Logarithmic Factors in the High-Energy Behavior of Quantum Electrodynamics*

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The perturbation series for electron-electron elastic scattering in quantum electrodynamics is studied in the limit of high energies. For this matrix element, in addition to the previously known terms which are proportional to s (the square of the c.m. energy) and hence lead to a constant total cross section at high energies, there are found terms of the orders of magnitudes $s \ln s$, $s(\ln s)^2$, $s(\ln s)^3$, etc. For $n=1, 2, 3, \ldots$, the coefficient of $s(\ln s)^n$ is a power series in the fine-structure constant α, where the leading term is proportional to $\alpha^{2(n+1)}$ and is due to Feynman diagrams with n closed electron loops. Physically, through the optical theorem, the presence of these terms is intimately related to the production of low-energy electron-positron pairs in high-energy electron-electron scattering, but is independent of whether the spin-1 particle is a photon with zero mass or massive neutral vector meson. These leading terms of order $\alpha^{2(n+1)}$ are explicitly found for all n, and are all imaginary, representing absorption. The procedure of summing the leading term is carried out, and the result demonstrates dramatically the importance of unitarity in the direct, or s, channel for high-energy processes. Generalization to two-body diffraction processes $a+b \to a'+b'$ is immediate.

1. INTRODUCTION

Over two years ago, a program to study high-energy amplitudes in various field theories was initiated. Even though field theory is a most interesting subject in itself, our purpose for pursuing this work is far more general. By studying high-energy problems in field theories, we hoped to gain a basic understanding of high-energy collision processes which may be applicable to hadron physics. We are therefore not so much trusting the quantitative significance of perturbative results, but rather using perturbation as a tool to extract some general behaviors exhibited by all high-energy processes. One of the principal motivations of this study stems from our belief that nature is much more imaginative than potential scattering, which has hitherto been relied on by many researchers with almost religious faith. By far the best theoretical "laboratory" we possess is still the field theory, which has the fundamentally important properties of relativistic invariance, crossing symmetry, and unitarity.

As a first step in this program of study, we have analyzed all the two-body elastic scattering in quantum electrodynamics.[1-5] It is found that to the orders considered $d\sigma/dt$ approaches a finite constant as s approaches infinity with fixed t, where as usual s is the square of the c.m. energy and $-t$ is the square of the momentum transfer. The first nonvanishing contribution, however, appears in different orders of perturbation for the processes: for example, sixth order for electron Compton scattering and eighth order for photon-photon scattering. As expected, these first results already contradict[1,2] the statements from the Regge-pole model without introducing complications; in particular, we find that the Pomeranchuk or vacuum trajectory cannot be a pole with factorizable residues. Moreover, these results also contradict that of the droplet model[6] in the most straightforward interpretation,[7] although there is some similarity such as the two-dimensional integration over the transverse variables. The reason for this disagreement with the droplet model is due almost entirely to the fact that the eikonal picture is applied directly to the incident photon in the droplet model, while we find that the eikonal picture must be applied to each member of the virtual electron-positron pair in the photon.[1,5,8] From this first step of the program emerges a natural picture of high-energy scattering processes.[8]

Out of this impact picture, we have formulated rules to calculate directly the limiting behavior at high energies of various matrix elements for elastic, diffraction, and inelastic scattering processes.[9,10] These rules of calculation are most efficiently expressed in terms of impact diagrams, and the results are of the form of the product of s with an integral which depends on t but not on s. We emphasize that *this method of impact diagrams* is applicable to all orders of perturbation theory, or in other words *takes care of all Feynman*

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[1] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969).
[2] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969).
[3] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1868 (1969).
[4] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1973 (1969).
[5] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1899 (1969).
[6] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966); T. T. Chou and C. N. Yang *ibid.* **170**, 1591 (1968); **175**, 1832 (1968); Phys. Rev. letters **20**, 1213 (1968).
[7] We disagree with the contrary claim of B. W. Lee [Comments Nucl. Particle Phys. **111**, 198 (1969)]. He reformulated the droplet model in q-number language with the guidance of our physical picture based on the rigorous calculation. Since the physical picture was the same, his results turned out to be identical to those of Ref. 8 which we had previously obtained.
[8] H. Cheng and T. T. Wu, Phys. Rev. Letters **23**, 670 (1969).
[9] H. Cheng and T. T. Wu, Phys. Rev. D **1**, 1069 (1970).
[10] H. Cheng and T. T. Wu, Phys. Rev. D **1**, 1083 (1970).

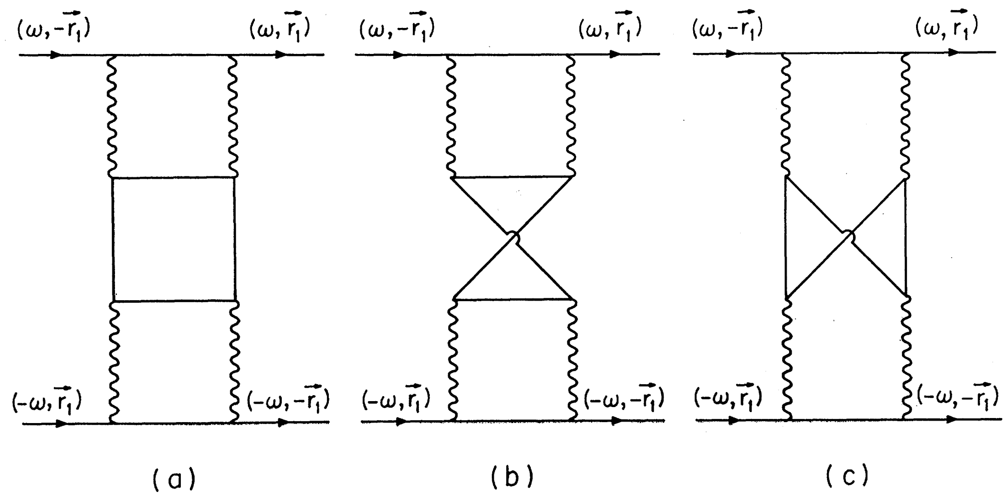

FIG. 1. Lowest-order Feynman diagrams that give rise to a logarithmic factor at high energies.

diagrams for the processes under consideration.[11] By this new method, our earlier results[1,4] can be reproduced with amazing ease; while it took us over a year of hard work before,[1] we can now obtain all the results in a few hours.

As mentioned above, the results of calculation with impact diagrams are essentially integrals which depend on t but not on s. To the lowest nontrivial orders, the integrals give the impact-factor representation.[1,2] How do the integrals behave in higher orders? If they are well defined to all orders, we can claim a satisfactory understanding of high-energy processes by the impact picture, and we can perhaps attempt to study the convergence or divergence of the perturbation series in this limit. Actually, nature is far more profound and interesting. As previously discussed,[8,12] the abovementioned integrals diverge logarithmically when the orders of perturbation are sufficiently high. For quantum electrodynamics, this logarithmic divergence first appears in connection with the diagrams[12] of Fig. 1. Although this divergence can clearly be interpreted as $\ln s$, its presence nevertheless raises many questions. For example, for large s the differential cross section $d\sigma/dt$ must now depend on $\ln s$ and the existence of $\lim_{s\to\infty} d\sigma/dt$ is accordingly in doubt.

Is it conceivable that this appearance of logarithmic divergence is a peculiarity of electrodynamics and hence irrelevant to hadron physics? We think that this is extremely unlikely. Some insight into this factor $\ln s$ can be obtained by the following consideration. Since this $\ln s$ appears in the imaginary part of the amplitude, we may apply the optical theorem to the diagrams of Fig. 1 and thus consider the diagrams of Fig. 2, which shows the production of a pair in electron-electron scattering. It is found that this $\ln s$ is associated with the production of low-energy pairs in the c.m. system. Such pairs, referred to as pionization products, have been studied in detail.[13,14] They are believed to have been observed in cosmic rays[15] in addition to the so-called fireballs.[16] For this reason these logarithmic factors seem to be of fundamental importance and neither the impact picture[8] nor the hypothesis of limiting fragmentation[17] gives the entire story.

This fundamental problem of the logarithmic factors is extremely difficult and challenging. We are still very far from arriving at the complete answer to the problem, and the aim of the present paper is to take the first step in that direction. To approach a problem of this magnitude, it is necessary to have some physical understanding which may serve as a guide. For this purpose, in Sec. 2, we first devote ourselves to the relatively simple task of obtaining the $s \ln s$ term to the lowest nontrivial order. Basically, the diagrams of interest are those of Fig. 1, although we shall approach

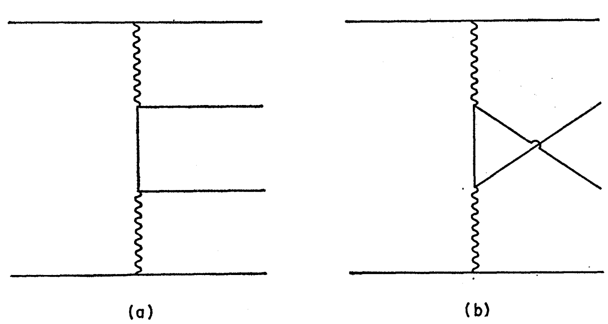

FIG. 2. Pionization diagrams.

[11] If we consider a scattering process, such as photon-electron backward scattering $e\gamma\to\gamma e$, where $d\sigma/dt$ approaches zero as some inverse power of s, then the method of impact diagrams developed so far merely gives the trivial answer 0.

[12] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 1405 (1969).

[13] H. Cheng and T. T. Wu, Phys. Rev. Letters **23**, 1311 (1969).

[14] H. Cheng and T. T. Wu (unpublished).

[15] *Proceedings of the Tenth International Conference on Cosmic Rays, Calgary, Canada, 1967* [Can. J. Phys. Suppl. **46**, (1968)].

[16] G. Cocconi, Phys. Rev. **111**, 1699 (1958); K. Niu, Nuovo Cimento **10**, 994 (1958); P. Coik, T. Coghen, J. Gierula, R. Holynski, A. Jurak, M. Miesowicz, T. Saniewska, and J. Pernegr, *ibid.* **10**, 741 (1958).

[17] J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. **188**, 2159 (1969).

the problem with impact diagrams. It is found that a certain function, called K and defined by (2.14), appears in the final answer for the coefficient of the $s\ln s$ term. The properties of this function are studied in Sec. 3, and Sec. 4 is devoted to the special case of forward scattering. After this, in Sec. 5, all the considerations of Sec. 2 for the $s\ln s$ term from the simplest electron-electron scattering diagrams with one fermion loop are generalized to the $s(\ln s)^n$ terms due to the simplest diagrams with n fermion loops. It is found that the same function K appears repeatedly, and the relevant properties of this K are further studied in Sec. 6, again for the special case of forward scattering. The results are discussed in some detail in Sec. 7. We also learn how important unitarity in the direct channel is, a fact which justifies our original choice of studying field theories.

2. ELECTRON-ELECTRON SCATTERING

The impact diagrams of interest are illustrated in Fig. 3. The diagrams in Figs. 3(a) and 3(b) describe the process in which the pair is produced by the first electron and scattered by the second one, while those in Figs. 3(c) and 3(d) describe the process in which the pair is produced by the second electron and scattered by the first one. Note that a solid line represents a fermion and a wavy line represents a photon or a neutral vector meson. Consider first diagram 3(a). We put[18]

$$\mathbf{P}=[(1-\beta_1-\beta_2)\omega,\ \mathbf{q}_\perp], \tag{2.1}$$

$$\mathbf{k}_1=[(\beta_1+\beta_2)\omega,\ -\mathbf{r}_1-\mathbf{q}_\perp], \tag{2.2}$$

$$\mathbf{k}_2=[(\beta_1+\beta_2)\omega,\ \mathbf{r}_1-\mathbf{q}_\perp], \tag{2.3}$$

$$\mathbf{p}_1=[\beta_1\omega,\ \mathbf{p}_\perp], \tag{2.4}$$

$$\mathbf{p}_2=[\beta_2\omega,\ -\mathbf{r}_1-\mathbf{q}_\perp-\mathbf{p}_\perp], \tag{2.5}$$

$$\mathbf{p}_3=[\beta_2\omega,\ \mathbf{q}_\perp'-\mathbf{q}_\perp-\mathbf{p}_\perp], \tag{2.6}$$

and

$$\mathbf{p}_4=[\beta_1\omega,\ \mathbf{r}_1-\mathbf{q}_\perp'+\mathbf{p}_\perp]. \tag{2.7}$$

In (2.1)–(2.7), the quantities entered in square brackets are the longitudinal and the transverse components, respectively, of the corresponding spatial momentum. We shall assume, without loss of generality, that the line carrying momentum $\mathbf{p}_1$ ($\mathbf{p}_2$) represents an electron (a positron).

The dominant contribution to the scattering amplitude comes from the region $\beta_1\ll1$, $\beta_2\ll1$. Applying the rules in Ref. 9 and making the approximation $\beta_1\ll1$, $\beta_2\ll1$, we obtain the scattering amplitude for Fig. 3(a) as

$$\begin{aligned}\mathfrak{M}_a\sim&-\tfrac{1}{2}\omega^2 im^{-2}\delta_{12}\delta_{1'2'}e^8(2\pi)^{-8}\int d\mathbf{p}_\perp d\mathbf{q}_\perp d\mathbf{q}_\perp'\int_0^1 d\beta_1\int_0^1 d\beta_2\\&\times\mathrm{Tr}[(\gamma_0-\gamma_3)(-\not{p}_2+m)\gamma_0(-\not{p}_3+m)(\gamma_0-\gamma_3)(\not{p}_4+m)\gamma_0(\not{p}_1+m)]\\&\times\{\beta_1[(\mathbf{r}_1+\mathbf{q}_\perp+\mathbf{p}_\perp)^2+m^2]+\beta_2(\mathbf{p}_\perp^2+m^2)\}^{-1}\\&\times\{\beta_1[(\mathbf{q}_\perp'-\mathbf{q}_\perp-\mathbf{p}_\perp)^2+m^2]+\beta_2[(\mathbf{r}_1-\mathbf{q}_\perp'+\mathbf{p}_\perp)^2+m^2]\}^{-1}\\&\times P_+(\mathbf{r}_1+\mathbf{q}_\perp')P_-(\mathbf{r}_1-\mathbf{q}_\perp')[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1}.\end{aligned} \tag{2.8}$$

In (2.8),[19] m and λ are the masses of the fermion and the vector meson, respectively, and δ_{12}, $\delta_{1'2'}$ are the Kronecker δ's in spin.[1] We now explicitly evaluate the trace in (2.8). Since

$$\not{p}_i\sim\beta_i\omega(\gamma_0-\gamma_3)-\mathbf{p}_{i\perp}\cdot\boldsymbol{\gamma}_\perp$$

and since

$$(\gamma_0-\gamma_3)^2=0,$$

we have

$$(\gamma_0-\gamma_3)\not{p}_i\sim(\gamma_0-\gamma_3)\not{p}_{i\perp},$$

where $\not{p}_{i\perp}=-\mathbf{p}_{i\perp}\cdot\boldsymbol{\gamma}_\perp$. Thus the trace in (2.8) is equal to

$$\begin{aligned}&\mathrm{Tr}[(\gamma_0-\gamma_3)(-\not{p}_{2\perp}+m)\gamma_0(-\not{p}_{3\perp}+m)(\gamma_0-\gamma_3)(\not{p}_{4\perp}+m)\gamma_0(\not{p}_{1\perp}+m)]\\&\qquad=2\,\mathrm{Tr}(\not{p}_{2\perp}+m)(-\not{p}_{3\perp}+m)(-\not{p}_{4\perp}+m)(\not{p}_{1\perp}+m).\end{aligned} \tag{2.9}$$

