.97) is thus seen to incorporate a considerable amount of presumably correct information, including the general current-algebra sum rule (3.89), the existence of the scaling limit $F_2(\rho)$ of νW_2, and the constant value of $F_2(\infty)$. It is also numerically accurate in that it implies the Adler-Weisberger relation for G_A/G_V and suggests a good value for w_2. It therefore seems desirable to explore further consequences of (3.97) and related relations.

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APPENDIX

Although in the text we considered only the Pomeranchuk trajectory with $\alpha=1$, our methods can be used for any α. We shall illustrate this here by outlining the analysis for a general scalar amplitude $W(\kappa,\nu)$. Thus we assume (i) the validity of the DGS representation

$$W(\kappa,\nu)=\int_0^\infty da\int_{-1}^{1}db\,\sigma(a,b)\delta(\kappa+2b\nu-a),\qquad\text{(A1)}$$

with $\sigma(a,b)$ vanishing rapidly for large a, (ii) a finite scaling limit

$$\nu W\underset{A}{\longrightarrow}F(\rho)<\infty,\qquad\text{(A2)}$$

and (iii) Regge asymptotic behavior of the form

$$W\underset{R}{\longrightarrow}w(\kappa)\nu^\alpha,\qquad\text{(A3)}$$

with α arbitrary in the range (for simplicity)

$$-1<\alpha<+1.\qquad\text{(A4)}$$

In the R limit, we have

$$W\underset{R}{\longrightarrow}(1/2\nu)\int da\,\sigma[a,(a-\kappa)/2\nu],\qquad\text{(A5)}$$

so that

$$\sigma(a,b)\sim|\,b\,|^{-1-\alpha}\sigma(a)\quad\text{for}\quad b\sim 0.\qquad\text{(A6)}$$

Then (A3) is satisfied with

$$w(\kappa)=\tfrac{1}{2}\int da\,\sigma(a)[2/(a-\kappa)]^{\alpha+1}.\qquad\text{(A7)}$$

Thus

$$w(\kappa)\underset{\kappa\to\infty}{\longrightarrow}w/(-\kappa)^{\alpha+1},\qquad\text{(A8)}$$

where

$$w=2^\alpha\int da\,\sigma(a).\qquad\text{(A9)}$$

In the A limit, (A2) is satisfied with

$$F(\rho)=\tfrac{1}{2}\int da\,\sigma(a,1/2\rho).\qquad\text{(A10)}$$

Thus

$$F(\rho)\underset{\rho\to\infty}{\longrightarrow}w\rho^{\alpha+1}.\qquad\text{(A11)}$$

We next assume a reduction representation

$$W(\kappa,\nu)=\int d^4x\,\exp(iq\cdot x)\hat{W}(x^2,x_0),\qquad\text{(A12)}$$

where

$$\hat{W}(x^2,x_0)=\langle p\,|\,[j(x),j(0)]\,|\,p\rangle,\qquad\text{(A13)}$$

with $j(x)$ a scalar current and $|\,p\rangle$ a one-scalar-particle state. We use the variables (3.4)–(3.11) and define (using causality)

$$f(\sigma,\lambda)\equiv\int d\mathbf{x}\,\hat{W}(4\sigma\lambda-\mathbf{x}^2,\sigma+\lambda)$$
$$=\pi\int_0^{4\sigma\lambda}dz\,\hat{W}(z,\sigma+\lambda),\qquad\text{(A14)}$$

so that

$$W\underset{A}{\longrightarrow}2\int d\sigma d\lambda\,\exp(2i\sigma\nu-i\lambda/2\rho)f(\sigma,\lambda)\qquad\text{(A15)}$$

and

$$W\underset{R}{\longrightarrow}2\int d\sigma d\lambda\,\exp(2i\sigma\nu+i\lambda\kappa/2\nu)f(\sigma,\lambda).\qquad\text{(A16)}$$

The finiteness of $F(\rho)$ dictates the leading possible σ singularity of $f(\sigma,\lambda)$:

$$2f(\sigma,\lambda)\xrightarrow[\sigma\to 0;\,\sigma\lambda\to 0]{}\theta(\sigma)\hat{f}(\lambda)+\text{L.S.}\qquad\text{(A17)}$$

for some (possibly vanishing) function $\hat{f}(\lambda)$. Then (A2) is satisfied with

$$F(\rho)=\tfrac{1}{2}i\int d\lambda\,\exp(-i\lambda/2\rho)\hat{f}(\lambda).\qquad\text{(A18)}$$

