$dP^{(n)}/d\Omega$ and $P^{(n)}$ into those for $d\sigma^{(n)}/d\Omega$ and $\sigma^{(n)}$ (the corresponding power cross sections), respectively.

The third quantity, the number cross section, appears the least useful from the experimental point of view yet has appeared in some of the standard papers on the subject, since it is most natural from the point of view of quantum-mechanical calculations. The differential number cross section in a given frame, $d\Sigma^{(n)}/d\Omega$, is defined as the number of nth harmonic photons detected in a solid angle $d\Omega$ divided by the number of (first-harmonic) incident photons all in some unit time interval. In a given frame, the relation is easily seen to be

$$d\Sigma^{(n)}/d\Omega=(\omega^0/\omega_{\text{scat}})(d\sigma^{(n)}/d\Omega)=(\omega^0/\omega_{\text{scat}})(1/I)(dP^{(n)}/d\Omega). \quad \text{(A1)}$$

In particular, in the L frame this relation becomes

$$\frac{d\Sigma_L^{(n)}}{d\Omega_L}=\frac{1}{n}[1+\tfrac{1}{2}q^2\sin^2(\tfrac{1}{2}\theta)]\frac{1}{I_L}\frac{dP_L^{(n)}}{d\Omega_L}. \quad \text{(A2)}$$

If we compare our results to those of Brown and Kibble,[3] we note that it is the number cross section derived from the power lost by the electron that agrees precisely with their results. They did not consider the extra time retardation that must be included when going from the power lost by the electron to the power observed in a given frame. It is the observed power rather than the power lost by the electron that is actually measured so that this extra retardation must be included when comparing theory to experiment.

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New Dynamical Group for the Relativistic Quantum Mechanics of Elementary Particles*

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Nonrelativistic Galilean quantum mechanics and the standard transition to relativistic Poincaré quantum mechanics is analyzed in terms of group theory. Special emphasis is given to the discussion of the relation between dynamics and geometry. Certain unsatisfactory features are pointed out and a new relativistic group $\mathcal{G}_5$ is suggested as the symmetry group of dynamics. $\mathcal{G}_5$ contains both the nonrelativistic Galilei group and the Poincaré group as subgroups, and it is a group extension of the restricted Lorentz group. For use in relativistic quantum mechanics, the central extension of $\mathcal{G}_5$ by a phase group must be employed. The Lie algebra of this relativistic quantum-mechanical Galilei group $\tilde{\mathcal{G}}_5$ contains an acceptable covariant space-time position operator and a nontrivial relativistic mass operator. The latter also serves to describe dynamical development. The irreducible unitary projective representations of $\tilde{\mathcal{G}}_5$ correspond to infinite towers of states with increasing spin.

I. GROUP-THEORETICAL ANALYSIS OF NONRELATIVISTIC QUANTUM MECHANICS

Undoubtedly, the most remarkable feature of relativistic dynamics is that the invariance group of the dynamical law coincides with the group of rigid motions (essentially the group of isometries) of the underlying geometrical manifold. In fact, the underlying geometrical manifold is the Minkowski space $E_{3,1}$ where the identity component of the group of isometries is the connected Poincaré group containing the identity, i.e., the inhomogeneous Lorentz group[1] $ISO_0(3,1)\equiv T_4\otimes\mathcal{L}_+^\uparrow$. At the same time, the laws of motion are required to be invariant under $ISO_0(3,1)$. The situation is very different in nonrelativistic physics. The underlying geometrical manifold is, to start with, the Euclidean space E_3, where the identity component of the group of isometries is the connected Euclidean group, i.e., the inhomogeneous rotation group $ISO(3)\equiv T_3\otimes SO(3)$. This space does not permit even the formulation of any dynamics. One therefore introduces the time as an additional kinematical variable and thereby changes the underlying manifold from E_3 to $E_3\times E_1$. Note that no metric is introduced into this Cartesian product space. Next one demands that the laws of motion be invariant under the connected component of the Galilei group. This group we shall denote in what follows by the symbol $\mathcal{G}_4$. The carrier space of $\mathcal{G}_4$ is $E_3\times E_1$ and the group is obtained by adjoining to the transformations of $ISO(3)$ the additional two sets of transformations

$$x_k\rightarrow x_k+v_kt, \quad \text{(1.1a)}$$

$$t\rightarrow t+\tau, \quad \text{(1.1b)}$$

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[1] For convenience, in this paper we shall use the symbol $SO_0(3,1)$ for the restricted Lorentz group $\mathcal{L}_+^\uparrow$, even though this notation is not quite standard.

with v_k and τ being parameters. Thus, the structure of $\mathcal{G}_4$ is[2]

$$\mathcal{G}_4=\{T_3{}^a\times T_1{}^\tau\}\otimes\{T_3{}^v\otimes SO(3)\}. \qquad (1.2)$$

Here $T_3{}^a$ is the translation group $x_k\to x_k+a_k$, $T_1{}^\tau$ the time translation group (1.1b), $T_3{}^v$ the velocity transformation (boost) group (1.1a). The symbol $\times$ stands for direct product and $\otimes$ for semidirect product. We emphasize that, since $E_3\times E_1$ is not endowed with metric, $\mathcal{G}_4$ has no geometrical significance. On the other hand, $\mathcal{G}_4$ does contain as a subgroup the $ISO(3)$ group of the *basic* geometry.

As was pointed out by Inönü and Wigner,[3] for the formulation of nonrelativistic *quantum* mechanics, the group $\mathcal{G}_4$ has to be further extended. Speaking somewhat loosely, the mathematical reason for this necessity is that the representation of the group operations in the Hilbert space of a quantized system is a ray representation, i.e., up to a phase factor, and for the Galilei group the classes of ray representations are not equivalent to the true representations. Now, because of the nontrivial phase factor, the generators are determined only up to an additive real multiple of the identity operator. By means of simple redefinitions and the use of the Jacobi identity, all such additive multiples can be eliminated, except for one. This will appear in the commutator of the P_k (the generators of $T_3{}^a$) with the G_l (the generators of the boost $T_3{}^v$), and we have[4,5]

$$[P_k, G_l]=-i\delta_{kl}M. \qquad (1.3)$$

Since the geometrical transformations $x_k\to x_k+a_k$ and $x_l\to x_l+v_lt$ evidently commute, it is clear that in quantum mechanics we are dealing not with $\mathcal{G}_4$ but with a larger group. A more detailed analysis[6–9] reveals that the group in question is the *central extension*[10] of the covering group of $\mathcal{G}_4$ by a phase group. This quantum-mechanical nonrelativistic Galilei group we shall denote in the following by $\tilde{\mathcal{G}}_4$. Its structure is given by[11]

$$\tilde{\mathcal{G}}_4=\{T_1{}^\alpha\times(T_3{}^a\times T_1{}^\tau)\}\otimes\{T_3{}^v\otimes SU(2)\}. \qquad (1.4)$$

Here $T_1{}^\alpha$ is the one-dimensional (Abelian) phase group responsible for the emergence of M. The $SU(2)$ in Eq. (1.4) appears as the covering group of $SO(3)$. For the reader's convenience, the complete algebra of $\tilde{\mathcal{G}}_4$ and some other simple related topics are summarized in Appendix B.