Just as in the case of Compton scattering discussed in Ref. 2, the amplitude $\mathfrak{M}_b$ for diagram 3(b) can be obtained from the right-hand side of (2.8) by setting $\mathbf{q}_\perp'=\mathbf{r}_1$ in the trace as well as the energy denominators. Thus

$$\begin{aligned}\mathfrak{M}_a+\mathfrak{M}_b\sim&\,\omega^2 im^{-2}\delta_{12}\delta_{1'2'}e^8(2\pi)^{-8}\int d\mathbf{p}_\perp d\mathbf{q}_\perp d\mathbf{q}_\perp'\int_0^1 d\beta_1\int_0^1 d\beta_2[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1}\\&\times P_+(\mathbf{r}_1+\mathbf{q}_\perp')P_-(\mathbf{r}_1-\mathbf{q}_\perp')\{\beta_1[(\mathbf{r}_1+\mathbf{q}_\perp+\mathbf{p}_\perp)^2+m^2]+\beta_2(\mathbf{p}_\perp^2+m^2)\}^{-1}\\&\times\left(\frac{(\mathbf{p}_\perp^2+m^2)\,\mathrm{Tr}[(\not{p}_{2\perp}+m)(-\not{p}_{2\perp}-2\not{r}_1+m)]}{\beta_1[(\mathbf{r}_1-\mathbf{q}_\perp-\mathbf{p}_\perp)^2+m^2]+\beta_2(\mathbf{p}_\perp^2+m^2)}-\frac{\mathrm{Tr}(\not{p}_{2\perp}+m)(-\not{p}_{3\perp}+m)(-\not{p}_{4\perp}+m)(\not{p}_{1\perp}+m)}{\beta_1[(\mathbf{q}_\perp'-\mathbf{q}_\perp-\mathbf{p}_\perp)^2+m^2]+\beta_2[(\mathbf{r}_1-\mathbf{q}_\perp'+\mathbf{p}_\perp)^2+m^2]}\right).\end{aligned} \tag{2.10}$$

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[18] See Sec. 3 of Ref. 9.

[19] Strictly speaking, in connection with the Feynman diagrams of Fig. 1, we can keep only the lowest-order terms in P_+ and P_- [defined by (5.2) of Ref. 9].

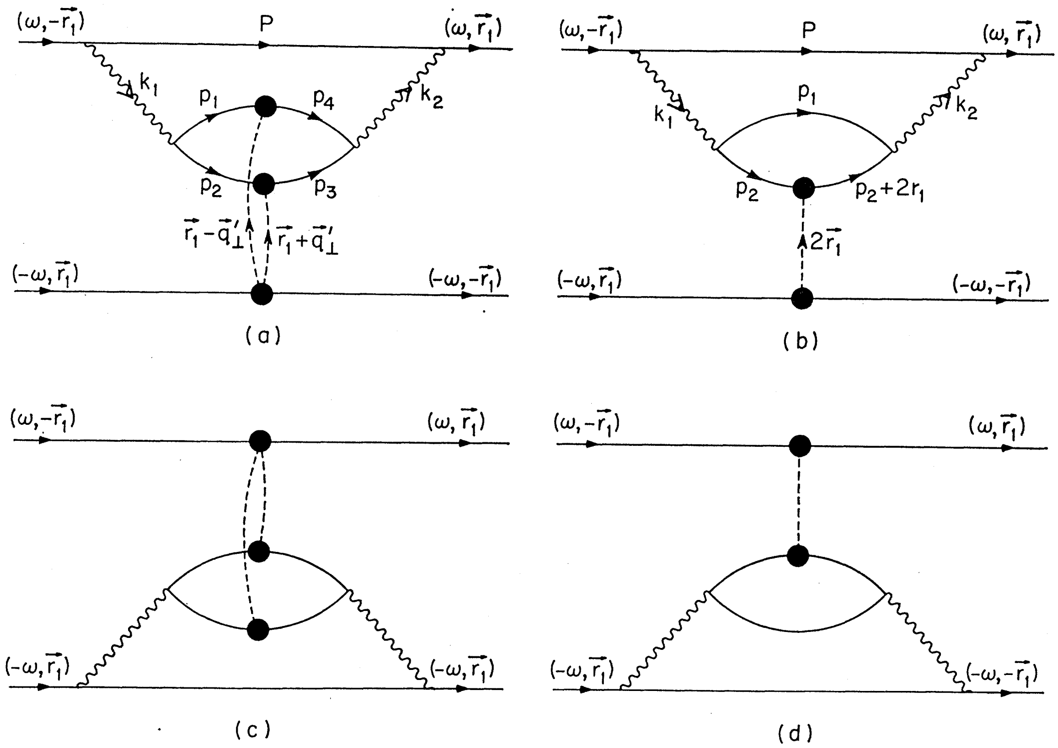

FIG. 3. Impact diagrams corresponding to Fig. 1.

Observe that the integration in (2.10) is not convergent at the end point $\beta_1=\beta_2=0$. This divergence is due to the fact that the method of impact diagrams is not applicable in the region $\omega\beta_1=O(1)$, $\omega\beta_2=O(1)$. This means that the integration over β_1 and β_2 must be cut off at ω^{-1}. Denoting

$$\beta_1=\beta x, \tag{2.11}$$

$$\beta_2=\beta(1-x), \tag{2.12}$$

and carrying out the integration over β by setting

$$\int_{\omega^{-1}}^{1} \beta^{-1}d\beta=\ln\omega,$$

we get

$$\mathfrak{M}_a+\mathfrak{M}_b\sim\tfrac{1}{2}is\ln s(2\pi)^{-4}\int d\mathbf{q}_\perp d\mathbf{q}_\perp'[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1}$$
$$\times\mathcal{I}^e(\mathbf{r}_1,\mathbf{q}_\perp)K(\mathbf{r}_1,\mathbf{q}_\perp,\mathbf{q}_\perp')\mathcal{I}^e(\mathbf{r}_1,\mathbf{q}_\perp')P_+(\mathbf{r}_1+\mathbf{q}_\perp')P_-(\mathbf{r}_1-\mathbf{q}_\perp'), \tag{2.13}$$

where

$$K(\mathbf{r}_1,\mathbf{q}_\perp,\mathbf{q}_\perp')=e^4(2\pi)^{-4}\int d\mathbf{p}_\perp\int_0^1 dx[x(\mathbf{r}_1+\mathbf{p}_\perp+\mathbf{q}_\perp)^2+(1-x)\mathbf{p}_\perp^2+m^2]^{-1}$$
$$\times\left(\frac{\mathrm{Tr}[(\not{p}_{2\perp}+m)(-\not{p}_{2\perp}-2\mathbf{r}_1+m)](\mathbf{p}_\perp^2+m^2)}{[x(\mathbf{r}_1-\mathbf{q}_\perp-\mathbf{p}_\perp)^2+(1-x)\mathbf{p}_\perp^2+m^2]}-\frac{\mathrm{Tr}[(\not{p}_{2\perp}+m)(-\not{p}_{3\perp}+m)(-\not{p}_{4\perp}+m)(\not{p}_{1\perp}+m)]}{[x(\mathbf{q}_\perp'-\mathbf{q}_\perp-\mathbf{p}_\perp)^2+(1-x)(\mathbf{r}_1-\mathbf{q}_\perp'+\mathbf{p}_\perp)^2+m^2]}\right). \tag{2.14}$$

In (2.13), $\mathcal{I}^e=\frac{1}{2}e^2m^{-1}\delta_{12}$ is the electron impact factor.

The scattering amplitude $\mathfrak{M}_c+\mathfrak{M}_d$ for diagrams 3(c) and 3(d) is equal to $\mathfrak{M}_a+\mathfrak{M}_b$. Thus

$$\mathfrak{M}=\mathfrak{M}_a+\mathfrak{M}_b+\mathfrak{M}_c+\mathfrak{M}_d$$
$$=is\ln s\,(2\pi)^{-4}\int d\mathbf{q}_\perp d\mathbf{q}_\perp'[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1}$$
$$\times\mathcal{I}^e(\mathbf{r}_1,\mathbf{q}_\perp)K(\mathbf{r}_1,\mathbf{q}_\perp,\mathbf{q}_\perp')\mathcal{I}^e(\mathbf{r}_1,\mathbf{q}_\perp')P_+(\mathbf{r}_1+\mathbf{q}_\perp')P_-(\mathbf{r}_1-\mathbf{q}_\perp'). \tag{2.15}$$

Equation (2.15) is the desired answer for this simple case. Sections 3 and 4 of this paper are devoted to a detailed study of this function $K(\mathbf{r}_1,\mathbf{q}_\perp,\mathbf{q}_\perp')$ as defined by (2.14).

3. SOME PROPERTIES OF $K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{q}_\perp')$

We obtain here some of the simple properties of $K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{q}_\perp')$, namely,

$$K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{q}_\perp') = K(\mathbf{r}_1, \mathbf{q}_\perp', \mathbf{q}_\perp) \tag{3.1}$$

and

$$K(\mathbf{r}_1, \pm\mathbf{r}_1, \mathbf{q}_\perp') = K(\mathbf{r}_1, \mathbf{q}_\perp, \pm\mathbf{r}_1) = 0. \tag{3.2}$$

The symmetry (3.1) is most easily proved by carrying out the x integration in (2.14). If the variable of integration $\mathbf{p}_\perp$ is everywhere replaced by $\mathbf{p}_\perp - \mathbf{q}_\perp$, we get[20]

$$\begin{aligned} K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{q}_\perp') &= e^4(2\pi)^{-4} \int d\mathbf{p}_\perp \int_0^1 dx[x(\mathbf{p}_\perp+\mathbf{r}_1)^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]^{-1} \\ &\quad \times \left\{ -\frac{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]\,\mathrm{Tr}(p_\perp+r_1-m)(p_\perp-r_1+m)}{x(\mathbf{p}_\perp-\mathbf{r}_1)^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2} \right. \\ &\quad \left. -\frac{\mathrm{Tr}(p_\perp+r_1-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'+r_1-m)(p_\perp-q_\perp+m)}{x(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2} \right\} \\ &= 4e^4(2\pi)^{-4} \int d\mathbf{p}_\perp \left\{ \frac{\mathbf{p}_\perp^2-\mathbf{r}_1^2+m^2}{(\mathbf{p}_\perp+\mathbf{r}_1)^2-(\mathbf{p}_\perp-\mathbf{r}_1)^2} \ln\frac{(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2}{(\mathbf{p}_\perp-\mathbf{r}_1)^2-m^2} \right. \\ &\quad -\frac{1}{4}\frac{\mathrm{Tr}(p_\perp+r_1-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'+r_1-m)(p_\perp-q_\perp+m)}{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]-[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]} \\ &\quad \left. \times \ln\frac{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]}{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]} \right\}. \end{aligned} \tag{3.3}$$

Equation (3.1) immediately follows from (3.3).

The relation

$$K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{r}_1) = 0 \tag{3.4}$$

is also a ready consequence of (3.3). However, in order to get the other relation

$$K(\mathbf{r}_1, \mathbf{q}_\perp, -\mathbf{r}_1) = 0, \tag{3.5}$$

it is necessary to note that the two terms on the right-hand side of (3.3), taken separately, contain no linearly divergent part. Equation (3.2) follows from (3.4), (3.5), and (3.1).

These two properties (3.1) and (3.2) are to be expected: (3.1) is due to the symmetry of the Feynman diagrams of Fig. 1 under the exchange of the two incoming electrons, and (3.2) is closely related to a property of the photon impact factor, first given by (3.6) of Ref. 2 and later used in discussing the relation between impact factors and form factors.[21] Rather, it is an advantage of the method of impact diagrams[9,10] that (3.1) and (3.2) can be easily derived. For instance, if we treat the Feynman diagrams of Fig. 1 directly in the most straightforward manner, the result takes a form that fails to exhibit the symmetry (3.1). The same problem appears in connection with the electrodynamics of scalar particles[22] and is discussed in detail in that context.[23]

We treat the special case $\mathbf{r}_1=0$ in detail in the next section. Although many of the considerations there can be generalized to all $\mathbf{r}_1$, the results become rather complicated, and hence the derivation is relegated to Appendix A.

4. FORWARD SCATTERING

A. Definition

Since both (2.14) and (3.3) are rather complicated, we restrict ourselves in this section to the case of forward scattering where $\mathbf{r}_1=0$. Let

$$K_0(\mathbf{q}_\perp, \mathbf{q}_\perp') = [4e^4(2\pi)^{-3}]^{-1} K(0, \mathbf{q}_\perp, \mathbf{q}_\perp'); \tag{4.1}$$

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[20] Note that $p_\perp^2 = -\mathbf{p}_\perp^2$, etc.

[21] H. Cheng and T. T. Wu, Phys. Rev. **184**, 1868 (1969).

[22] H. Cheng and T. T. Wu (unpublished).

[23] See Appendix B of Ref. 22.

then the two expressions for K_0 are

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\tfrac{1}{4}(2\pi)^{-1}\int d\mathbf{p}_\perp\int_0^1 dx[x\mathbf{p}_\perp^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]^{-1}$$

$$\times\left\{-\frac{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]\,\mathrm{Tr}[(p_\perp-m)(p_\perp+m)]}{x\mathbf{p}_\perp^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2}-\frac{\mathrm{Tr}[(p_\perp-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'-m)(p_\perp-q_\perp+m)]}{x(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2}\right\} \quad (4.2)$$

and

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\left\{1-\frac{1}{4}\frac{\mathrm{Tr}[(p_\perp-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'-m)(p_\perp-q_\perp+m)]}{(\mathbf{p}_\perp^2+m^2)[(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2]-[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}\right.$$
$$\left.\times\ln\frac{(\mathbf{p}_\perp^2+m^2)[(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2]}{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}\right\}. \quad (4.3)$$

These two expressions can be usefully combined in the form

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\int_0^1 dx$$
$$\times\left\{1-\frac{1}{4}\frac{\mathrm{Tr}(p_\perp-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'-m)(p_\perp-q_\perp+m)}{[x\mathbf{p}_\perp^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][x(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2]}\right\}. \quad (4.4)$$

If the trace of the γ matrices is explicitly written out, the result is

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\int_0^1 dx$$
$$\times\left\{1-\frac{(\mathbf{p}_\perp^2+m^2)^2-(\mathbf{p}_\perp^2+m^2)[2\mathbf{p}_\perp\cdot(\mathbf{q}_\perp+\mathbf{q}_\perp')-(\mathbf{q}_\perp^2+\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+\mathbf{q}_\perp'^2)]+2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)(\mathbf{p}_\perp\cdot\mathbf{q}_\perp')-(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)\mathbf{q}_\perp'^2-(\mathbf{p}_\perp\cdot\mathbf{q}_\perp')\mathbf{q}_\perp^2}{[x\mathbf{p}_\perp^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][x(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2]}\right\}. \quad (4.5)$$

This is the starting point of the present investigation.