As argued in the text, the limit (A17) also controls the R' limit, so that

$$W\underset{R'}{\longrightarrow}\frac{i}{2\nu}\int d\lambda\,\exp(i\lambda\kappa/2\nu)\hat{f}(\lambda).\qquad\text{(A19)}$$

Thus (A3) requires that

$$\hat{f}(\lambda)=fg\,|\,\lambda\,|^\alpha+\text{L.S.}\qquad\text{(A20)}$$

Equation (A19) then has the form (A3) with $w(\kappa)$ for some (possibly vanishing) constant f and

$$g^{-1}=-2(\sin\tfrac{1}{2}\alpha\pi)\Gamma(\alpha+1).\qquad\text{(A21)}$$

satisfying (4.8) provided we make the identification

$$w=2^\alpha if.\qquad\text{(A22)}$$

Finally, use of (A20)–(A22) in (A18) gives precisely the asymptotic behavior (A11). We therefore obtain complete agreement with the first analysis in terms of the DGS representation.

Because of causality, the limit in (A17) is the light-cone limit $x^2=4\sigma\lambda-\mathbf{x}^2\to 0$. It indeed follows from (A14) and (A17) that

$$\pi\hat{W}(z,\lambda)\sim\delta(z)\tfrac{1}{2}\epsilon(\lambda)\hat{f}(\lambda) \qquad \text{(A23)}$$

for $z\sim 0$. We therefore have determined the leading singularity of $\hat{W}$ on the light cone and, by (A20), the large-λ behavior of its coefficient. The behavior of W near the light cone can also be determined from the configuration-space form

$$\pi\hat{W}(x^2,x_0)=-\tfrac{1}{2}i\int dadb\,\sigma(a,b)\,\exp(-ibx_0)\Delta(x;a+b^2) \qquad \text{(A24)}$$

of (A1) and the behavior (3.58) of Δ. These give

$$\pi\hat{W}(z,\lambda)\sim\delta(z)L(\lambda) \qquad \text{(A25)}$$

for $z\sim 0$, where

$$L(\lambda)=-[i\epsilon(\lambda)/4\pi]\int dadb\,\sigma(a,b)\,\exp(-ib\lambda). \qquad \text{(A26)}$$

Equations (A6) and (A9) give

$$L(\lambda)\underset{\lambda\to\infty}{\longrightarrow}-2^{-\alpha-1}igw\epsilon(\lambda)\,|\lambda|^\alpha. \qquad \text{(A27)}$$

In view of (A22), Eqs. (A25) and (A27) are in perfect agreement with (A23) and (A20).

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Three-Dimensional Formulation of the Relativistic Two-Body Problem and Infinite-Component Wave Equations

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A relativistic quasipotential equation is derived from the conventional Hamiltonian formalism and old-fashioned "noncovariant" off-energy-shell perturbation theory in a similar way to that by which the four-dimensional Bethe-Salpeter equation is obtained from the off-mass-shell Feynman rules. The three-dimensional equation for the (off-energy-shell) scattering amplitude appears as a straightforward generalization of the nonrelativistic Lippmann-Schwinger equation. The corresponding homogeneous equation for the bound-state wave function and the normalization condition for its solutions are derived from the equation for the complete four-point Green's function. In order to obtain a solvable model, we consider a simplified version of the quasipotential equation which still reproduces correctly the on-shell scattering amplitude and is consistent with the elastic unitarity condition. It involves a "local" approximation to the potential $V(p-q)$ which defines the kernel of our integral equation (the integration being carried over a two-sheeted hyperboloid in the energy-momentum space). It is shown that for the scalar Coulomb potential $V(p-q)=\alpha/(p-q)^2$, our model equation is equivalent to a simple infinite-component wave equation of the type considered by Nambu, Barut, and Fronsdal. The energy eigenvalues for the bound-state problem are calculated explicitly in this case and are found to be $O(4)$ degenerate (just as in the nonrelativistic Coulomb problem and in Wick and Cutkosky's treatment of the Bethe-Salpeter equation in the same approximation).

I. INTRODUCTION

THE purpose of this paper is to show the relationship between a modification of the quasipotential approach to the relativistic two-body problem developed in Refs. 1–3 and the infinite-component wave equations for the "relativistic hydrogen atom" of the type considered in Refs. 4–6.

A three-dimensional relativistic quasipotential equation for the two-particle scattering amplitude and for the bound-state wave function was first proposed by Logunov and Tavkhelidze.[7] It was derived in the framework of the Bethe-Salpeter equation using the nonuniqueness of the off-shell extrapolation of the scattering amplitude.

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