We now summarize the "gain in physics" that is achieved when going from $ISO(3)$ to $\tilde{\mathcal{G}}_4$. The underlying geometrical manifold E_3 permits the definition of linear and angular momentum (P_k and J_k) only. These are suitable for the specification of a *given* state of motion. When we make the extension to $\tilde{\mathcal{G}}_4$, we are permitted to introduce the *energy H* [generator of (1.1b)], which is an obviously dynamical variable. Furthermore, by setting

$$X_k\equiv M^{-1}G_k, \qquad (1.5)$$

we obtain a dynamical definition of the *position operator* which is consistent with the Heisenberg rules of quantization [as seen from (1.3)] as well as with other requirements.[7] Further, a *velocity operator*

$$V_k\equiv i[H, X_k] \qquad (1.6)$$

can be defined. Evaluating the commutator with Eq. (B3f), we get $V_k=MP_k$. This relation then permits us to interpret M as *mass* and we see that

$$V_k=\dot{X}_k. \qquad (1.7)$$

Comparing (1.7) and (1.6), we then conclude that development with respect to the kinematical time variable is expressed by commutation with H. Thus, the latter assumes the role of a *Hamiltonian*. Instead of being constrained [as we were in the $ISO(3)$ framework] to talk about a fixed state, we now can consider a *family* of states whose members differ from each other by the eigenvalues of H. The *relation between the dynamical notion of energy and the kinematical notion of momentum* is borne out by noting that

$$\mathcal{B}=\mathbf{P}^2/2M-H$$

is a Casimir operator of $\tilde{\mathcal{G}}_4$. Thus, selecting the representation of $\tilde{\mathcal{G}}_4$ characterized by $\mathcal{B}'=0$, we have

$$\mathbf{P}^2/2M-H=0. \qquad (1.8)$$

In fact, this relation between the energy and the momentum of an elementary particle is true for *any* representation with $\mathcal{B}'\neq0$, because H occurs only *inside* the commutators of the $\tilde{\mathcal{G}}_4$ algebra, so that we may redefine H to be $H+\mathcal{B}'$. (A more rigorous justification follows from the circumstance that the representations with different $\mathcal{B}$ eigenvalues are equivalent; cf. Ref. 8.)

At this point we can clearly summarize, in terms of group invariants, the above-emphasized transition from the sole consideration of a *single*, fixed state to the consideration of a *family* of states. The Casimir operator of $ISO(3)$ which corresponds to $\mathcal{B}$ of $\tilde{\mathcal{G}}_4$ is of course just $\mathcal{C}\equiv\mathbf{P}^2$. Hence, in the "predynamical" stage we have, instead of (1.8), the equation (in the proper reference frame)

$$\mathbf{P}^2=0. \qquad (1.9)$$

Denoting the eigenvalues of P_k and H by p_k and E,

[2] There are ten other isomorphic forms, but (1.2) is best suited for the study of representations.
[3] E. Inönü and E. P. Wigner, Nuovo Cimento **9**, 705 (1952).
[4] A simple discussion of this topic can be found in T. F. Jordan, *Linear Operators for Quantum Mechanics* (Wiley, New York, 1969), Chap. VII.
[5] Throughout this paper we use natural units $\hbar=c=1$.
[6] V. Bargmann, Ann. Math. **59**, 1 (1954).
[7] M. Hamermesh, Ann. Phys. (N.Y.) **9**, 518 (1960).
[8] J. -M. Lévy-Leblond, J. Math. Phys. **4**, 776 (1963).
[9] J. Voisin, J. Math. Phys. **6**, 1519 (1965).
[10] For the definition of group extension, see Appendix A.
[11] Again, there are ten other isomorphic forms. See Ref. 2.

respectively, we thus have in the $ISO(3)$ framework

$$\mathbf{p}^2=0, \tag{1.10a}$$

and in the $\tilde{\mathcal{G}}_4$ framework[12]

$$\mathbf{p}^2/2M-E=0. \tag{1.10b}$$

Equation (1.10a) characterizes a *single* possible state, viz., a particle at rest, whereas Eq. (1.10b) characterizes a *family* of states, with arbitrary energy (the spectrum of E is continuous) and with a corresponding state of motion ($\mathbf{p}^2$ being *determined* by E).

Actually, we have a further enrichment in physics. The second Casimir operator of $\tilde{\mathcal{G}}_4$ [cf. Eq. (B5)] turns out to be related to intrinsic *spin*, which has no place in the $ISO(3)$ background.[13] Thus, the "family" of states is differentiated by spin, too. Finally, since M commutes with all generators, we also have a *superselection rule* for states with different mass.[6,8,4]

In order to extract detailed statements from the above-sketched $\tilde{\mathcal{G}}_4$ characterization of quantum dynamics, it is best to construct a representation[14] of the $\tilde{\mathcal{G}}_4$ algebra in the Hilbert space $\mathfrak{H}(E_3\times E_1)$ built upon the carrier space $E_3\times E_1$ of $\tilde{\mathcal{G}}_4$. The realization of the operators in this $\mathfrak{H}(E_3\times E_1)$ is given in Eq. (B6). Then the Schrödinger equation is nothing but the realization of (1.8), i.e., the relation

$$\mathcal{B}\psi(\mathbf{x};t)=0. \tag{1.11a}$$

In detail,

$$(\tfrac{1}{2}M^{-1}\Delta+i\partial_t)\psi(\mathbf{x};t)=0. \tag{1.11b}$$

At this point we note that (1.11b) has separable solutions. Setting

$$\psi(\mathbf{x};t)=\varphi(\mathbf{x})\chi(t), \tag{1.12}$$

we obtain

$$\chi(t)=\exp(-iEt) \tag{1.13}$$

and

$$(\tfrac{1}{2}M^{-1}\Delta+E)\varphi(\mathbf{x})=0. \tag{1.14}$$

Here E appears as a *separation constant.* Equation (1.14) is now an eigenfunction problem *in the Hilbert space* $\mathfrak{H}(E_3)$ *built upon the "predynamical" underlying geometrical manifold* E_3 (and not upon the dynamical carrier space $E_3\times E_1$). Actually, *by separating variables we lost* $\tilde{\mathcal{G}}_4$ *invariance*: Equation (1.14) is invariant not under $\tilde{\mathcal{G}}_4$ but only under $ISO(3)$. [In fact, $-\Delta\sim\mathbf{P}^2$ is the Casimir operator of $ISO(3)$, so that (1.14) tells us to pick a representation of *this* group.] The loss of $\tilde{\mathcal{G}}_4$ invariance is compensated for by having now a statement to the effect that, instead of the $\mathcal{A}\equiv\mathbf{P}^2=0$ "predynamical" representation of $ISO(3)$, we must choose the $\mathcal{A}\equiv\mathbf{P}^2=2ME$ representation. Thus, naturally, we did not lose physical information by separating the variables in (1.11a). On the other hand, if one started with an $ISO(3)$ equation $\mathcal{A}=2ME$ [of which (1.14) is a realization in $\mathfrak{H}(E_3)$], the energy E would appear *not* as a dynamical variable [i.e., not as the eigenvalue of an operator in the Lie algebra of $ISO(3)$] but rather as a label of a representation[15] or, equivalently, as the eigenvalue of an operator in the *enveloping* algebra of $ISO(3)$. The *necessity* of a dynamical eigenvalue problem for E would not even arise: *Any* representation label $\mathcal{A}$ is as good as any other. The state of affairs is even more transparent if one considers not a free particle but one under the influence of an interaction. Then the Schrödinger equation is (in the spinless case)

$$[\tfrac{1}{2}M^{-1}\Delta-V(\mathbf{x})+i\partial_t]\psi(\mathbf{x};t)=0. \tag{1.15}$$