B. Second Feynman Parameter

The form (4.5) fails to exhibit the symmetry property (3.1). To restore this symmetry explicitly, we introduce a second Feynman parameter y to combine the two denominators:

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\int_0^1 dx\int_0^1 dy\ \ (1-N_0/D_0^2), \quad (4.6)$$

where N_0 is the numerator that appears in (4.5) and D_0 is given by

$$D_0=xy\mathbf{p}_\perp^2+(1-x)y(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+x(1-y)(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(1-y)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp')^2+m^2$$
$$=\mathbf{p}_\perp^2-2(1-x)\mathbf{p}_\perp\cdot\mathbf{q}_\perp-2(1-y)\mathbf{p}_\perp\cdot\mathbf{q}_\perp'+(1-x)y\mathbf{q}_\perp^2+x(1-y)\mathbf{q}_\perp'^2+(1-x)(1-y)(\mathbf{q}_\perp+\mathbf{q}_\perp')^2+m^2. \quad (4.7)$$

We reverse the order of integration and rewrite (4.6) in the form

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int_0^1 dx\int_0^1 dy\int d\mathbf{p}_\perp(1-N_0/D_0^2). \quad (4.8)$$

Because of (4.7), we write

$$\mathbf{p}_\perp=\delta\mathbf{p}_\perp+\mathbf{p}_\perp', \quad (4.9)$$

where

$$\delta\mathbf{p}_\perp=(1-x)\mathbf{q}_\perp+(1-y)\mathbf{q}_\perp'. \quad (4.10)$$

In changing to this new variable $\mathbf{p}_\perp'$, the fact that

$$\int d\mathbf{p}_\perp(1-N_0/D_0^2)$$

is linearly divergent must be taken into account. More precisely, this shift gives the contribution

$$-\int_0^1 dx\int_0^1 dy\,\delta\mathbf{p}_\perp\cdot[(\mathbf{q}_\perp+\mathbf{q}_\perp')-2\delta\mathbf{p}_\perp]=\int_0^1 dx\int_0^1 dy[(1-x)(1-2x)\mathbf{q}_\perp^2+(1-y)(1-2y)\mathbf{q}_\perp'^2]$$
$$=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2). \quad (4.11)$$

With this contribution properly included, symmetric integration over $\mathbf{p}_\perp'$ yields, after a tedious calculation,

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)-(2\pi)^{-1}\int_0^1 dx\int_0^1 dy\int d\mathbf{p}_\perp'[\mathbf{p}_\perp'^2+x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2]^{-2}$$
$$\times\{(\mathbf{p}_\perp'^2+m^2)[(1-6x+6x^2)\mathbf{q}_\perp^2+2(1-2x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+(1-6y+6y^2)\mathbf{q}_\perp'^2]$$
$$+m^2[2x(1-x)\mathbf{q}_\perp^2-(1-2x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+2y(1-y)\mathbf{q}_\perp'^2]$$
$$-x(1-x)(1-2x)(1-2y)\mathbf{q}_\perp^2(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')-[x(1-x)+y(1-y)]\mathbf{q}_\perp^2\mathbf{q}_\perp'^2$$
$$+4x(1-x)y(1-y)(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2-y(1-y)(1-2x)(1-2y)\mathbf{q}_\perp'^2(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')\}. \quad (4.12)$$

At this stage, simplification can be achieved by noticing that the denominator is not changed by the replacement $x\to 1-x$. Thus a number of terms in (4.12) do not contribute, and

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)-(2\pi)^{-1}\int_0^1 dx\int_0^1 dy\int d\mathbf{p}_\perp'[\mathbf{p}_\perp'^2+x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2]^{-2}$$
$$\times\{(\mathbf{p}_\perp'^2+m^2)[(1-6x+6x^2)\mathbf{q}_\perp^2+(1-6y+6y^2)\mathbf{q}_\perp'^2]+2m^2[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2]$$
$$-[x(1-x)+y(1-y)]\mathbf{q}_\perp^2\mathbf{q}_\perp'^2+4x(1-x)y(1-y)(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2\}. \quad (4.13)$$

The $\mathbf{p}_\perp'$ integral on the right-hand side of (4.13) is still logarithmically divergent. This divergence does not cause any trouble because

$$\int_0^1 dx(1-6x+6x^2)=0. \quad (4.14)$$

It is now straightforward to carry out the integration over $\mathbf{p}_\perp'$ to get

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)+\frac{1}{2}\int_0^1 dx\int_0^1 dy\{[(1-6x+6x^2)\mathbf{q}_\perp^2+(1-6y+6y^2)\mathbf{q}_\perp'^2]$$
$$\times\ln[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2]+[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2]^{-1}$$
$$\times\{[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2][(1-6x+6x^2)\mathbf{q}_\perp^2+(1-6y+6y^2)\mathbf{q}_\perp'^2-2m^2]$$
$$+[x(1-x)+y(1-y)]\mathbf{q}_\perp^2\mathbf{q}_\perp'^2-4x(1-x)y(1-y)(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2\}\}$$
$$=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)-\int_0^1 dx\int_0^1 dy[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2]$$
$$+\frac{1}{2}\int_0^1 dx\int_0^1 dy[x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2]^{-1}$$
$$\times\{-x(1-x)(1-2x)^2(\mathbf{q}_\perp^2)^2-y(1-y)(1-2y)^2(\mathbf{q}_\perp'^2)^2+[x(1-x)\mathbf{q}_\perp^2+y(1-y)q_\perp'^2]$$
$$\times[(1-2x)^2\mathbf{q}_\perp^2+(1-2y)^2\mathbf{q}_\perp'^2]+[x(1-x)+y(1-y)]\mathbf{q}_\perp^2\mathbf{q}_\perp'^2-4x(1-x)y(1-y)(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2\}. \quad (4.15)$$

Accordingly we get the desired answer

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\int_0^1 dx\int_0^1 dy\frac{[x(1-x)+y(1-y)]\mathbf{q}_\perp^2\mathbf{q}_\perp'^2-2x(1-x)y(1-y)[2\mathbf{q}_\perp^2\mathbf{q}_\perp'^2+(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2]}{x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2}. \quad (4.16)$$

This form exhibits explicitly the properties

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=K_0(\mathbf{q}_\perp',\mathbf{q}_\perp) \quad (4.17)$$

and

$$K_0(\mathbf{q}_\perp,0)=K_0(0,\mathbf{q}_\perp')=0, \quad (4.18)$$

which are the special cases of (3.1) and (3.2) for $\mathbf{r}_1=0$.

C. Explicit Integration

The function K_0 can be expressed explicitly in terms of Clausen's integral,[24] which has been tabulated.[24,25] The integrals that we need are

$$\int_0^1 d\bar{x}\int_0^1 d\bar{y}[(1-\bar{x}^2)a^2+(1-\bar{y}^2)a'^2+1]^{-1}$$
$$=\tfrac{1}{2}(aa')^{-1}[f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')], \quad (4.19)$$

[24] T. Clausen, J. Reine Angew. Math. **8**, 298 (1832).

[25] Clausen's original tabulation in Ref. 24 is quite extensive. A short table can be found in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, D.C., 1964), pp. 1005 and 1006.

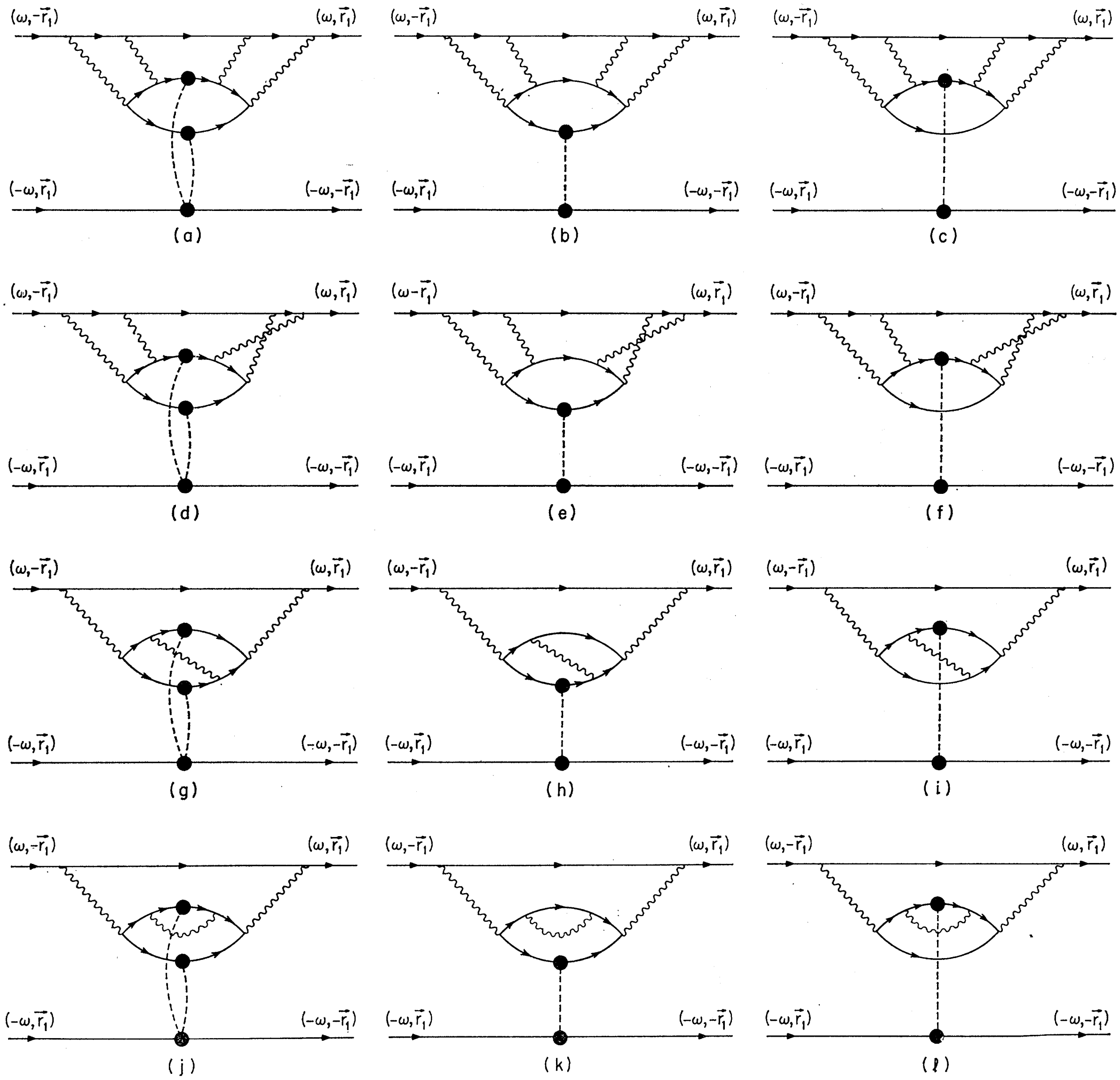

FIG. 4. Some examples of impact diagrams with one loop.

$$\int_0^1 d\bar{x}\int_0^1 d\bar{y}\,\bar{x}^2[(1-\bar{x}^2)a^2+(1-\bar{y}^2)a'^2+1]^{-1}$$
$$=(2a^3a')^{-1}\{\tfrac{1}{2}(a^2+a'^2+1)[f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')]$$
$$+a'(a^2+1)^{1/2}\sinh^{-1}a-a(a'^2+1)^{1/2}\sinh^{-1}a'-aa'\}, \quad (4.20)$$

and

$$\int_0^1 d\bar{x}\int_0^1 d\bar{y}\,\bar{x}^2\bar{y}^2[(1-\bar{x}^2)a^2+(1-\bar{y}^2)a'^2+1]^{-1}$$
$$=(2aa')^{-3}\{\tfrac{1}{2}(a^2+a'^2+1)^2[f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')]$$
$$-(a^2-a'^2+1)a'(a^2+1)^{1/2}\sinh^{-1}a-(-a^2+a'^2+1)a(a'^2+1)^{1/2}\sinh^{-1}a'-aa'(a^2+a'^2-1)\}, \quad (4.21)$$

where a and a' are two non-negative real numbers,

$$\hat{a}=\sin^{-1}[a/(a^2+a'^2+1)^{1/2}], \qquad \hat{a}'=\sin^{-1}[a'/(a^2+a'^2+1)^{1/2}], \quad (4.22)$$

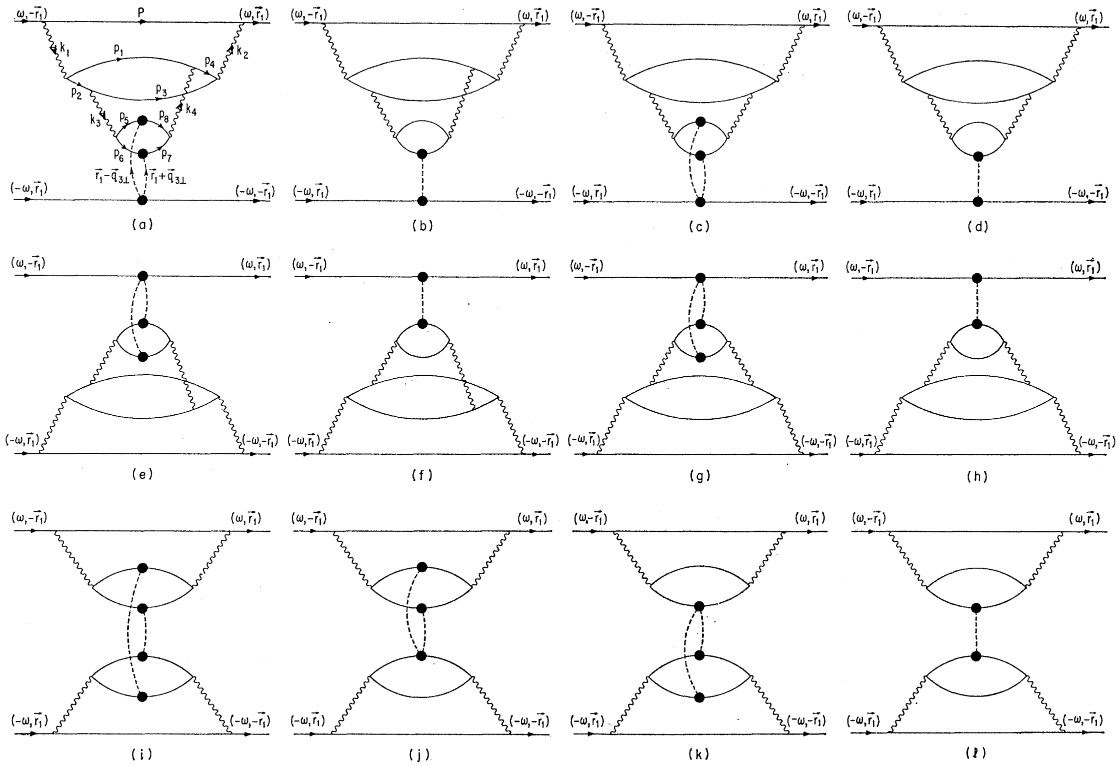

FIG. 5. Lowest-order impact diagrams that give rise to two logarithmic factors at high energies.

and f is Clausen's integral[24] defined by

$$f(\theta)=-\int_0^\theta \ln(2\sin\tfrac{1}{2}t)\,dt=\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2}\,. \tag{4.23}$$

Note that, since $\hat{a}\geq 0$, $\hat{a}'\geq 0$, and $\hat{a}+\hat{a}'\leq\frac{1}{2}\pi$, the arguments of all Clausen's integrals in (4.19)–(4.21) are in the range 0–π.

In order to apply these integrals, let

$$x=\tfrac{1}{2}(1+\bar{x})\quad\text{and}\quad y=\tfrac{1}{2}(1+\bar{y}) \tag{4.24}$$

in (4.16):

$$\begin{aligned}
K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=&\int_0^1 d\bar{x}\int_0^1 d\bar{y}\,\frac{(2-\bar{x}^2-\bar{y}^2)\mathbf{q}_\perp^2\mathbf{q}_\perp'^2-(1-\bar{x}^2)(1-\bar{y}^2)[\mathbf{q}_\perp^2\mathbf{q}_\perp'^2+\frac{1}{2}(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2]}{(1-\bar{x}^2)\mathbf{q}_\perp^2+(1-\bar{y}^2)\mathbf{q}_\perp'^2+4m^2}\\
=&\tfrac{1}{2}(|\mathbf{q}_\perp||\mathbf{q}_\perp'|)^{-1}[\mathbf{q}_\perp^2\mathbf{q}_\perp'^2-\tfrac{1}{2}(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2][f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')\\
&+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')]\\
&+\tfrac{1}{2}(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2\{\tfrac{1}{4}(|\mathbf{q}_\perp||\mathbf{q}_\perp'|)^{-1}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2+4m^2)[(\mathbf{q}_\perp^2)^{-1}+(\mathbf{q}_\perp'^2)^{-1}]\\
&\times[f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')]\\
&+\tfrac{1}{2}[(\mathbf{q}_\perp^2)^{-1}-(\mathbf{q}_\perp'^2)^{-1}][|\mathbf{q}_\perp|^{-1}(\mathbf{q}_\perp^2+4m^2)^{1/2}\sinh^{-1}(\tfrac{1}{2}|\mathbf{q}_\perp|/m)\\
&-|\mathbf{q}_\perp'|^{-1}(\mathbf{q}_\perp'^2+4m^2)^{1/2}\sinh^{-1}(\tfrac{1}{2}|\mathbf{q}_\perp'|/m)]-\tfrac{1}{2}[(\mathbf{q}_\perp^2)^{-1}+(\mathbf{q}_\perp'^2)^{-1}]\}\\
&-[\mathbf{q}_\perp^2\mathbf{q}_\perp'^2+\tfrac{1}{2}(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2](2|\mathbf{q}_\perp||\mathbf{q}_\perp'|)^{-3}\{\tfrac{1}{2}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2+4m^2)[f(\tfrac{1}{2}\pi+\hat{a}-\hat{a}')\\
&+f(\tfrac{1}{2}\pi-\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi+\hat{a}+\hat{a}')-f(\tfrac{1}{2}\pi-\hat{a}-\hat{a}')]-(\mathbf{q}_\perp^2-\mathbf{q}_\perp'^2+4m^2)|\mathbf{q}_\perp'|(\mathbf{q}_\perp^2+4m^2)^{1/2}\sinh^{-1}(\tfrac{1}{2}|\mathbf{q}_\perp|/m)\\
&-(-\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2+4m^2)|\mathbf{q}_\perp|(\mathbf{q}_\perp'^2+4m^2)^{1/2}\sinh^{-1}(\tfrac{1}{2}|\mathbf{q}_\perp'|/m)-|\mathbf{q}_\perp||\mathbf{q}_\perp'|(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2-4m^2)\}\,,
\end{aligned} \tag{4.25}$$

where

$$\hat{a}=\sin^{-1}[\mathbf{q}_\perp^2/(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2+4m^2)]$$

and

$$\hat{a}'=\sin^{-1}[\mathbf{q}_\perp'^2/(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2+4m^2)]. \tag{4.26}$$

This explicit formula (4.25) is rather complicated and cumbersome, and hence useful mainly for numerical computations. For analytical purposes, the integral representation (4.16) is much more useful, as we shall see in Sec. 6.

5. TWO OR MORE LOOPS

We have seen that uncanceled $s\ln s$ terms exist in the electron-electron scattering amplitude. In fact, the lowest-order impact diagrams which give an uncanceled $s\ \ln s$ term in the electron-electron scattering amplitude are those illustrated in Fig. 3, and all of them have one electron loop. It is now natural to pose the question whether uncanceled $s(\ln s)^2$ terms, or more generally uncanceled terms of the order of $s(\ln s)^n$, $n>1$, exist. To answer this question we must examine higher-order impact diagrams. A number of such diagrams are illustrated in Fig. 4. All of these diagrams have one electron loop, and it can be shown that they give uncanceled $s\ \ln s$ terms only. The lowest-order impact diagrams which give uncanceled $s(\ln s)^2$ terms are those illustrated in Fig. 5. The corresponding Feynman diagrams are illustrated in Fig. 6. The diagrams in Fig. 5 are the lowest-order impact diagrams which have two electron loops. In this section we show that the sum of these diagrams gives the amplitude

$$\tfrac{1}{2}is(\ln s)^2(2\pi)^{-6}\int d\mathbf{q}_{1\perp}d\mathbf{q}_{2\perp}d\mathbf{q}_{3\perp}[(\mathbf{r}_1+\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1-\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1+\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1-\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1+\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1-\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}K(\mathbf{r}_1,\mathbf{q}_{1\perp},\mathbf{q}_{2\perp})\times K(\mathbf{r}_1,\mathbf{q}_{2\perp},\mathbf{q}_{3\perp})\mathcal{g}^e(\mathbf{r}_1,\mathbf{q}_{1\perp})\mathcal{g}^e(\mathbf{r}_1,\mathbf{q}_{3\perp}), \quad (5.1)$$

where $K(\mathbf{r}_1,\mathbf{q}_{1\perp},\mathbf{q}_{2\perp})$ is precisely the function defined by (2.14).

Analogous to (2.1)–(2.6), the various momenta that appear in Fig. 5(a) are taken to be

$$\begin{aligned}
\mathbf{k}_1&=[(\beta_1+\beta_2)\omega,\ -\mathbf{r}_1-\mathbf{q}_{1\perp}],\\
\mathbf{k}_2&=[(\beta_1+\beta_2)\omega,\ \mathbf{r}_1-\mathbf{q}_{1\perp}],\\
\mathbf{k}_3&=[(\gamma_1+\gamma_2)\omega,\ -\mathbf{r}_1-\mathbf{q}_{2\perp}],\\
\mathbf{k}_4&=[(\gamma_1+\gamma_2)\omega,\ \mathbf{r}_1-\mathbf{q}_{2\perp}],\\
\mathbf{P}&=[(1-\beta_1-\beta_2)\omega,\ \mathbf{q}_{1\perp}],\\
\mathbf{p}_1&=[\beta_1\omega,\ \mathbf{p}_{1\perp}],\\
\mathbf{p}_2&=[\beta_2\omega,\ -\mathbf{r}_1-\mathbf{q}_{1\perp}-\mathbf{p}_{1\perp}],\\
\mathbf{p}_3&=[(\beta_2-\gamma_1-\gamma_2)\omega,\ -\mathbf{q}_{1\perp}+\mathbf{q}_{2\perp}-\mathbf{p}_{1\perp}],\\
\mathbf{p}_4&=[(\beta_1+\gamma_1+\gamma_2)\omega,\ \mathbf{r}_1-\mathbf{q}_{2\perp}+\mathbf{p}_{1\perp}],\\
\mathbf{p}_5&=[\gamma_1\omega,\ \mathbf{p}_{5\perp}],\\
\mathbf{p}_6&=[\gamma_2\omega,\ -\mathbf{r}_1-\mathbf{q}_{2\perp}-\mathbf{p}_{5\perp}],\\
\mathbf{p}_7&=[\gamma_2\omega,\ -\mathbf{q}_{2\perp}+\mathbf{q}_{3\perp}-\mathbf{p}_{5\perp}],
\end{aligned}$$

and

$$p_8=[\gamma_1\omega,\ \mathbf{r}_1-\mathbf{q}_{3\perp}+\mathbf{p}_{5\perp}]. \quad (5.2)$$

The region which contributes to the $s(\ln s)^2$ terms is that region where β_1, β_2, γ_1, and γ_2 are all small and, furthermore, $\gamma_1+\gamma_2\ll\beta_1+\beta_2$. We shall therefore concentrate on this region. As before, the lower limits of integration for $\beta_1+\beta_2$ and $\gamma_1+\gamma_2$ will be understood to be cut off at ω^{-1}. The scattering amplitude corresponding to Fig. 5(a) is then given by

$$\tfrac{1}{8}i\omega^2m^{-2}e^{12}(2\pi)^{-14}\int\prod_{i=1}^{3}d\mathbf{q}_{i\perp}d\mathbf{p}_{1\perp}d\mathbf{p}_{5\perp}\times\int_0^1 d\beta_1d\beta_2d\gamma_1d\gamma_2\theta(1-\beta_\perp-\beta_2)\theta(\beta_2-\gamma_1-\gamma_2)\times\mathrm{Tr}[(\gamma_0-\gamma_3)(-p_2+m)\gamma_\mu(-p_3+m)(\gamma_0-\gamma_3)(p_4+m)\times\gamma_\nu(p_1+m)]\,\mathrm{Tr}[\gamma_\mu(-p_6+m)\gamma_0(-p_7+m)\gamma_\nu(p_8+m)\times\gamma_0(p_5+m)][(\mathbf{r}_1+\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1+\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}P_-(\mathbf{r}_1+\mathbf{q}_{3\perp})\times P_+(\mathbf{r}_1-\mathbf{q}_{3\perp})[\beta_1\mathbf{p}_{2\perp}^2+\beta_2\mathbf{p}_{1\perp}^2+(\beta_1+\beta_2)m^2]^{-1}\times[\beta_1\mathbf{p}_{3\perp}^2+\beta_2\mathbf{p}_{4\perp}^2+(\beta_1+\beta_2)m^2]^{-1}\times[\gamma_1\mathbf{p}_{6\perp}^2+\gamma_2\mathbf{p}_{5\perp}^2+(\gamma_1+\gamma_2)m^2]^{-1}\times[\gamma_1\mathbf{p}_{7\perp}^2+\gamma_2\mathbf{p}_{8\perp}^2+(\gamma_1+\gamma_2)m^2]^{-1}\delta_{12}\delta_{1'2'}. \quad (5.3)$$

To evaluate (5.3), we must note the following two points: (i) We may approximate $\gamma_\mu\cdots\gamma_\mu$ and $\gamma_\nu\cdots\gamma_\nu$ in (5.3) by $\frac{1}{2}(\gamma_0+\gamma_3)\cdots(\gamma_0-\gamma_3)$, where $(\gamma_0+\gamma_3)$ is to be inserted in the first trace. This is because

$$\gamma_\mu\cdots\gamma_\mu=\tfrac{1}{2}(\gamma_0+\gamma_3)\cdots(\gamma_0-\gamma_3)+\tfrac{1}{2}(\gamma_0-\gamma_3)\cdots(\gamma_0+\gamma_3)+\boldsymbol{\gamma}_\perp\cdots\boldsymbol{\gamma}_\perp,$$

and the last two terms in the above equation can be shown to give terms of the order of $s\ \ln s$ only. (ii) We make the change of variables $\beta_1=\rho x$, $\beta_2=\rho(1-x)$, $\gamma_1=\rho'x'$, and $\gamma_2=\rho'(1-x')$ in (5.3), and carry out the integration over ρ and ρ', obtaining

$$\int_{\omega^{-1}}^1\rho^{-1}d\rho\int_{\omega^{-1}}^{\rho}\rho'^{-1}d\rho'=\tfrac{1}{2}(\ln\omega)^2\sim\tfrac{1}{8}(\ln s)^2. \quad (5.4)$$

After these manipulations (5.3) becomes

$$i\omega^2(\ln\omega)^2e^8(2\pi)^{-14}\int\prod_{i=1}^{3}d\mathbf{q}_{i\perp}d\mathbf{p}_{1\perp}d\mathbf{p}_{5\perp}\int_0^1dx\int_0^1dx'\times\mathrm{Tr}[(-p_{2\perp}+m)(-p_{3\perp}+m)(p_{4\perp}+m)(p_{1\perp}+m)]\times\mathrm{Tr}[(-p_{6\perp}+m)(-p_{7\perp}+m)(p_{8\perp}+m)(p_{5\perp}+m)]\times[(\mathbf{r}_1+\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{1\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1+\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{2\perp})^2+\lambda^2]^{-1}\times[(\mathbf{r}_1+\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}\times[x\mathbf{p}_{2\perp}^2+(1-x)\mathbf{p}_{1\perp}^2+m^2]^{-1}\times[x\mathbf{p}_{3\perp}^2+(1-x)\mathbf{p}_{4\perp}^2+m^2]^{-1}\times[x'\mathbf{p}_{7\perp}^2+(1-x')\mathbf{p}_{8\perp}^2+m^2]^{-1}\times[x'\mathbf{p}_{6\perp}^2+(1-x')\mathbf{p}_{5\perp}^2+m^2]^{-1}\times\mathcal{g}^e(\mathbf{r}_1,\mathbf{q}_{1\perp})\mathcal{g}^e(\mathbf{r}_1,\mathbf{q}_{3\perp}). \quad (5.5)$$

In the above, we have replaced P_+P_- in (5.3) by its lowest-order term $[(\mathbf{r}_1+\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{3\perp})^2+\lambda^2]^{-1}$ (see Sec. 5 of Ref. 9), since we shall be interested in only the lowest-order term in Eq. (5.3). Equation (5.5) is merely half of the contribution for diagram 5(a). This is because the virtual photon of momentum $\mathbf{k}_3$ may be emitted from the electron of momentum $\mathbf{p}_1$ instead of from the positron of momentum $\mathbf{p}_2$. Further-

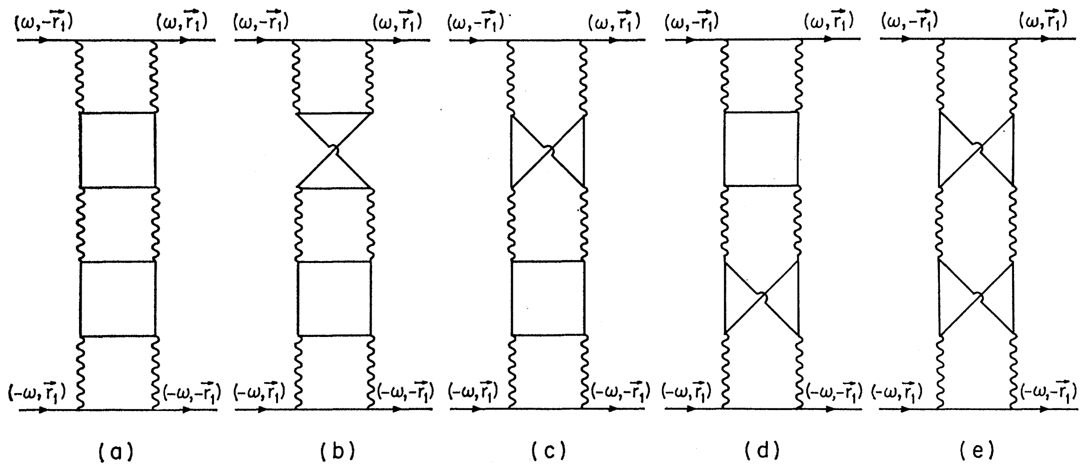

FIG. 6. Feynman diagrams corresponding to Fig. 5.

more, the amplitude for diagram 5(e) is exactly equal to that for diagram 5(a), while that for diagram 5(i) is twice of that for diagram 5(a). The latter is because for diagram 5(i), we have, instead of (5.4),

$$\int_{\omega-1}^{1}\frac{d\rho}{\rho}\int_{\omega-1}^{1}\frac{d\rho'}{\rho'}=(\ln\omega)^2. \qquad (5.6)$$

Thus we must multiply (5.5) by a factor of 8. Then (5.5) is equal to (5.1), with K replaced by K_a, which is the second term in (2.14), i.e., the contribution of diagram 3(a) to K.