Here $\mathbf{x}$ must be interpreted as a relative coordinate (the c.m. motion has been separated off), M is the reduced mass, and V depends on $r=\sqrt{(\mathbf{x}^2)}$ only, so that Eq. (1.15) is still $\tilde{\mathcal{G}}_4$ invariant.[16] After separation we get

$$[\tfrac{1}{2}M^{-1}\Delta-V(\mathbf{x})+E]\varphi(\mathbf{x})=0. \tag{1.16}$$

Naturally, $\tilde{\mathcal{G}}_4$ invariance is again lost, but now (1.16) cannot be interpreted at all as an equation in the $ISO(3)$ enveloping algebra, selecting a representation. This is in spite of the fact that, obviously, (1.16) is $ISO(3)$ invariant. The *dynamical* origin of E is now well emphasized, as opposed to its previous role of simply labeling an $ISO(3)$ representation.

II. STANDARD TRANSITION TO RELATIVISTIC QUANTUM MECHANICS AND CRITIQUE

In the previous, somewhat lengthy, section we elaborated on generally quite familiar topics (although, perhaps, in an unusual presentation and with particular emphasis on certain points). The purpose of this analysis was to prepare the ground. The present section serves the same purpose: We shall analyze, from our particular point of view, the standard transition from nonrelativistic to relativistic quantum mechanics.

As is well known, the first step in this transition is to define the underlying geometrical manifold to be

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[12] It is amusing to observe that, as was shown by Hamermesh (Ref. 7), the first Casimir invariant of the *original* $\mathcal{G}_4$ group is not $\mathbf{P}^2/2M-H$ but rather just $\mathbf{P}^2$, the same as the one for $ISO(3)$. Thus, even for the present purpose of physical interpretation of E, the use of $\tilde{\mathcal{G}}_4$ (rather than $\mathcal{G}_4$) is crucial.

[13] Nor in the $\mathcal{G}_4$ background. The second Casimir operator of $\mathcal{G}_4$ is $\mathbf{P}\cdot\mathbf{J}$, rather than $\mathbf{F}^2$ given by (B5).

[14] The systematic construction and detailed study of the irreducible unitary projective representations of $\tilde{\mathcal{G}}_4$ was first given by Lévy-Leblond, Ref. 8, on the basis of the Bargmann paper, Ref. 6. See also Ref. 9.

[15] Actually, the label is $2ME$, and there is no possibility of giving a separate meaning to M and E.

[16] Equation (1.16) arises from the study of the decomposition of the tensor product of two one-particle representations $(m^1\,|\,\mathcal{B}^1,s^1)$ and $(m^2\,|\,\mathcal{B}^2,s^2)$; cf. Ref. 8, especially Eqs. (VI-5) and (VI-6). Incidentally, the same analysis shows that even though (as stated above) $\mathcal{B}$ may be taken equal to zero for the one-particle representation, yet it cannot be altered simultaneously to zero in all the $(M\,|\,\mathcal{B},s)$ representations which occur in the decomposition of the tensor product. It is in this context that $\mathcal{B}$ assumes the nontrivial role of "internal energy" of the compound systems.

the Minkowski space $E_{3,1}$. This means that in the carrier space $E_3 \times E_1$ of $\tilde{\mathcal{G}}_4$ one introduces the pseudo-Euclidean metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \text{with } g_{00} = -g_{kk} = 1;\ g_{\mu\nu} = 0 \quad (\mu \neq \nu). \tag{2.1}$$

The group of isometries in this space is the Poincaré group. The next step is crucial: One declares, by *fiat*, the identity component of *the very same group* [i.e., the $ISO_0(3, 1)$ group[1]] to be the invariance group of dynamics. This means that one has (as before, in the $\tilde{\mathcal{G}}_4$ dynamics[17]) ten basic dynamical observables (P_μ and $J_{\mu\nu}$) but with a very different algebra. The relation of this $ISO_0(3, 1)$ algebra to the $\tilde{\mathcal{G}}_4$ algebra (and hence the relation of the relativistic dynamical ovservables to the corresponding nonrelativistic ones) is revealed, as is well known,[18] by the procedure of *contraction*, performing the limit $c \rightarrow \infty$. In this context we only point out that the nonrelativistic mass M is defined as the contracted limit of P_0/c, and thus one obtains (1.3). This implies that it is indeed $\mathcal{G}_4$ (rather than $\mathcal{G}_4$) which arises from $ISO_0(3, 1)$ upon contraction.

Since $ISO_0(3, 1)$ has been declared to be the dynamical invariance group, the equation of motion is obtained by selecting a representation corresponding to an *arbitrary* value of the Casimir operator $\mathcal{C} \equiv P_\mu P^\mu$. Thus, the dynamical equation is

$$P_\mu P^\mu = m^2, \tag{2.2}$$

and the relativistic mass makes its appearance simply as a *representation label*. In other words, the mass operator is *not* an observable contained in the Lie algebra. Unlike the case of $\tilde{\mathcal{G}}_4$ dynamics, the equation of motion now coincides with the selection of an arbitrary representation of the *kinematical* (purely geometric) group. Equation (2.2) describes one *single* kind of state, that of a particle with *fixed* mass: In any irreducible unitary representation of the Poincaré group the mass is a fixed constant.

When constructing a representation of the $ISO(3, 1)$ algebra in the Hilbert space $\mathcal{H}(E_{3,1})$ built upon the *geometrical* background manifold, the familiar realization of (2.2) becomes the Klein-Gordon equation

$$(\square + m^2)\psi(x) = 0. \tag{2.3}$$

Obviously, this is *not* an eigenvalue equation for m^2. Let us once again point out that we have no operator for the relativistic mass in the Lie algebra. Formally, this is related to the circumstance that in the Poincaré algebra there is no analog of (1.3), i.e., no multiple of the identity operator appears in any of the commutators. (This is so because, as is well known,[6] all ray representations of the Poincaré group are equivalent to the faithful representations of its covering group.) For the very same reason, we have no analog of the position operator (1.5) in the Lie algebra.