The sum of diagrams of Figs. 5(d), 5(h), and 5(l) gives an amplitude equal to (5.1) with K replaced by K_b, the first term in (2.14), and the sum of diagrams of Figs. 5(b), 5(c), 5(f), 5(g), 5(j), and 5(k) gives an amplitude equal to (5.1) with KK replaced by $K_aK_b+K_bK_a$. Thus the scattering amplitude for the sum of the twelve diagrams in Fig. 5 is equal to (5.1).

Before going on to study higher powers of $\ln s$, we attempt to rewrite (5.1) in a somewhat neater form. Let

$$\begin{aligned}\mathcal{K}(\mathbf{r}_1,\mathbf{q}_{1\perp},\mathbf{q}_{2\perp})&=[(\mathbf{r}_1+\mathbf{q}_{1\perp})^2+\lambda^2]^{-1/2}\\&\times[(\mathbf{r}_1-\mathbf{q}_{1\perp})^2+\lambda^2]^{-1/2}[(\mathbf{r}_1+\mathbf{q}_{2\perp})^2+\lambda^2]^{-1/2}\\&\times[(\mathbf{r}_1-\mathbf{q}_{2\perp})^2+\lambda^2]^{-1/2}K(\mathbf{r}_1,\mathbf{q}_{1\perp},\mathbf{q}_{2\perp}),\end{aligned} \qquad (5.7)$$

and define the corresponding operator $\mathcal{K}$ by

$$(\mathcal{K}F)(\mathbf{r}_1,\mathbf{q}_{1\perp})=(2\pi)^{-2}\int d\mathbf{q}_{2\perp}\mathcal{K}(\mathbf{r}_1,\mathbf{q}_{1\perp},\mathbf{q}_{2\perp})F(\mathbf{r}_1,\mathbf{q}_{2\perp}). \qquad (5.8)$$

It is also convenient to use the notation of scalar products

$$(F_1,F_2)=(2\pi)^{-2}\int d\mathbf{q}_\perp F_1(\mathbf{r}_1,\mathbf{q}_\perp)F_2(\mathbf{r}_1,\mathbf{q}_\perp), \qquad (5.9)$$

which is a function of $\mathbf{r}_1$. In (5.9) we have omitted complex conjugations because we are dealing with real functions. Finally, let

$$J^e(\mathbf{r}_1,\mathbf{q}_\perp)=[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1/2}\times[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1/2}g^e(\mathbf{r}_1,\mathbf{q}_\perp). \qquad (5.10)$$

In this notation, the matrix element (2.15) due to the diagrams of Fig. 3 is simply

$$is\ \ln s(J^e,\mathcal{K}J^e), \qquad (5.11)$$

provided that only the lowest-order terms in P_+ and P_- are kept, while the result (5.1) is

$$\tfrac{1}{2}is(\ln s)^2(J^e,\mathcal{K}^2J^e), \qquad (5.12)$$

where $\mathcal{K}^2=\mathcal{K}\mathcal{K}$.

We now proceed to discuss the lowest-order impact diagrams with n electron loops. They are also the lowest-order impact diagrams which yield uncanceled $s(\ln s)^n$ terms. Some typical diagrams of this kind are illustrated in Fig. 7. Instead of (5.3), we have for Fig. 6(a)

$$\int_{\omega-1}^{1}\frac{d\rho_1}{\rho_1}\int_{\omega-1}^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_{\omega-1}^{\rho_{n-1}}\frac{d\rho_n}{\rho_n}=(n!)^{-1}(\ln\omega)^n\sim(1/n!)(\tfrac{1}{2})^n(\ln s)^n. \qquad (5.13)$$

We also have the following factors of 2: (i) a factor $(2^{-2})^n$ from replacing $\gamma_\mu\cdots\gamma_\mu$ by $\frac{1}{2}(\gamma_0+\gamma_3)\cdots(\gamma_0-\gamma_3)$; (ii) a factor $(2^3)^n$ from contracting the $(\gamma_0+\gamma_3)$ and $(\gamma_0-\gamma_3)$ factors in the traces; (iii) a factor 2^n to take care of the fact that both the electron and the positron in a loop can emit a photon for the creation of the next loop; (iv) a factor of $(2^{-4})^n$ from rule 7 of Ref. 9 for the virtual electrons in the loop; (v) a factor $(2^2)^n$ from rule 7 of Ref. 9 for the denominator factors connected with the loops. These five factors completely cancel each other, and we are left with a numerical factor $(\frac{1}{2})^n/n!$ from (5.13).

For Fig. 7(b), we have, instead of (5.13),

$$\left(\int_{\omega-1}^{1}\frac{d\rho_1}{\rho_1}\int_{\omega-1}^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_{\omega-1}^{\rho_{n-2}}\frac{d\rho_{n-1}}{\rho_{n-1}}\right)\left(\int_{\omega-1}^{1}\frac{d\rho_n}{\rho_n}\right)\sim[(n-1)!]^{-1}(\tfrac{1}{2})^n(\ln s)^n, \qquad (5.14)$$

and similarly for other n-loop impact diagrams. Adding up the amplitude from diagrams 7(a) and 7(b) as well as those from all other n-loop diagrams of this kind, we get

$$(\tfrac{1}{2})^n(n!)^{-1}\{1+n+[n(n-1)/2!]+\cdots+1\} = (\tfrac{1}{2})^n(n!)(1+1)^n=(n!)^{-1}. \quad (5.15)$$

For the purpose of consistency, we shall again replace $P_+(\mathbf{r}_1+\mathbf{q}_{i\perp})P_-(\mathbf{r}_1-\mathbf{q}_{i\perp})$ from the dashed lines of Fig. 7 by its lowest approximation

$$[(\mathbf{r}_1+\mathbf{q}_{i\perp})^2+\lambda^2]^{-1}[(\mathbf{r}_1-\mathbf{q}_{i\perp})^2+\lambda^2]^{-1}.$$

Then the sum of the amplitudes from all the impact diagrams of the type in Fig. 7 gives

$$is(\ln s)^n(n!)^{-1}(2\pi)^{-2n-2}\int\prod_1^{n+1}d\mathbf{q}_{i\perp}\prod_1^{n+1}[(\mathbf{r}_1+\mathbf{q}_{i\perp})^2+\lambda^2]^{-1} \times[(\mathbf{r}_1-\mathbf{q}_{i\perp})^2+\lambda^2]^{-1}\prod_1^{n}K(\mathbf{q}_{i\perp},\mathbf{q}_{(i+1)\perp},\mathbf{r}_1) \times\mathcal{G}^e(\mathbf{r}_1,\mathbf{q}_{1\perp})\mathcal{G}^e(\mathbf{r}_1,\mathbf{q}_{(n+1)\perp}). \quad (5.16)$$

By the notation of (5.8)–(5.10), this complicated looking (5.16) is simply

$$i(n!)^{-1}s(\ln s)^n(J^e,\mathcal{K}^nJ^e). \quad (5.17)$$

We may easily generalize the above results to other diffractive processes in quantum electrodynamics. For the process $a+b\to a'+b'$, the lowest-order impact diagrams which give uncanceled $s(\ln s)^n$ terms are those with n more electron loops than the lowest-order impact diagrams which gives terms proportional to s. The sum of these amplitudes is equal to (5.16), with $\mathcal{G}^e\mathcal{G}^e$ replaced by $\mathcal{G}^{aa'}\mathcal{G}^{bb'}$.[26] If, analogously to (5.10), we define $J^{aa'}$ by

$$J^{aa'}(\mathbf{r}_1,\mathbf{q}_\perp)=[(\mathbf{r}_1+\mathbf{q}_\perp)^2+\lambda^2]^{-1/2} \times[(\mathbf{r}_1-\mathbf{q}_\perp)^2+\lambda^2]^{-1/2}\mathcal{G}^{aa'}(\mathbf{r}_1,\mathbf{q}_\perp), \quad (5.18)$$

then the amplitude mentioned above is

$$i(n!)^{-1}s(\ln s)^n(J^{aa'},\mathcal{K}^nJ^{bb'}). \quad (5.19)$$

We emphasize that the operator $\mathcal{K}$ does not depend on what a, a', b, and b' are.

6. SPECTRUM OF $\mathcal{K}$ IN FORWARD DIRECTION

A. Formulation

Because of (5.17) and (5.19), where various iterations of the kernel $\mathcal{K}(\mathbf{r},\mathbf{q}_\perp,\mathbf{q}_\perp')$ appear, it is desirable to study the spectrum of this operator $\mathcal{K}$ as defined by (5.8). We have been able to carry out such an analysis only for the forward direction, where $\mathbf{r}_1=0$. As discussed in Sec. 7, there are serious difficulties even in the understanding of the result for the forward direction. For this reason, we do not consider the generalization to other directions to be the most urgent problem.

When $\mathbf{r}_1=0$, there is rotational invariance for electron-electron scattering. Let θ be the angle between the two-dimensional vectors $\mathbf{q}_\perp$ and $\mathbf{q}_\perp'$, then the K_0 of (4.16) can be written in the form

$$K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')=\mathbf{q}_\perp^2\mathbf{q}_\perp'^2\int_0^1dx\int_0^1dy\frac{[x(1-x)+y(1-y)]-2x(1-x)y(1-y)(2+\cos^2\theta)}{x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2+m^2}. \quad (6.1)$$

As seen from (5.10), we are only interested in applying this kernel to functions that are rotationally invariant. We can therefore average over θ in (6.1) to get

$$\langle K_0(\mathbf{q}_\perp,\mathbf{q}_\perp')\rangle_\theta=zz'\int_0^1dx\int_0^1dy\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}, \quad (6.2)$$

where

$$z=\mathbf{q}_\perp^2 \quad\text{and}\quad z'=\mathbf{q}_\perp'^2. \quad (6.3)$$

Analogous to (5.7), define the kernal

$$\mathcal{K}_0(z,z')=(z+\lambda^2)^{-1}(z'+\lambda^2)^{-1}\langle K_0(q_\perp,q_\perp')\rangle_\theta =\frac{z}{z+\lambda^2}\frac{z'}{z'+\lambda^2}\int_0^1dx\int_0^1dy\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'+m^2}, \quad (6.4)$$

and the corresponding operator $\mathcal{K}_0$ by[27]

$$(\mathcal{K}_0f)(z)=\int_0^\infty dz'\mathcal{K}_0(z,z')f(z'). \quad (6.5)$$

For the sake of mathematical rigor, we let $f(z)$ be elements of the L_2 space, i.e., we consider those $f(z)$ that satisfy

$$\int_0^\infty|f(z)|^2dz<\infty. \quad (6.6)$$

We shall see that $\mathcal{K}_0$ is a bounded operator on L_2. Therefore, as a consequence of

$$\mathcal{K}_0(z,z')=\mathcal{K}_0(z',z), \quad (6.7)$$

the spectrum $\mathcal{S}$ of $\mathcal{K}_0$ is a real, bounded, closed set. Let μ_0 be the lowest upper bound of this set. It is the purpose of this section to calculate μ_0.

B. Result

Our result here is simply

$$\mu_0=11\pi^3/64, \quad (6.8)$$

independent of m and λ.

[26] H. Cheng and T. T. Wu, Phys. Rev. D 1, 459 (1970).
[27] Note that $\mathcal{K}_0$ is not a Fredholm operator.

The remainder of this section is devoted to a derivation of (6.8). The procedure followed is roughly as follows. In Sec. 6 C we study in detail the special case $m=\lambda=0$. This case can be exactly solved by Mellin transformation or, equivalently, Fourier transformation. We can thus verify (6.8) directly for this special case, and this implies that $\mu_0 \leq 11\pi^3/64$ for all m and λ. In Sec. 6 D we show, by a variational principle, that $\mu_0 \geq 11\pi^3/64$ from which (6.8) follows.

The variational principle also yields the additional result that this end point $11\pi^3/64$ is not a point spectrum and hence belongs to the continuum.

At first glance, the lack of dependence on m and λ may seem peculiar. In Appendix B we give an explicit mathematical example where this happens. It is hoped that this example may make the result (6.8) appear more natural.

C. Case $m=\lambda=0$

We first study the solvable special case $m=\lambda=0$, where

$$\mathcal{K}_0(z,z')=\int_0^1 dx\int_0^1 dy \times\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)z+y(1-y)z'}. \quad (6.9)$$

Let

$$z=e^{\xi}, \qquad z'=e^{\xi'}, \quad (6.10)$$

and

$$f(z)=e^{-\xi/2}g(\xi). \quad (6.11)$$

The reason for using (6.11) is the fact that (6.6) is equivalent to

$$\int_{-\infty}^{\infty} |g(\xi)|^2 d\xi<\infty. \quad (6.12)$$

In the ξ space, we need to study the kernel

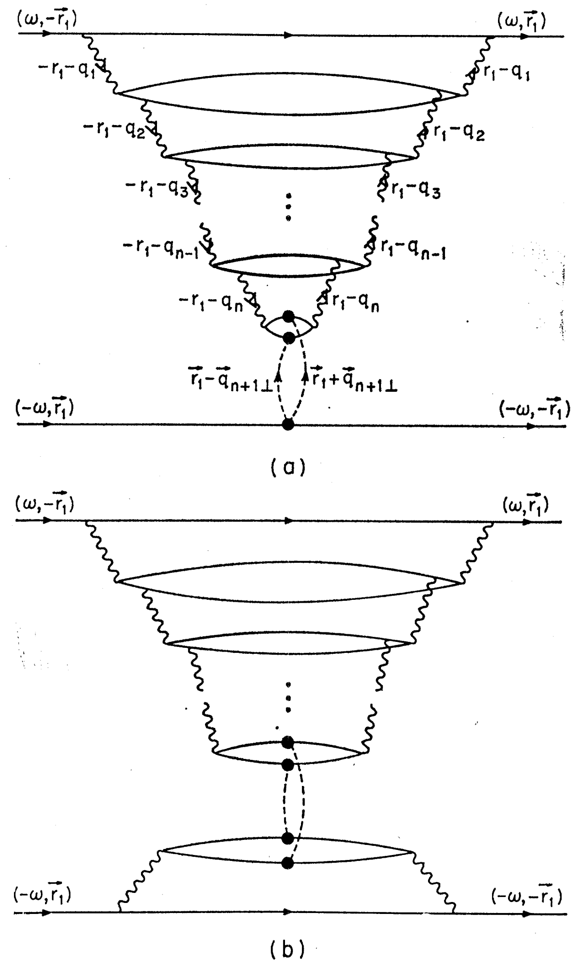

FIG. 7. Examples of lowest-order impact diagrams that give rise to n logarithmic factors at high energies.

$$\exp[(\xi+\xi')/2]\int_0^1 dx\int_0^1 dy\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)e^{\xi}+y(1-y)e^{\xi'}}$$

$$=\int_0^1 dx\int_0^1 dy\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)\exp[(\xi-\xi')/2]+y(1-y)\exp[-(\xi-\xi')/2]}, \quad (6.13)$$

which is a function of $\xi-\xi'$ only. The Fourier transform of (6.13) is

$$\int_{-\infty}^{\infty} d\xi \exp(i\xi\tau)\int_0^1 dx\int_0^1 dy\frac{x(1-x)+y(1-y)-5x(1-x)y(1-y)}{x(1-x)e^{\xi/2}+y(1-y)e^{-\xi/2}}$$

$$=\pi\,\mathrm{sech}\pi\tau\int_0^1 dx\int_0^1 dy[x(1-x)+y(1-y)-5x(1-x)y(1-y)]$$

$$\times[x(1-x)y(1-y)]^{-1/2}\exp\{-i\tau[\ln x(1-x)-\ln y(1-y)]\}$$

$$=\pi\,\mathrm{sech}\pi\tau\left\{2\,\mathrm{Re}\frac{[\Gamma(\tfrac{3}{2}-i\tau)]^2[\Gamma(\tfrac{1}{2}+i\tau)]^2}{\Gamma(3-2i\tau)\Gamma(1+2i\tau)}-5\frac{[\Gamma(\tfrac{3}{2}-i\tau)]^2[\Gamma(\tfrac{3}{2}+i\tau)]^2}{\Gamma(3-2i\tau)\Gamma(3+2i\tau)}\right\}$$

$$=\frac{(\pi\,\mathrm{sech}\pi\tau)^3}{2\tau\pi\,\mathrm{csch}2\pi\tau}\left(\tfrac{1}{2}\,\mathrm{Re}\frac{1-2i\tau}{2-2i\tau}-\frac{5}{16}\frac{1+4\tau^2}{4+4\tau^2}\right)$$

$$=\frac{\pi^2}{64}\frac{11+12\tau^2}{1+\tau^2}\frac{\sinh\pi\tau}{\tau\cosh^2\pi\tau}. \quad (6.14)$$

For real values of τ, the right-hand side of (6.14) takes on all real values between 0 and $11\pi^3/64$. Since this is bounded, it follows from (6.12) that, when $m=\lambda=0$, the spectrum of $\mathcal{K}_0$ is the closure of the values taken by the right-hand side of (6.14); i.e., the spectrum of $\mathcal{K}_0$ is

$$[0, 11\pi^3/64]. \tag{6.15}$$

D. Proof of (6.8)

We first note that

$$x(1-x)+y(1-y)-5x(1-x)y(1-y)>0 \tag{6.16}$$

for $0<x<1$ and $0<y<1$. Accordingly, it follows from (6.15) that

$$\int_0^\infty |(\mathcal{K}_0 f)(z)|^2 dz \leq (11\pi^3/64)^2 \int_0^\infty |f(z)|^2 dz \tag{6.17}$$

and hence

$$\mu_0 \leq 11\pi^3/64 \tag{6.18}$$

for all m and λ. Equation (6.17) further implies that $\mathcal{K}_0$ is a bounded operator on L_2.

On the other hand, since

$$\mu_0 = \sup \frac{\int_0^\infty dz \int_0^\infty dz' f(z) f(z') \mathcal{K}_0(z, z')}{\int_0^\infty [f(z)]^2 dz}, \tag{6.19}$$

over all real, nonzero $f(z)$ that satisfies (6.6), we can obtain a lower bound for μ_0 by trying some $f(z)$. In particular, we choose, for all $\Lambda>1$,[28]

$$\begin{aligned} f(z, \Lambda) &= z^{-3/2}(z+\lambda^2) \qquad \text{for} \quad 1 \leq z \leq \Lambda \\ &= 0 \qquad \text{otherwise}. \end{aligned} \tag{6.20}$$

Then

$$\mu_0 \geq \sup_\Lambda [\ln\Lambda + 2\lambda^2(1-\Lambda^{-1}) + \tfrac{1}{2}\lambda^4(1-\Lambda^{-2})]^{-1} \times \int_0^\infty dz \int_0^\infty dz' f(z, \Lambda) f(z', \Lambda) \mathcal{K}_0(z, z'). \tag{6.21}$$

The reason for this choice of $f(z, \Lambda)$ is as follows. Since the right-hand side of (6.14) has a maximum at $\tau=0$, the eigenfunction that corresponds to μ_0 is, by (6.10) and (6.11), simply $z^{-1/2}$, which of course does not satisfy (6.6) and holds only for $m=\lambda=0$. The $f(z, \Lambda)$ of (6.20) is essentially the product of this function $z^{-1/2}$ with the inverse of the factor $z/(z+\lambda^2)$ that appears in (6.4). By explicit calculation, it is easy to verify that

$$\lim_{\Lambda\to\infty} (\ln\Lambda)^{-1} \int_1^\Lambda dz \int_1^\Lambda dz' z^{-1/2} z'^{-1/2} \mathcal{K}_0(z, z') \big|_{m=\lambda=0} = 11\pi^3/64. \tag{6.22}$$

[28] The choice of this 1 is completely arbitrary.

If (6.22) is substituted into (6.21), we get in particular

$$\mu_0 \geq 11\pi^3/64 - \lim_{\Lambda\to\infty} (\ln\Lambda)^{-1} I(\Lambda, m), \tag{6.23}$$

where, by (6.4),

$$\begin{aligned} I(\Lambda, m) = \int_1^\Lambda dz \int_1^\Lambda dz' z^{-1/2} z'^{-1/2} \int_0^1 dx \int_0^1 dy \\ \times [x(1-x)+y(1-y)-5x(1-x)y(1-y)] \\ \times \{[x(1-x)z+y(1-y)z']^{-1} \\ -[x(1-x)z+y(1-y)z'+m^2]^{-1}\}. \end{aligned} \tag{6.24}$$

Note that $I(\Lambda, m)$ does not depend on λ and that the integrand of (6.24) is non-negative because of (6.16).

Suppose for the moment that

$$\begin{aligned} I(m) = \int_1^\infty dz \int_1^\infty dz' z^{-1/2} z'^{-1/2} \int_0^1 dx \int_0^1 dy \\ \times [x(1-x)+y(1-y)-5x(1-x)y(1-y)] \\ \times \{[x(1-x)z+y(1-y)z']^{-1} \\ -[x(1-x)z+y(1-y)z'+m^2]^{-1}\} \end{aligned} \tag{6.25}$$

exists; then

$$\lim_{\Lambda\to\infty} I(\Lambda, m) = I(m),$$

and hence by (6.23)

$$\mu_0 \geq 11\pi^3/64. \tag{6.26}$$

The required answer (6.8) then follows from (6.18) and (6.26).

Therefore it only remains to show that the $I(m)$ of (6.25) exists. For this purpose, change the variables x, y according to (4.24) and the variables z, z' by

$$z=u^2, \qquad z'=u'^2; \tag{6.27}$$

then

$$\begin{aligned} I(m) = 4\int_0^1 d\bar{x} \int_0^1 d\bar{y} [(1-\bar{x}^2)+(1-\bar{y}^2) \\ -(5/4)(1-\bar{x}^2)(1-\bar{y}^2)] \int_1^\infty du \int_1^\infty du' \\ \times \{[(1-\bar{x}^2)u^2+(1-\bar{y}^2)u'^2]^{-1} \\ -[(1-\bar{x}^2)u^2+(1-\bar{y}^2)u'^2+4m^2]^{-1}\}. \end{aligned} \tag{6.28}$$

By symmetry, we can integrate over the region $\bar{x}>\bar{y}$ so that

$$1-\bar{x}^2 < 1-\bar{y}^2. \tag{6.29}$$

Therefore

$$I(m)=8\int_0^1 d\bar{x}\int_0^{\bar{x}} d\bar{y}[(1-\bar{x}^2)+(1-\bar{y}^2)-(5/4)(1-\bar{x}^2)(1-\bar{y}^2)](1-\bar{x}^2)^{-1/2}(1-\bar{y})^{-1/2}\times\int_{(1-\bar{x}^2)^{1/2}}^{\infty} du\int_{(1-\bar{y}^2)^{1/2}}^{\infty} du'\times[(u^2+u'^2)^{-1}-(u^2+u'^2+m^2)^{-1}]$$

$$\leq 8\int_0^1 d\bar{x}\int_0^{\bar{x}} d\bar{y}(1-\bar{x}^2)^{-1/2}(1-\bar{y}^2)^{-1/2}\times[(1-\bar{x}^2)+(1-\bar{y}^2)-(5/4)(1-\bar{x}^2)(1-\bar{y}^2)]\times\int_{u^2+u'^2>1-\bar{x}^2} dudu'[(u^2+u'^2)^{-1}-(u^2+u'^2+4m^2)^{-1}]$$

$$=2\pi\int_0^1 d\bar{x}\int_0^{\bar{x}} d\bar{y}(1-\bar{x}^2)^{-1/2}(1-\bar{y}^2)^{-1/2}\times[(1-\bar{x}^2)+(1-\bar{y}^2)-(5/4)(1-\bar{x}^2)(1-\bar{y}^2)]\times\ln[1+4m^2/(1-\bar{x}^2)]<\infty. \quad (6.30)$$

This completes the proof.

7. CONCLUSIONS AND DISCUSSIONS

As already stated in the Introduction, there are, in the perturbation series for quantum electrodynamics, terms of the matrix elements proportional to $s(\ln s)^n$, $n=1, 2, 3, \ldots$, when $s\to\infty$ with fixed t. For electron-electron elastic scattering discussed in detail in this paper, these terms first appear in the $4(n+1)$th order, i.e., the coefficient is proportional to $e^{4(n+1)}$, where $n=1, 2, 3, \ldots$. Moreover, these coefficients, to this leading order, are explicitly given in Sec. 5. For electron-photon scattering and photon-photon scattering, the corresponding orders are, respectively, $4n+6$ and $4n+8$. We emphasize that *the appearance of these* $s(\ln s)^n$ *terms holds for all* $t\neq0$, *for both massive and massless photons*. These logarithmic factors are therefore not related to, but rather in addition to, the more familiar logarithmic factors due to the massless nature of the photon, an example being the factor in the total pair-production cross section.[29]

We are only beginning to realize the existence and importance of these terms, and there is as yet no satisfactory understanding of them. We must, for the time being, be content with the most elementary properties of these terms, discussed in Secs. 7 A and 7 B.

A. Leading Coefficients

Although some of the results can be easily generalized to nonforward directions, we shall restrict ourselves to the forward direction $\mathbf{r}_1=0$, where a more complete discussion is possible. By Sec. 6 A, we write down the spectral decomposition[30] of $\mathcal{K}$ for $\mathbf{r}_1=0$:

$$\mathcal{K}_0(z,z')=\int_S \mu d\mu\, \phi(z,\mu)\phi(z',\mu). \quad (7.1)$$

By (5.8), (5.7), (4.1), (6.4), and (6.5), the eigenvalues of $\mathcal{K}$ are, when $\mathbf{r}_1=0$,

$$2(\alpha/\pi)^2\mu, \quad (7.2)$$

where $\alpha=e^2/(4\pi)$ is the fine-structure constant. Accordingly, by (6.8), the lowest upper bound for the spectrum of $\mathcal{K}$ at $\mathbf{r}_1=0$ is

$$\tfrac{11}{32}\alpha^2\pi. \quad (7.3)$$

By (5.10), define the coefficients

$$a(\mu)=(2\pi)^{-2}\int d\mathbf{q}_\perp\phi(\mathbf{q}_\perp^2,\mu)(\mathbf{q}_\perp^2+\lambda^2)^{-1}, \quad (7.4)$$

then the matrix element (5.17) is

$$i(n!)^{-1}s(\ln s)^n g^e g^e \times\left\{[2(\alpha/\pi)^2]^n\int_S \mu^n d\mu[a(\mu)]^2\right\}. \quad (7.5)$$

Note that (7.5), when divided by i, is positive for all n.

An important property of the $a(\mu)$ of (7.4) is that

$$\bar{a}(\mu)=\lim_{\mu\to\mu_0}(1-\mu/\mu_0)^{-1/2}a(\mu)\neq0, \quad (7.6)$$

where μ_0 has been defined in Sec. 6 A to be the lowest upper bound of S. First, $a(\mu)$ is finite because, from Sec. 6 D, as $z\to\infty$

$$\phi(z,\mu)=O(z^{-1/2}). \quad (7.7)$$

Also note the fact that, because $\mathcal{K}_0(z,z')>0$ from (6.4), $\phi(\mathbf{q}_\perp^2,\mu_0)$ is either non-negative or non-positive, depending on the choice.

This inequality (7.6) makes it possible to calculate the asymptotic behavior of (7.5) for large n. It follows immediately from (6.8) that

$$(5.17)\sim i[(n+2)!]^{-1}s(\ln s)^n g^e g^e(11\pi^3/64)\times[\tfrac{11}{32}\alpha^2\pi]^n[\bar{a}(\mu_0)]^2, \quad (7.8)$$

as $n\to\infty$.

B. Sum of Leading Terms

It is tempting to sum (5.17) or, equivalently, (7.5) over all n. Such a calculation is sometimes referred to as summing the leading terms, and has been discussed in great detail[31] before. Such a procedure has no mathematical basis, but was used with great success over a decade ago in a number of many-body problems. Much later, attempts were made to apply similar considerations to field-theoretic problems, renormalizable[32]

[29] R. Jost, J. M. Luttinger, and M. Slotnick, Phys. Rev. **80**, 189 (1950).