At this point, we are prepared to raise the following rather unconventional question: *Is the standard transition from the nonrelativistic to the relativistic quantum dynamics* (as outlined above) *the best possible one*? It is not difficult to conceive of reasons why the answer could lie in the negative.

First, it would be desirable to posses a dynamical relativistic mass operator in the Lie algebra of the dynamical group. As we pointed out above, this is not the case. Hence, mass is an "unquantized" parameter. Even if we combined the dynamical Poincaré group with some internal semisimple Lie group, the celebrated O'Raifeartaigh theorem[19] still prevents the emergence of a nontrivial (discrete) mass spectrum. From another viewpoint, the failure in obtaining a mass spectrum can be traced to the Flato-Sternheimer theorem,[20] according to which every extension of the Poincaré algebra by a semisimple Lie algebra is trivial (i.e., it is the direct sum of the two algebras), implying commutativity of the "mass operator" $P_\mu P^\mu$ with all generators of the internal symmetry group.

The second reason why we may be discontent with the standard transition from nonrelativistic to relativistic quantum dynamics is the following. We would like to have in the dynamical Lie algebra a relativistic position operator X_μ. Again, as indicated above, this is not the case. It is true, of course, that several attempts have been made[21] to define, in a somewhat artificial manner, some kind of such operators. However, for several reasons, even these *constructs* are not entirely satisfactory objects. On the other hand, it is possible to define satisfactory operators for the *spatial* position only.[22] However, the existence of such objects is not what we are looking for in the present context.

Summarizing our misgivings in somewhat different formulation, we may say that the conventional transition from the nonrelativistic to the relativistic quantum dynamics is disappointing because (a) no new quantity becomes quantized and (b) actually we seem to lose something in the process, such as the Heisenberg relation $[P_i, X_k] = -i\delta_{ik}$ in the dynamical Lie algebra.

Apart from these physical considerations, one might also view with suspicion the mathematical procedure itself. The Poincaré group does *not* contain the $\tilde{\mathcal{G}}_4$ group as a subgroup, as one would expect if it were

[17] The nonrelativistic mass operator M is not counted in this context.

[18] E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **39**, 510 (1953). See also E. J. Saletan, J. Math. Phys. **2**, 1 (1961).

[19] L. O'Raifeartaigh, Phys. Rev. **139**, B1052 (1965). See also P. Roman and C. J. Koh, Nuovo Cimento **39**, 1015 (1965).

[20] M. Flato and D. Sternheimer, J. Math. Phys. **7**, 1932 (1966).

[21] See, for example, T. O. Philips, Ph.D. thesis, Princeton University, 1963 (unpublished). Further developments are given by H. Bacry, Phys. Letters **5**, 37 (1963); A. Sankaranarayanan and R. H. Good, Phys. Rev. **140**, B509 (1965).

[22] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 300 (1949). For a review of further developments, see A. S. Wightman, *ibid.* **34**, 845 (1962).

a straightforward enlargement of the dynamical framework. Instead, it is the nonrelativistic *kinematical* (geometric) $ISO(3)$ group which is a subgroup of the supposedly *dynamical* relativistic group $ISO_0(3, 1)$. Of course it is true that $ISO_0(3, 1)$ and $\tilde{\mathcal{G}}_4$ are related by contraction.[18] However, this can be interpreted only by saying that nonrelativistic dynamics is a limiting case of the relativistic dynamics. The converse notion of extension fails, since the "expansion" of $\tilde{\mathcal{G}}_4$ is *not unique*.[23]

These last remarks give us a hint of a possible procedure that might lead to a nonconventional passage from nonrelativistic to relativistic quantum mechanics. As we shall see, our proposed structure may be viewed either as a direct generalization of the nonrelativistic Galilei quantum mechanics to a relativistic enlargement, or, alternatively, as an extension of the Poincaré quantum mechanics [in the same sense as the $\tilde{\mathcal{G}}_4$ dynamics is the enlargement of the $ISO(3)$ kinematical framework]. In the development of our proposed structure we shall stress the second point of view.

III. NEW DYNAMICAL GROUP

In this section we shall construct an enlarged framework for relativistic quantum mechanics. Our procedure will closely parallel the transition from $ISO(3)$ to $\tilde{\mathcal{G}}_4$ which was discussed in Sec. I. The reader is asked to pay special attention to the analogies, even when they are not explicitly pointed out.

Our first step in passing from nonrelativistic to relativistic physics is the same as in the standard procedure: We change the manifold $E_3 \times E_1$ into the geometrical manifold $E_{3,1}$ by introducing the usual Minkowski metric (2.1). The group of isometries is the Poincaré group, with the Lorentz transformations

$$x^\mu \to \Lambda_\nu{}^\mu x^\nu \qquad (\Lambda^\nu{}_\mu \Lambda_\nu{}^\rho = g_\mu{}^\rho), \tag{3.1}$$

and the translations

$$x^\mu \to x^\mu + a^\mu. \tag{3.2}$$

However, we do not consider this group (or rather, its connected component) to be the full dynamical invariance group. Instead, following closely the pattern which leads from nonrelativistic kinematics to nonrelativistic dynamics, we introduce now an additional kinematical variable, to be denoted by u. The nature and physical meaning of u is left unspecified at this point.[24] We thus change the underlying manifold from $E_{3,1}$ to the product space $E_{3,1} \times E_1$. No metric is introduced into this manifold, but we consider it as the *carrier space of a new group*. The transformations of this group consist of the Poincaré transformations (3.1) and (3.2) as well as the additional transformations

$$x^\mu \to x^\mu + b^\mu u \tag{3.3}$$

and

$$u \to u + \sigma. \tag{3.4}$$

Here we remark that for future convenience, we take u to have the dimension of length (like all x^μ), so that the four (real, unrestricted) parameters b^μ in (3.3) are dimensionless. The transformations (3.3) are the generalization of the nonrelativistic boost transformations (1.1a). To avoid confusion, we shall call (3.3) the "relativistic Galilean boost" (RG-boost) transformations. In Eq. (3.4), the parameter σ is an unrestricted real number with the dimension of length. The Abelian set (3.4) is the analog of the Galilean time translation (1.1b).

It is easy to verify that the transformations (3.1)–(3.4) indeed form a 15-parameter group over the carrier space $E_{3,1} \times E_1$. The composition law, as well as the Lie algebra, is given in Appendix C. *Our group contains the original* (geometrical) *Poincaré group as a subgroup*. Furthermore, it is a natural and straightforward generalization of the nonrelativistic Galilei group. We shall denote the connected component[25] of our group by $\mathcal{G}_5$. Evidently, $\mathcal{G}_4$ *is a subgroup of* $\mathcal{G}_5$. (One obtains $\mathcal{G}_4$ from $\mathcal{G}_5$ if one restricts the parameters by setting $\Lambda_\nu{}^0 = \Lambda_0{}^k = a^0 = b^0 = 0$ and formally identifies[26] u with t.)