[30] Strictly speaking, the spectral decomposition should be written in the form of a Stieltjes intergral. See, for sample, F. Riesz and B. Nagy, *Functional Analysis* (Ungar, New York, 1955), p. 275.

[31] T. T. Wu, Phys. Rev. **149**, 380 (1966). The procedure of summing the leading terms is discussed in Sec. 8(4).

[32] See, for example, M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962); J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

or otherwise.[33] The physical meaning of these attempts is not clear. About three years ago, as a possible check on the validity of this procedure of summing the leading terms, the soluble problem of the two-dimensional Ising model[34] was studied from this point of view. It was found[31] that this procedure gives *a finite but wrong answer*. Thus summing the leading terms does not assure us of the correct answer.

With these reservations in mind, we sum (5.17) over n:

$$\sum_{n=0}^{\infty} i(n!)^{-1}s(\ln s)^n(J^e, \mathcal{K}^n J^e) = is[J^e, \exp(\mathcal{K}\ln s)J^e] = is(J^e, s^{\mathcal{K}}J^e). \quad (7.9)$$

While (7.9) holds for all momentum transfers on which $\mathcal{K}$ depends, specialization to the forward direction $\mathbf{r}_1=0$ gives more specifically

$$(7.9)\,|_{\mathbf{r}_1=0} = is\mathcal{G}^e\mathcal{G}^e\int_S d\mu[a(\mu)]^2\exp[2(\alpha/\pi)^2\mu\ln s] \sim \tfrac{1}{4}i\mathcal{G}^e\mathcal{G}^e(\pi/\alpha)^4(\ln s)^{-2}[\bar{a}(\mu_0)]^2 s^{1+11\alpha^2\pi/32} \quad (7.10)$$

for large s. Note that the power of s is larger than 1. For sufficiently small momentum transfers at least, the value of $\bar{a}(\mu_0)$ remains different from zero and the power of s remains larger than 1. We therefore reach the conclusion that, for the present problem, *the procedure of summing the leading terms gives an answer that violates s-channel unitarity.*

That the sum of leading terms gives such a large answer has far-reaching consequences. First, it raises doubts about the usual derivation[32] of Regge poles from field theory. Indeed, the procedure of obtaining (7.10) here is just the operator generalization of the usual derivation[32,35] of Regge poles. In our opinion, therefore, much work is needed to justify the existence of Regge poles in relativistic field theory.

Secondly, we note that renormalization plays a very minor role in the present consideration. Accordingly, the failure of the result (7.10) to satisfy unitarity cannot possibly be interpreted as a piece of evidence for the breakdown of quantum electrodynamics.

Finally, we emphasize the important role played by unitarity in the direct channel. Precisely on the basis of this unitarity we conclude that the answer (7.10) is incorrect. Unitarity can be partially restored by including the iterations in the s channel of the diagrams considered here, for example those of Fig. 1. This has been studied before,[36] and it is found that, if we start with the diagrams of Fig. 1 with the contribution (2.15) of the order s ln$s$, the sum of all iterations is smaller and of order s ln(ln$s$). Moreover, if we start with the sum (7.10), the sum of iterations[36] saturates the Froissart bound[37] and no longer violates s-channel unitarity. However, it is not certain that this process of taking iterations into account solves the problem. We think that the most important problem now is how to have some understanding of unitarity conditions and in particular of their role in determining the high-energy behavior of scattering amplitudes.

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APPENDIX A

In this appendix, we generalize the procedure of Secs. 4 A and 4 B to the case $\mathbf{r}_1\neq 0$. Let

$$K_1(\mathbf{q}_\perp, \mathbf{q}_\perp') = [4e^4(2\pi)^{-3}]^{-1}K(\mathbf{r}_1, \mathbf{q}_\perp, \mathbf{q}_\perp'), \quad (A1)$$

then the generalizations of (4.2) and (4.3) are

$$K_1(\mathbf{q}_\perp, \mathbf{q}_\perp') = \tfrac{1}{4}(2\pi)^{-1}\int d\mathbf{p}_\perp \int_0^1 dx[x(\mathbf{p}_\perp+\mathbf{r}_1)^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]^{-1} \times\left\{-\frac{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2]\,\mathrm{Tr}(\not{p}_\perp+\not{r}_1-m)(\not{p}_\perp-\not{r}_1+m)}{x(\mathbf{p}_\perp-\mathbf{r}_1)^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2} - \frac{\mathrm{Tr}(\not{p}_\perp+\not{r}_1-m)(\not{p}_\perp-\not{q}_\perp'+m)(\not{p}_\perp-\not{q}_\perp-\not{q}_\perp'+\not{r}_1-m)(\not{p}_\perp-\not{q}_\perp+m)}{x(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2}\right\} \quad (A2)$$

[33] See, for example, T. D. Lee, Phys. Rev. **128**, 899 (1962); G. Feinberg and A. Pais, *ibid.* **131**, 2724 (1963).
[34] E. Ising, Z. Physik **31**, 253 (1925); L. Onsager, Phys. Rev. **65**, 117 (1944).
[35] Regge poles correspond to the point spectrum (absent here) of the operator $\mathcal{K}$.
[36] H. Cheng and T. T. Wu, Phys. Rev. **186**, 1611 (1969).
[37] M. Froissart, Phys. Rev. **123**, 1053 (1961).

and

$$K_1(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\left\{\frac{\mathbf{p}_\perp^2-\mathbf{r}_1^2+m^2}{(\mathbf{p}_\perp+\mathbf{r}_1)^2-(\mathbf{p}_\perp-\mathbf{r}_1)^2}\ln\frac{(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2}{(\mathbf{p}_\perp-\mathbf{r}_1)^2+m^2}\right.$$
$$-\frac{1}{4}\frac{\mathrm{Tr}(p_\perp+r_1-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'+r_1-m)(p_\perp-q_\perp+m)}{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]-[(\mathbf{p}_\perp-q_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}$$
$$\left.\times\ln\frac{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]}{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}\right\}. \quad \text{(A3)}$$

To save some writing, we shall deal instead with the quantity [compare (4.3)]

$$K_D(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp$$
$$\times\left\{1-\frac{\frac{1}{4}\mathrm{Tr}(p_\perp+r_1-m)(p_\perp-q_\perp'+m)(p_\perp-q_\perp-q_\perp'+r_1-m)(p_\perp-q_\perp+m)}{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]-[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}\right.$$
$$\left.\times\ln\frac{[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2]}{[(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+m^2][(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+m^2]}\right\}. \quad \text{(A4)}$$

Since the first term in the integrand of (A3) is independent of both $\mathbf{q}_\perp$ and $\mathbf{q}_\perp'$, we have

$$K_D(\mathbf{q}_\perp,\mathbf{r}_1)=K_D(\mathbf{r}_1,\mathbf{q}_\perp')=K_D(\mathbf{r}_1,\mathbf{r}_1) \quad \text{(A5)}$$

and

$$K_1(\mathbf{q}_\perp,\mathbf{q}_\perp')=K_D(\mathbf{q}_\perp,\mathbf{q}_\perp')-K_D(\mathbf{r}_1,\mathbf{r}_1). \quad \text{(A6)}$$

That K_D is logarithmically divergent cannot cause any trouble.

If a second Feynman parameter y is introduced via (A2), K_D can be expressed in the form

$$K_D(\mathbf{q}_\perp,\mathbf{q}_\perp')=(2\pi)^{-1}\int d\mathbf{p}_\perp\int_0^1 dx\int_0^1 dy(1-N_1/D_1^2), \quad \text{(A7)}$$

where N_1 is the numerator that appears in (A4) and

$$D_1=xy(\mathbf{p}_\perp+\mathbf{r}_1)^2+(1-x)y(\mathbf{p}_\perp-\mathbf{q}_\perp)^2+x(1-y)(\mathbf{p}_\perp-\mathbf{q}_\perp')^2+(1-x)(1-y)(\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2$$
$$=\mathbf{p}_\perp^2-2\mathbf{p}_\perp\cdot[(1-x)\mathbf{q}_\perp+(1-y)\mathbf{q}_\perp'-(1-x-y+2xy)\mathbf{r}_1]+xy\mathbf{r}_1^2+(1-x)y\mathbf{q}_\perp^2+x(1-y)\mathbf{q}_\perp'^2$$
$$+(1-x)(1-y)(\mathbf{q}_\perp+\mathbf{q}_\perp'-\mathbf{r}_1)^2+m^2. \quad \text{(A8)}$$

We can again use (4.9), but here, instead of (4.10),

$$\delta\mathbf{p}_\perp=(1-x)\mathbf{q}_\perp+(1-y)\mathbf{q}_\perp'-(1-x-y+2xy)\mathbf{r}_1. \quad \text{(A9)}$$

In terms of $\mathbf{p}_\perp'$,

$$D_1=\mathbf{p}_\perp'^2+A+m^2, \quad \text{(A10)}$$

where

$$A=x(1-x)\mathbf{q}_\perp^2+y(1-y)\mathbf{q}_\perp'^2-2x(1-x)(1-2y)\mathbf{r}_1\cdot\mathbf{q}_\perp-2y(1-y)(1-2x)\mathbf{r}_1\cdot\mathbf{q}_\perp'$$
$$+(x+y-2xy)(1-x-y+2xy)\mathbf{r}_1^2. \quad \text{(A11)}$$

For $\mathbf{r}_1\neq 0$, this quantity A is quite complicated. It is therefore useful to introduce

$$\mathbf{Q}=\mathbf{q}_\perp-(1-2y)\mathbf{r}_1 \quad \text{and} \quad \mathbf{Q}'=\mathbf{q}_\perp'-(1-2x)\mathbf{r}_1. \quad \text{(A12)}$$

Except for a factor of 2, these $\mathbf{Q}$'s are essentially the same as the $\mathbf{Q}$ previously encountered in (4.13) of Ref. 5. In terms of these $\mathbf{Q}$'s, it follows from (A11) that

$$A=x(1-x)\mathbf{Q}^2+y(1-y)\mathbf{Q}'^2+4x(1-x)y(1-y)\mathbf{r}_1^2, \quad \text{(A13)}$$

and the $\delta\mathbf{p}_\perp$ of (A9) is

$$\delta\mathbf{p}_\perp=(1-x)\mathbf{Q}+(1-y)\mathbf{Q}'+(1-2x-2y+2xy)\mathbf{r}_1. \quad \text{(A14)}$$

We next turn our attention to the numerator N_1. Let

$$\mathbf{a}_1=\delta\mathbf{p}_\perp+\mathbf{r}_1=(1-x)\mathbf{Q}+(1-y)\mathbf{Q}'+2(1-x)(1-y)\mathbf{r}_1,$$
$$\mathbf{a}_2=\delta\mathbf{p}_\perp-\mathbf{q}_\perp=-x\mathbf{Q}+(1-y)\mathbf{Q}'-2x(1-y)\mathbf{r}_1, \tag{A15}$$
$$\mathbf{a}_3=\delta\mathbf{p}_\perp-\mathbf{q}_\perp-\mathbf{q}_\perp'+\mathbf{r}_1=-x\mathbf{Q}-y\mathbf{Q}'+2xy\mathbf{r}_1,$$

and

$$\mathbf{a}_4=\delta\mathbf{p}_\perp-\mathbf{q}_\perp'=(1-x)\mathbf{Q}-y\mathbf{Q}'-2(1-x)y\mathbf{r}_1,$$

then

$$N_1=\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_\perp'+\not{a}_1-m)(\not{p}_\perp'+\not{a}_4+m)(\not{p}_\perp'+\not{a}_3-m)(\not{p}_\perp'+\not{a}_2+m)$$
$$=[(\mathbf{p}_\perp'+\mathbf{a}_1)\cdot(\mathbf{p}_\perp'+\mathbf{a}_2)+m^2][(\mathbf{p}_\perp'+\mathbf{a}_3)\cdot(\mathbf{p}_\perp'+\mathbf{a}_4)+m^2]+[(\mathbf{p}_\perp'+\mathbf{a}_2)\cdot(\mathbf{p}_\perp'+\mathbf{a}_3)+m^2]$$
$$\times[(\mathbf{p}_\perp'+\mathbf{a}_1)\cdot(\mathbf{p}_\perp'+\mathbf{a}_4)+m^2]-[(\mathbf{p}_\perp'+\mathbf{a}_1)\cdot(\mathbf{p}_\perp'+\mathbf{a}_3)+m^2][(\mathbf{p}_\perp'+\mathbf{a}_2)\cdot(\mathbf{p}_\perp'+\mathbf{a}_4)+m^2]. \tag{A16}$$

With symmetric integration over the two-dimensional vector $\mathbf{p}_\perp'$, this numerator N_1 can be replaced by its symmetric part

$$N_1'=(\mathbf{p}'^2+m^2)^2+C'(\mathbf{p}'^2+m^2)+C+m^2C'', \tag{A17}$$

where

$$C=(\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_3\cdot\mathbf{a}_4)+(\mathbf{a}_2\cdot\mathbf{a}_3)(\mathbf{a}_1\cdot\mathbf{a}_4)-(\mathbf{a}_1\cdot\mathbf{a}_3)(\mathbf{a}_2\cdot\mathbf{a}_4), \tag{A18}$$
$$C'=(\mathbf{a}_1+\mathbf{a}_3)\cdot(\mathbf{a}_2+\mathbf{a}_4), \tag{A19}$$

and

$$C''=-(\mathbf{a}_1\cdot\mathbf{a}_3+\mathbf{a}_2\cdot\mathbf{a}_4). \tag{A20}$$

From the explicit formula (A15), the computation of C' and C'' is relatively straightforward:

$$C'=(1-2x)^2\mathbf{Q}^2+(1-2y)^2\mathbf{Q}'^2+2(1-2x)(1-2y)\mathbf{Q}\cdot\mathbf{Q}'$$
$$+2(1-2x)^2(1-2y)\mathbf{r}_1\cdot\mathbf{Q}+2(1-2x)(1-2y)^2\mathbf{r}_1\cdot\mathbf{Q}'-[1-(1-2x)^2(1-2y)^2]\mathbf{r}_1^2$$
$$=(1-2x)^2\mathbf{Q}^2+(1-2y)^2\mathbf{Q}'^2+2(1-2x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-[1+(1-2x)^2(1-2y)^2]\mathbf{r}_1^2 \tag{A21}$$

and

$$C''=2x(1-x)\mathbf{Q}^2+2y(1-y)\mathbf{Q}'^2-(1-2x)(1-2y)\mathbf{Q}\cdot\mathbf{Q}'+4x(1-x)(1-2y)\mathbf{r}_1\cdot\mathbf{Q}$$
$$+4(1-2x)y(1-y)\mathbf{r}_1\cdot\mathbf{Q}'-8x(1-x)y(1-y)\mathbf{r}_1^2$$
$$=2x(1-x)\mathbf{Q}^2+2y(1-y)\mathbf{Q}'^2-(1-2x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+(1-2y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$+(1-2x)\mathbf{r}_1\cdot\mathbf{q}_\perp'-[1-8x(1-x)y(1-y)]\mathbf{r}_1^2. \tag{A22}$$