The structure of our $\mathcal{G}_5$ can be represented as[27,28]

$$\mathcal{G}_5 = \{T_4{}^a \times T_1{}^\sigma\} \otimes \{T_4{}^b \otimes SO_0(3, 1)\}. \tag{3.5}$$

Here $T_4{}^a$ is the space-time translation group (3.2), $T_1{}^\sigma$ is the u-translation group (3.4), $T_4{}^b$ is the RG-boost group (3.3), and $SO_0(3, 1)$ is the restricted Lorentz group.

The Lie algebra (C5) tells us that $T_4{}^a \times T_1{}^\sigma \times T_4{}^b$ is an invariant subgroup. Equation (3.5) then reveals that we have the isomorphism

$$SO_0(3, 1) \approx \mathcal{G}_5 / T_4{}^a \times T_1{}^\sigma \times T_4{}^b. \tag{3.6}$$

Hence, *the group* $\mathcal{G}_5$ *is an extension*[10] *of the restricted Lorentz group*.[29] This statement is, of course, more powerful than our previous observation that $\mathcal{G}_5 \supset ISO_0(3, 1)$, and it will have important consequences regarding the representation theory. However, it must be emphasized that $\mathcal{G}_5$ is *not* an extension of the Poincaré group $ISO_0(3, 1)$. On the other hand, we wish to point out here that the Poincaré group $ISO_0(3, 1)$ is itself an extension of the restricted

[23] By "expansion" we mean the procedure inverse to contraction. See, for example, J. Rosen, Nuovo Cimento **35**, 1234 (1965).

[24] This is also true in nonrelativistic kinematics when t is introduced. The only function played by t is that of a parameter labeling the dynamical sequence of states.

[25] Obtained by restricting the Lorentz subgroup (3.1) to the transformations with det $\Lambda = +1$, $\Lambda_0{}^0 \geq +1$.

[26] This observation provides a partial interpretation of the physical role played by u; in the nonrelativistic limit it assumes the role of ordinary time.

[27] It is instructive to compare (3.5) with (1.2).

[28] There are ten other isomorphic forms, but (3.5) is best suited for the study of representations.

[29] The situation is analogous to the case of the nonrelativistic Galilei group, where $\mathcal{G}_4$ is a group extension of $SO(3)$.

Lorentz group $SO_0(3, 1)$, because we have the isomorphism

$$SO_0(3, 1) \approx ISO_0(3, 1)/T_4^a,$$

with T_4^a being an invariant subgroup. Now, as we just said above, our $\mathcal{G}_5$ is also an extension of $SO_0(3, 1)$. Hence, both the customary relativistic dynamical group $ISO_0(3, 1)$ and our new proposed relativistic dynamical group $\mathcal{G}_5$ "grow out" as extensions from $SO_0(3, 1)$, which, in turn, can be looked upon as the group that determines the metric of Minkowski space. We find this observation interesting, because it sheds light on the rather natural emergence of $\mathcal{G}_5$ as a reasonable generalization of $ISO_0(3, 1)$.

For the reasons elucidated above, we make it our *dynamical postulate* that *the laws of dynamics should be invariant under the* $\mathcal{G}_5$ *group*. Naturally, this automatically implies Poincaré invariance, but the latter is considered as only a *kinematical* symmetry (as, similarly, $ISO(3)$ is only a kinematical nonrelativistic symmetry). As we shall see later, the true dynamical development of the system will be associated with the progress according to the new variable u.

For use in *quantum* mechanics, however, we must make a further extension. The reason is the same as in the case of the nonrelativistic Galilei group: The up-to-a-phase representations in Hilbert space determine the generators only up to an additive multiple of the identity operator. When writing down the algebra (C5), we ignored all such additive terms. If one, however, keeps them, it turns out that by simple redefinitions and by the use of the Jacobi identity all but one of these multiples of the identity can be eliminated. Denoting the Lorentz generators of (3.1) by $J_{\mu\nu}$, the translation generators of (3.2) by P_μ, the *RG*-boost generators of (3.3) by Q_μ, and the u-translation generator of (3.4) by S, we actually find the following algebra:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\mu\sigma}J_{\rho\nu} + g_{\nu\sigma}J_{\rho\mu}), \quad (3.7)$$

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), \quad (3.8)$$

$$[P_\mu, P_\nu] = [Q_\mu, Q_\nu] = [J_{\mu\nu}, S] = [P_\mu, S] = 0, \quad (3.9)$$

$$[P_\mu, Q_\nu] = -ig_{\mu\nu}l^{-1}, \quad (3.10)$$

$$[J_{\mu\nu}, Q_\rho] = i(g_{\nu\rho}Q_\mu - g_{\mu\rho}Q_\nu), \quad (3.11)$$

$$[S, Q_\mu] = iP_\mu. \quad (3.12)$$

The departure from (C5) is the relation (3.10). Here the constant l has the dimension of length.

A closer inspection tells us that we are dealing with the central extension[10] of the covering group of $\mathcal{G}_5$ by a phase group. We shall denote this group in what follows by $\tilde{\mathcal{G}}_5$. Its structure is given by[30]

$$\tilde{\mathcal{G}}_5 = \{T_1^\theta \times (T_4^a \times T_1^\sigma)\} \otimes \{T_4^b \otimes SL(2, C)\}. \quad (3.13)$$

Here T_1^θ is the one-dimensional (Abelian) phase group connected with the appearance of l^{-1}. The $SL(2, C)$ in Eq. (3.13) appears as the covering group of $SO_0(3, 1)$. It is readily seen that $\tilde{\mathcal{G}}_5$ can be looked upon[31] as *a group extension of* $SL(2, C)$.

The invariants of $\tilde{\mathcal{G}}_5$, as well as a brief discussion of its representations, are given in Appendix C.

IV. SOME IMMEDIATE CONSEQUENCES

We now try to extricate the basic physical consequences of our $\tilde{\mathcal{G}}_5$ invariance group.

To start with, Eq. (3.10) has an important implication. It permits us to introduce, in a completely natural way, the *relativistic space-time position operators*

$$X_\mu \equiv -lQ_\mu. \quad (4.1)$$

This identification is substantiated not only by the Heisenberg relations (3.10), but also by (3.9) (i.e., $[X_\mu, X_\nu]=0$), and by (3.11), which tells us that X_μ behaves as a four-vector under Lorentz transformations. We shall come back to some properties of X_μ later.

The appearance of l in (3.10) has yet another welcome consequence. It allows, in fact it forces on us, the introduction of a *universal length*, in a completely natural and covariant way. Since $l1$ commutes with everything, we have a *superselection rule*: Systems with different fundamental length are incoherent and do not communicate.[32]

The next question that arises is to find the physical meaning of the generator S. This is achieved by observing that

$$\mathfrak{D} \equiv P_\mu P^\mu + 2l^{-1}S \quad (4.2a)$$

is a Casimir operator of $\tilde{\mathcal{G}}_5$. [It is the analog of $\mathfrak{G}$, the Casimir operator of $\tilde{\mathcal{G}}_4$; cf. Eq. (B4).] Thus, selecting a representation of $\tilde{\mathcal{G}}_5$ characterized by the eigenvalue $\mathfrak{D}'=0$, we have

$$P_\mu P^\mu + 2l^{-1}S = 0. \quad (4.2b)$$

This shows that

$$\mathfrak{M}^2 \equiv -2l^{-1}S \quad (4.3)$$

can be defined as the *relativistic mass-squared operator*. Note that $\mathfrak{M}^2$ lies *in the Lie algebra* of $\tilde{\mathcal{G}}_5$. The definition (4.3) is consistent with the commutation relations (3.9): The mass so defined is a translation-invariant Lorentz scalar, as it should be.

This interpretation of S remains valid also if one chooses a representation with $\mathfrak{D}' \neq 0$. This is so because S occurs only *inside* the commutators of the $\tilde{\mathcal{G}}_5$ algebra, so that we may redefine S to be $S - \mathfrak{D}'$. A more precise justification of this statement is that,

[30] There are ten other isomorphic forms. See Ref. 28.

[31] Similarly as $\tilde{\mathcal{G}}_4$ can be looked upon as an extension of $SU(2)$.

[32] Let us note here that the particular choice $l^{-1}=0$ would change (3.10) to $[P_\mu, Q_\nu]=0$, so that it would bring us back to the original $\mathcal{G}_5$ algebra; i.e., for $l^{-1}=0$ our central extension becomes a trivial extension, a simple direct product. We shall always take $l^{-1} \neq 0$.

as mentioned at the end of Appendix C, all unitary irreducible (one-particle) representations of $\tilde{\mathcal{G}}_5$ which differ only in the value of $\mathfrak{D}$ are equivalent.

Next we observe that, apart from being the mass-squared operator, S has also a second, different role.[33] From Eq. (3.12) and the identification (4.1), we obtain

$$i[S, X_\mu]=lP_\mu, \tag{4.4}$$

so that we can define the *four-velocity operator* U_μ by setting

$$U_\mu\equiv (i/lm)[S, X_\mu]. \tag{4.5}$$

[The evaluation of the right-hand side by means of (4.4) gives P_μ/m. Here m is the mass *eigenvalue*.] Since, on the other hand, the four-velocity is the derivative of position with respect to proper time, Eq. (4.5) tells us that $l^{-1}m^{-1}S$ *is the evolution operator with respect to proper time*. In view of the fact that S is the generator of the u translations (3.4), we now see that the new parameter u serves the role of labeling the sequence of intrinsic dynamical development, as we already suggested in Sec. III.

Since, by (3.9), P_μ and S commute, we have, as a generalization of (4.5),

$$d\Omega/du=i[S, \Omega] \tag{4.6}$$

for every operator Ω that is a function (polynomial) in X_μ and P_ν. The integrated form of (4.6) is

$$\Omega(u)=\exp(iSu)\,\Omega(0)\,\exp(-iSu). \tag{4.7}$$

This displays the *intrinsic development* of Ω from its "initial value" $\Omega(0)$ to an arbitrary "u instance." In particular, if an observable is a *dynamical constant of motion*, it must obey the relation[34]

$$[S, \Omega]=0. \tag{4.8}$$

We emphasize that, owing to (3.9), $J_{\mu\nu}$ and P_μ are constants of motion, as it should be. Furthermore, since S is a translation-invariant scalar, (4.8) is a Poincaré-invariant relation. Finally, in view of (4.3) and (4.8), $\mathfrak{M}^2$ is trivially a dynamical constant of motion, as expected.

Some further insight into the structure of our theory is obtained if we note that the second and third Casimir operators of $\tilde{\mathcal{G}}_5$ are constructed (see Appendix C) from the tensor

$$T_{\mu\nu}\equiv J_{\mu\nu}-lM_{\mu\nu}, \tag{4.9a}$$

where

$$M_{\mu\nu}\equiv P_\mu Q_\nu-P_\nu Q_\mu. \tag{4.9b}$$

[33] This dual role of S is analogous to the dual role of H in $\tilde{\mathcal{G}}_4$, which, on the one hand, is the nonrelativistic energy operator and, on the other hand, is the evolution operator (Hamiltonian) with respect to nonrelativistic time.

[34] Note that in the standard Poincaré-invariant theory there is no intrinsic dynamical development operator. In particular, $[P_0, \Omega]=0$ is *not* a sufficient condition for Ω to be a constant of motion, nor is it a covariant relation. The intrinsic dynamical development in the standard theory is described covariantly by taking the derivative with respect to a spacelike surface, and the condition for Ω to be a constant of motion is $\delta\Omega/\delta\sigma(x)=0$.

The operator

$$K_{\mu\nu}\equiv -lM_{\mu\nu}=P_\mu X_\nu-P_\nu X_\mu \tag{4.10}$$

is interesting inasmuch as it can be looked upon as an "internal" counterpart of $J_{\mu\nu}$. The commutators of $K_{\mu\nu}$ with $J_{\mu\nu}$, P_μ, Q_μ, and S are the same as the corresponding ones for $J_{\mu\nu}$ itself, and also the $[K_{\mu\nu}, K_{\rho\sigma}]$ commutator has the same structure as $[J_{\mu\nu}, J_{\rho\sigma}]$. In particular,

$$K_{ik}=P_iX_k-P_kX_i \tag{4.11}$$

must be interpreted as *internal spin*. This is consistent with the identification of X_μ as position operator.[35]

Additional insight can be gained by specifying an explicit realization of the $\tilde{\mathcal{G}}_5$ algebra in the Hilbert space $\mathcal{H}(E_{3,1}\times E_1)$ defined over the carrier space $E_{3,1}\times E_1$. This is given in Appendix C, Eqs. (C12a)–(C12d). With this realization the position operator (4.1) can be written in the rather remarkable form

$$X_\mu=x_\mu-luP_\mu. \tag{4.12}$$

When $u=0$ [i.e., at the beginning of the dynamical development, cf. (4.7)], the position operator coincides with the geometrical coordinate x_μ. The dynamical development renders the position "nonlocal." The position is "washed out" over a region characterized by the fundamental length l. Thus, the discrepancy between x_μ and X_μ is clearly a microscopic, quantal effect.

It may be also worthwhile to write down the explicit realization of the "total angular momentum" T_{ik} [defined by (4.9a) and (4.9b)]. In the realization (C12), we get

$$T_{ik}=i(x_i\partial_k-x_k\partial_i)+i\Sigma_{ik}+i(\partial_ix_k-\partial_kx_i)=i\Sigma_{ik}. \tag{4.13}$$

Not unexpectedly, in our "one-particle realization" T_{ik} reduces to its *intrinsic* spin part. Some further comments on the spin content of our theory will be given in Appendix C.