It is seen from (A21) that it is convenient to use both the $\mathbf{Q}$'s and the $\mathbf{q}_\perp$'s. The corresponding calculation for C is much more complicated. We first write down

$$\mathbf{a}_1\cdot\mathbf{a}_2=-x(1-x)\mathbf{Q}^2+(1-y)^2\mathbf{Q}'^2+(1-2x)(1-y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-(1-y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$+(1-2x)(1-y)\mathbf{r}_1\cdot\mathbf{q}_\perp'-[(1-2x)^2(1-y)+4x(1-x)y(1-y)]\mathbf{r}_1^2, \tag{A23}$$
$$\mathbf{a}_3\cdot\mathbf{a}_4=-x(1-x)\mathbf{Q}^2+y^2\mathbf{Q}'^2-(1-2x)y\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+y\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$+(1-2x)y\mathbf{r}_1\cdot\mathbf{q}_\perp'-[(1-2x)^2y+4x(1-x)y(1-y)]\mathbf{r}_1^2, \tag{A24}$$
$$\mathbf{a}_2\cdot\mathbf{a}_3=x^2\mathbf{Q}^2-y(1-y)\mathbf{Q}'^2-x(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+x(1-2y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$+x\mathbf{r}_1\cdot\mathbf{q}_\perp'-[x(1-2y)^2+4x(1-x)y(1-y)]\mathbf{r}_1^2, \tag{A25}$$
$$\mathbf{a}_1\cdot\mathbf{a}_4=(1-x)^2\mathbf{Q}^2-y(1-y)\mathbf{Q}'^2+(1-x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+(1-x)(1-2y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$-(1-x)\mathbf{r}_1\cdot\mathbf{q}_\perp'-[(1-x)(1-2y)^2+4x(1-x)y(1-y)]\mathbf{r}_1^2, \tag{A26}$$
$$\mathbf{a}_1\cdot\mathbf{a}_3=-x(1-x)\mathbf{Q}^2-y(1-y)\mathbf{Q}'^2-(x+y-2xy)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-(x-y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$+(x-y)\mathbf{r}_1\cdot\mathbf{q}_\perp'-[-x-y+2xy+4x(1-x)y(1-y)]\mathbf{r}_1^2, \tag{A27}$$

and

$$\mathbf{a}_2\cdot\mathbf{a}_4=-x(1-x)\mathbf{Q}^2-y(1-y)\mathbf{Q}'^2+(1-x-y+2xy)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-(1-x-y)\mathbf{r}_1\cdot\mathbf{q}_\perp$$
$$-(1-x-y)\mathbf{r}_1\cdot\mathbf{q}_\perp'-[-1+x+y-2xy+4x(1-x)y(1-y)]\mathbf{r}_1^2. \tag{A28}$$

The substitution of (A23)–(A28) into (A18) then yields

$$\begin{aligned}C=&\,x^2(1-x)^2(\mathbf{Q}^2)^2+y^2(1-y)^2(\mathbf{Q}'^2)^2+16x^2(1-x)^2y^2(1-y)^2(\mathbf{r}_1^2)^2\\&+4x(1-x)y(1-y)[(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2-(\mathbf{r}_1\cdot\mathbf{q}_\perp)^2-(\mathbf{r}_1\cdot\mathbf{q}_\perp')^2]+[-x(1-x)-y(1-y)+2x(1-x)y(1-y)]\mathbf{Q}^2\mathbf{Q}'^2\\&+x(1-x)[(1-2x)^2+8x(1-x)y(1-y)]\mathbf{Q}^2\mathbf{r}_1^2+y(1-y)[(1-2y)^2+8x(1-x)y(1-y)]\mathbf{Q}'^2\mathbf{r}_1^2\\&-x(1-x)(1-2x)(1-2y)\mathbf{Q}^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-(1-2x)y(1-y)(1-2y)\mathbf{Q}'^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp'\\&+x(1-x)(1-2y)\mathbf{Q}^2\mathbf{r}_1\cdot\mathbf{q}_\perp-x(1-x)(1-2x)\mathbf{Q}^2\mathbf{r}_1\cdot\mathbf{q}_\perp'-y(1-y)(1-2y)\mathbf{Q}'^2\mathbf{r}_1\cdot\mathbf{q}_\perp\\&+(1-2x)y(1-y)\mathbf{Q}'^2\mathbf{r}_1\cdot\mathbf{q}_\perp'-4x(1-x)(1-2x)y(1-y)(1-2y)\mathbf{r}_1^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp'\\&+4x(1-x)y(1-y)(1-2y)\mathbf{r}_1^2\mathbf{r}_1\cdot\mathbf{q}_\perp+4x(1-x)(1-2x)y(1-y)\mathbf{r}_1^2\mathbf{r}_1\cdot\mathbf{q}_\perp'.\end{aligned}\tag{A29}$$

Analogous to (4.11), there is a contribution to K_D due to the shift from $\mathbf{p}_\perp$ to $\mathbf{p}_\perp'$:

$$\begin{aligned}\frac{1}{2}\int_0^1 dx\int_0^1 dy\,\delta\mathbf{p}_\perp\cdot(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4)&=\int_0^1 dx\int_0^1 dy[(1-x)\mathbf{q}_\perp+(1-y)\mathbf{q}_\perp'-(1-x-y+2xy)\mathbf{r}_1]\cdot[(1-2x)\mathbf{q}_\perp\\&\qquad+(1-2y)\mathbf{q}_\perp'-(1-2x)(1-2y)\mathbf{r}_1]\\&=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}'^2)+\mathbf{r}_1^2/18.\end{aligned}\tag{A30}$$

Substitution into (A7) then yields

$$\begin{aligned}K_D(\mathbf{q}_\perp,\mathbf{q}_\perp')-[\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)+\mathbf{r}_1^2/18]&=(2\pi)^{-1}\int_0^1 dx\int_0^1 dy\int d\mathbf{p}_\perp'(1-N_1/D_1^2)\\&=(2\pi)^{-1}\int_0^1 dx\int_0^1 dy\int d\mathbf{p}_\perp'\frac{(2A-C')(\mathbf{p}_\perp'^2+m^2)+(A^2-C-m^2C'')}{(\mathbf{p}_\perp'^2+A+m^2)^2}\\&=\frac{1}{2}\int_0^1 dx\int_0^1 dy\left((2A-C')\ln\frac{\Lambda}{A+m^2}+\frac{-A^2+AC'-C-m^2C''}{A+m^2}\right),\end{aligned}\tag{A31}$$

where Λ is a large cutoff whose presence indicates the logarithmic divergence of K_D. Note first that, from (A13) and (A21),

$$\begin{aligned}2A-C'=&-(1-6x+6x^2)\mathbf{Q}^2-(1-6y+6y^2)\mathbf{Q}'^2-2(1-2x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'\\&+2[1-2x(1-x)-2y(1-y)+12x(1-x)y(1-y)]\mathbf{r}_1^2,\end{aligned}\tag{A32}$$

and hence

$$\int_0^1 dx\int_0^1 dy(2A-C')=\tfrac{4}{3}\mathbf{r}_1^2\tag{A33}$$

is independent of $\mathbf{q}_\perp$ and $\mathbf{q}_\perp'$. Also note that, from (A22),

$$\frac{1}{2}\int_0^1 dx\int_0^1 dy\,C''=\tfrac{1}{6}(\mathbf{q}_\perp^2+\mathbf{q}_\perp'^2)-5\mathbf{r}_1^2/18\tag{A34}$$

and hence (A30) and (A34) differ only by a term proportional to $\mathbf{r}_1^2$. By (A32),

$$\begin{aligned}\int_0^1 dx\int_0^1 dy(2A-C')\ln(A+m^2)=&\,\frac{1}{3}\int_0^1 dy\,\mathbf{r}_1^2\ln\{[y(1-y)(\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2][y(1-y)(\mathbf{q}_\perp'-\mathbf{r}_1)^2+m^2]\}\\&+\frac{1}{3}\int_0^1 dx\,\mathbf{r}_1^2\ln\{[x(1-x)(\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2][x(1-x)(\mathbf{q}_\perp-\mathbf{r}_1)^2+m^2]\}+\int_0^1 dx\int_0^1 dy(A+m^2)^{-1}(\partial A/\partial x)\\&\times\{x(1-x)(1-2x)\mathbf{Q}^2+x(1-x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+(1-2x)[\tfrac{1}{3}(1-x+x^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\}\\&+\int_0^1 dx\int_0^1 dy(A+m^2)^{-1}(\partial A/\partial y)\{y(1-y)(1-2y)\mathbf{Q}'^2+(1-2x)y(1-y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'\\&+(1-2y)[\tfrac{1}{3}(1-y+y^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\}.\end{aligned}\tag{A35}$$

If we substitute (A33)–(A35) into (A31), the result is

$$K_D(\mathbf{q}_\perp,\mathbf{q}_\perp')+(2/9)\mathbf{r}_1^2-(2/3)\mathbf{r}_1^2\ln\Lambda +\tfrac{1}{6}\mathbf{r}_1^2\int_0^1 dx\ln\{[x(1-x)(\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2][x(1-x)(\mathbf{q}_\perp-\mathbf{r}_1)^2+m^2] \times[x(1-x)(\mathbf{q}_\perp'+\mathbf{r}_1)^2+m^2][x(1-x)(\mathbf{q}_\perp'-\mathbf{r}_1)^2+m^2]\}=\int_0^1 dx\int_0^1 dy\frac{\mathfrak{N}}{A+m^2},\quad (\mathrm{A36})$$

where

$$2\mathfrak{N}=-A^2+A(C'+C'')-C-[(1-2x)\mathbf{Q}^2+4y(1-y)\mathbf{r}_1\cdot\mathbf{q}_\perp']\{x(1-x)(1-2x)\mathbf{Q}^2 +x(1-x)(1-2y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+(1-2x)[\tfrac{1}{3}(1-x+x^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\} -[(1-2y)\mathbf{Q}'^2+4x(1-x)\mathbf{r}_1\cdot\mathbf{q}_\perp]\{y(1-y)(1-2y)\mathbf{Q}'^2+(1-2x)y(1-y)\mathbf{q}_\perp\cdot\mathbf{q}_\perp' +(1-2y)[\tfrac{1}{3}(1-y+y^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\}.\quad (\mathrm{A37})$$

By (A13), (A21), (A22), and (A27), this is explicitly

$$\mathfrak{N}=-2x(1-x)y(1-y)[2+(1-2x)^2(1-2y)^2](\mathbf{r}_1^2)^2-2x(1-x)y(1-y)[(\mathbf{q}_\perp\cdot\mathbf{q}_\perp')^2-(\mathbf{r}_1\cdot\mathbf{q}_\perp)^2-(\mathbf{r}_1\cdot\mathbf{q}_\perp')^2] +[x(1-x)+y(1-y)-4x(1-x)y(1-y)]\mathbf{Q}^2\mathbf{Q}'^2 +\{-\tfrac{1}{6}[1+7x(1-x)-20x^2(1-x)^2]+3x(1-x)(1-2x)^2y(1-y)\}\mathbf{Q}^2\mathbf{r}_1^2 +\{-\tfrac{1}{6}[1+7y(1-y)-20y^2(1-y)^2]+3x(1-x)y(1-y)(1-2y)^2\}\mathbf{Q}'^2\mathbf{r}_1^2 +\tfrac{1}{2}x(1-x)(1-2x)(1-2y)\mathbf{Q}^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp'+\tfrac{1}{2}(1-2x)y(1-y)(1-2y)\mathbf{Q}'^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp' +x(1-x)(1-2x)(1-2y+2y^2)\mathbf{Q}^2\mathbf{r}_1\cdot\mathbf{q}_\perp'+(1-2x+2x^2)y(1-y)(1-2y)\mathbf{Q}'^2\mathbf{r}_1\cdot\mathbf{q}_\perp +4x(1-x)(1-2x)y(1-y)(1-2y)\mathbf{r}_1^2\mathbf{q}_\perp\cdot\mathbf{q}_\perp'-2x(1-x)(1-2y)[\tfrac{1}{3}(1-y+y^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\mathbf{r}_1\cdot\mathbf{q}_\perp -2(1-2x)y(1-y)[\tfrac{1}{3}(1-x+x^2)+2x(1-x)y(1-y)]\mathbf{r}_1^2\mathbf{r}_1\cdot\mathbf{q}_\perp'-2x(1-x)(1-2x)y(1-y)\mathbf{r}_1\cdot\mathbf{q}_\perp\mathbf{q}_\perp\cdot\mathbf{q}_\perp' -2x(1-x)y(1-y)(1-2y)\mathbf{r}_1\cdot\mathbf{q}_\perp'\mathbf{q}_\perp\cdot\mathbf{q}_\perp'.\quad (\mathrm{A38})$$

Finally, the desired answer is, from (A6),

$$K_1(\mathbf{q}_\perp,\mathbf{q}_\perp')=-\tfrac{1}{6}\mathbf{r}_1^2\int_0^1 dx\ln\frac{[x(1-x)(\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2][x(1-x)(\mathbf{q}_\perp-\mathbf{r}_1)^2+m^2]}{m^2[4x(1-x)\mathbf{r}_1^2+m^2]} +(\text{previous term with } \mathbf{q}_\perp \text{ replaced by } \mathbf{q}_\perp') +\int_0^1 dx\int_0^1 dy\frac{\mathfrak{N}}{x(1-x)\mathbf{Q}^2+y(1-y)\mathbf{Q}'^2+4x(1-x)y(1-y)\mathbf{r}_1^2+m^2}-(\text{previous term with } \mathbf{q}_\perp=\mathbf{q}_\perp'=\mathbf{r}_1).\quad (\mathrm{A39})$$

APPENDIX B

In this appendix, we give a trivial example where a change in the kernel does not change the spectrum. First consider the kernel

$$(z+z')^{-1},\quad (\mathrm{B1})$$

on the L_2 space defined by (6.6). This kernel can be treated in the same way as that of (6.9) in Sec. 6 C. The spectrum is

$$[0,\pi].\quad (\mathrm{B2})$$

Next consider the kernel

$$(z+z'+m^2)^{-1}.\quad (\mathrm{B3})$$

Let

$$v=1+2z/m^2 \quad\text{and}\quad v'=1+2z'/m^2,\quad (\mathrm{B4})$$

then, to find the spectrum of (B3), we need to study the integral equation

$$\int_1^\infty dv'\varphi(v')(v+v')^{-1}=\mu\varphi(v)\quad (\mathrm{B5})$$

for $v\geq 1$. But (B5) is a known integral equation, whose solutions are,[38] for ζ real,

$$\mu=\pi\,\mathrm{sech}\pi\zeta\quad (\mathrm{B6})$$

and

$$\varphi(v)=P_{i\zeta-1/2}(v),\quad (\mathrm{B7})$$

where P is the Legendre function of the first kind. From (B6), the spectrum of this kernel (B3) is also given by (B2).

[38] F. G. Mehler, Math. Ann. **18**, 161 (1881).