V. MASS SPECTRUM

In order to perform explicit calculations, it is best to write down the realization of the equation of motion, Eq. (4.2b), in terms of the differential operators as given by (C12). We obtain

$$(\Box-2l^{-1}i\partial_u)\psi(x;u)=0, \tag{5.1}$$

with $\Box=\partial_\mu\partial^\mu$. This is the analog of the nonrelativistic Schrödinger equation (1.11b). Equation (5.1) is invariant[36] under $\tilde{\mathcal{G}}_5$.

[35] We relegate the discussion of the spin, as well as that of the other quantum numbers associated with the representations of $\tilde{\mathcal{G}}_5$, to Appendix C.

[36] This follows from the way the equation was obtained, but it can be also directly verified by explicit calculation, in a way analogous to the explicit proof of the Galilean invariance of the ordinary Schrödinger equation; cf. Ref. 8.

We now separate coordinates, putting

$$\psi(x;u)=\varphi(x)\chi(u), \tag{5.2}$$

and obtain[37]

$$\chi(u)=\exp(i\tfrac{1}{2}lm^2u) \tag{5.3}$$

and

$$(\Box+m^2)\varphi(x)=0. \tag{5.4}$$

Thus, we obtain the Klein-Gordon (KG) equation. $\tilde{\mathcal{G}}_5$ invariance is lost; we have only Poincaré invariance. The constant m^2 in (5.4) made its appearance as a *separation constant.* Hence, (5.4) has now a different interpretation than in the usual theory: *It is an eigenvalue equation for* m^2. It is the relativistic analog of (1.14). Equation (5.4) is an eigenvalue problem in the Hilbert space $\mathcal{H}(E_{3,1})$, built upon the underlying *geometrical* (not dynamical) mainfold $E_{3,1}$. In the customary interpretation, as we pointed out following Eq. (2.2), the KG equation describes a *single* state. In contrast, in our framework (5.4) describes a *family* of states, with all possible permitted eigenvalues m^2, i.e., with all possible masses. The situation is similar to the one which occurs when we pass from the kinematical $ISO(3)$ group to the dynamical $\tilde{\mathcal{G}}_4$ group; cf. our discussion in the latter part of Sec. I.

Of course, the solution of the eigenvalue problem (5.4) is trivial. The admissible (i.e., "square integrable" over the $E_{3,1}$ space) solutions (with positive energy) are the plane waves

$$\psi(x)=(2\omega_k)^{-1/2}\exp[i(\omega_k x_0-\mathbf{k}\cdot\mathbf{x})],$$

with

$$\omega_k=(\mathbf{k}^2+m^2)^{1/2},$$

and thus m can be any real positive number.

However, suppose we wish to study not a free but rather an interacting $\tilde{\mathcal{G}}_5$-invariant system. The interaction ought to be described by a phenomenological potential which represents the interaction between the two particles. It depends[38] on x only through $x^2=x_0^2-\mathbf{x}^2$. Here x must be interpreted as a relative coordinate, the relativistic center-of-mass motion of the particles having been already separated off. The $\tilde{\mathcal{G}}_5$-invariant equation that replaces (5.1) thus becomes

$$[\Box+V(x^2)-2l^{-1}i\partial_u]\psi(x;u)=0. \tag{5.5}$$

[This is the analog of (1.15).] Separating (5.2) now leads to

$$[\Box+V(x^2)+m^2]\varphi(x)=0. \tag{5.6}$$

Once again, $\tilde{\mathcal{G}}_5$ invariance is lost and reduced to $ISO_0(3,1)$ invariance. Equation (5.6) is a nontrivial eigenvalue problem for m^2. We now discuss the solutions of this problem.

It is best to introduce a biharmonic coordinate system

$$\begin{aligned} x^0&=s\sinh\alpha,\\ x^1&=s\cosh\alpha\sin\theta\cos\phi,\\ x^2&=s\cosh\alpha\sin\theta\sin\phi,\\ x^3&=s\cosh\alpha\cos\theta,\end{aligned} \tag{5.7a}$$

where

$$0\leq s<\infty,\quad -\infty<\alpha<+\infty,\quad 0\leq\theta\leq\pi,\quad 0\leq\phi\leq 2\pi. \tag{5.7b}$$

The coordinate system is so chosen that it corresponds to spacelike points. This is necessary because in our problem x stands for the relative coordinate of the two particles which form a composite system. Thus, corresponding pairs $x^{(1)}$ and $x^{(2)}$ along the world lines of the two particles always belong to a spacelike surface.[39]

For further reference we note that in terms of (5.7) the invariant inner product of two functions f and g becomes

$$(f,g)=\int f^*(s,\alpha,\theta,\phi)g(s,\alpha,\theta,\phi)s^3\cosh^2\alpha \times\sin\theta\, d\phi\, d\theta\, d\alpha\, ds. \tag{5.8}$$

In terms of the new coordinates, (5.6) becomes

$$\{-s^{-3}\partial_s s^3\partial_s+s^{-2}[\cosh^{-2}\alpha\,(\partial_\alpha\cosh^2\alpha\,\partial_\alpha-\partial_\theta{}^2-\cot\theta\,\partial_\theta -\sin^{-2}\theta\,\partial_\phi{}^2)]+V(s)+m^2\}\varphi(s,\alpha,\theta,\phi)=0. \tag{5.9}$$

This equation can be separated by putting

$$\varphi=A(s)B(\alpha)C(\theta)D(\phi), \tag{5.10}$$

and we get

$$(\partial_\phi{}^2+\kappa^2)D(\phi)=0, \tag{5.11a}$$

$$[\partial_\theta{}^2+\cot\theta\,\partial_\theta-\kappa^2/\sin^2\theta+\lambda(\lambda+1)]C(\theta)=0, \tag{5.11b}$$

$$[\cosh^{-2}\alpha\,\partial_\alpha\cosh^2\alpha\,\partial_\alpha+\lambda(\lambda+1)/\cosh^2\alpha -\mu(\mu+2)]B(\alpha)=0, \tag{5.11c}$$

$$[-s^{-3}\partial_s s^3\partial_s+\mu(\mu+2)/s^2+V(s)+m^2]A(s)=0. \tag{5.11d}$$

We are looking for regular, square-integrable solutions. Hence, (5.11a) yields

$$D(\phi)=\exp(i\kappa\phi),\qquad |\kappa|=0,1,2,\cdots. \tag{5.12}$$

Therefore, the regular solution of (5.11b) becomes

$$C(\theta)=P_\lambda{}^\kappa(\cos\theta),\qquad \lambda=0,1,2,\cdots,\text{ and }|\kappa|\leq\lambda. \tag{5.13}$$

Proceeding, the solution of (5.11c) is found to be[40]

$$B(\alpha)=\cosh\alpha\, P_\lambda{}^{\mu+1}(\tanh\alpha). \tag{5.14}$$

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[37] Since $-2l^{-1}i\partial_u$ is the realization of the operator $\mathfrak{M}^2=-2l^{-1}S$, the separation constant is correctly denoted by m^2.

[38] We do not consider u-dependent potentials, since they would not allow for states that are stationary with respect to u development.

[39] Putting it in another way, we may say that when $x_0^{(1)}=x_0^{(2)}$, we have $x^2=-(\mathbf{x}^{(1)}-\mathbf{x}^{(2)})^2=-\mathbf{x}^2$, so that in this system we clearly have $x^2<0$. Lorentz covariance then leads to $x^2<0$ in general.

[40] This can be seen by setting $B=(1-z^2)^{-1/2}\hat{B}(z)$, with $z=\tanh\alpha$, which leads to the standard Legendre equation for $\hat{B}(z)$.

Polynomial behavior of $P_\lambda^{\mu+1}$ requires that $\lambda-\mu-1$ be a non-negative integer. Since we already know that λ is a non-negative integer, this implies that μ is an *integer,* and we have the relation

$$\lambda=\mu+1+n, \qquad n=0, 1, 2, \cdots. \tag{5.15}$$

Furthermore, we have the constraint $\mu+1\leq\lambda$, so that, for given λ, the new quantum number μ has the range

$$\mu=-1, 0, 1, \cdots, \lambda-2, \lambda-1. \tag{5.16}$$

It can now be checked that our function

$$R_{\kappa\lambda\mu}(\phi, \theta, \alpha)\equiv D(\phi)C(\theta)B(\alpha)$$

is square integrable on the unit hyperboloid $s^2=-1$, with respect to the measure $da\equiv\cosh^2\alpha\,\sin\theta\,d\phi\,d\theta\,d\alpha$ [as implied by (5.8)]. Actually, with a normalization factor supplied, we have the orthonormality relation

$$(R_{\kappa\lambda\mu}, R_{\kappa'\lambda'\mu'})=\delta_{\kappa\kappa'}\delta_{\lambda\lambda'}\delta_{\mu\mu'}. \tag{5.17}$$

Thus, when we now turn to Eq. (5.11d), we only have to guarantee the square integrability of $A(s)$ with respect to the measure $db\equiv s^3ds$.

To solve (5.11d), we set

$$A(s)=s^{-3/2}g(s), \tag{5.18}$$

which transforms it to

$$[\partial_s^2-\mu'(\mu'+1)/s^2-V(s)-m^2]g(s)=0, \tag{5.19a}$$

where

$$\mu'=\mu+\tfrac{1}{2}. \tag{5.19b}$$

Equation (5.19a) has exactly the same structure as the familiar radial equation in nonrelativistic quantum mechanics. We are looking for regular solutions with $m^2>0$. Unfortunately, there are only a few types of "potential" $V(s)$ which are known to permit an exact solution. We consider first

$$V(s)=-v/s, \qquad v>0 \text{ const.} \tag{5.20}$$

The square-integrable solution is

$$A(s)=s^{\mu-1}e^{-ms}L_{\nu+2\mu+1}{}^{2\mu+2}(2ms), \tag{5.21a}$$

with

$$\nu=1, 2, 3, \cdots. \tag{5.21b}$$

The corresponding eigenvalues are

$$m_{\nu,\mu}{}^2=v^2/4(\nu+\mu+\tfrac{1}{2})^2. \tag{5.21c}$$

Another example is afforded by

$$\begin{aligned} V(s)&=0 \quad \text{for} \quad 0\leq s\leq s_0\\ &=\infty \quad \text{for} \quad s>s_0. \end{aligned} \tag{5.22}$$

A square-integrable solution exists only if $\mu\neq-1$, and it is given by

$$A(s)=s^{-2}J_{\mu+1}(ms) \quad \text{for} \quad 0\leq s\leq s_0 \tag{5.23a}$$

and $A(s)=0$ for $s>s_0$. Hence, the mass spectrum is given by the (nonzero) roots of the equation

$$J_{\mu+1}(m_{\nu,\mu}s_0)=0, \tag{5.23b}$$

with $\nu=1, 2, 3, \cdots$ labeling the successive zeros. Unlike in the previous example, the spectrum is not bounded from above, and the higher mass values tend to be equally spaced and to be closer to each other than the lower mass levels are. These are quite agreeable features. Naturally, neither example gives a truly acceptable hadron mass spectrum.

Of course, the generation of a mass spectrum by some guessed "potential" should not be taken seriously and serves only an illustration of having a nontrivial spectrum. A more realistic approach to the mass-spectrum problem would be to combine our space-time group $\tilde{\mathcal{G}}_5$ with some internal symmetry group [like $SU(3)$] and investigate the ensuing structure in relation to the mass operator S. We shall attempt to carry out this program at a later time. Here we only note that, since $\tilde{\mathcal{G}}_5$ is *not* an extension of the Poincaré group, the Flato-Sternheimer theorem[20] will not apply when $\tilde{\mathcal{G}}_5$ is further extended by internal symmetries, so that there is no reason why this extension should be trivial. Hence, the investigation of the emergence of a nontrivial mass spectrum appears appealing.

VI. CONCLUDING REMARKS

In this paper we have proposed a new relativistic space-time group which seems eminently suitable for the quantum-mechanical description of elementary particles. From the heuristic point of view, we find it intriguing that our group arises from the metric-specifying Lorentz group in the same way as the nonrelativistic Galilei group arises from the corresponding metric-specifying rotation group; cf. our discussion following Eq. (3.6). Once we accept the group $\tilde{\mathcal{G}}_5$, we are immediately in possession of a natural space-time position operator X_μ and a mass-squared operator $-2l^{-1}S$, both being members of the Lie algebra. We also have a way to specify covariantly internal dynamical development, with the help of the S operator itself. Finally, as will be briefly discussed in Appendix C, our symmetry group leads directly to the emergence of towers of states with increasing spin. We find these features very interesting.

In conclusion, we wish to briefly touch upon the following problem: Is it possible to obtain our dynamical group $\tilde{\mathcal{G}}_5$ by the *process of contraction* from some geometrical group of isometries, in a manner analogous to the construction of the nonrelativistic dynamical group $\tilde{\mathcal{G}}_4$ from the covering of the connected Poincaré group? The answer lies in the affirmative. It is not difficult to see that $\tilde{\mathcal{G}}_5$ is the contracted limit of the covering of the connected component of $ISO(3, 2)$, the inhomogeneous de Sitter group, which is the group of isometries in $E_{3,2}$. The contraction parameter is g_{44}, defined by $d\sigma^2=g_{ab}dx^adx^b$ (a, $b=0, 4, 1, 2, 3$). The relationship between $\tilde{\mathcal{G}}_5$ and $ISO(3, 2)$ will be investigated elsewhere. [Cf. J. Math.

Phys. (to be published).] We only mention that this study will shed additional light on the properties of the RG-boost operator Q_μ and of the mass operator S. It may also give rise to a cosmological interpretation.

The relationships between the Euclidean, nonrelativistic Galilean, Poincaré, relativistic Galilean, and inhomogeneous de Sitter groups can be well visualized by the following diagram:

$$\begin{array}{ccccc} ISO(3) & \subset & ISO(3,1) & \subset & ISO(3,2) \\ \cap & \swarrow & \cap & \swarrow & \\ \tilde{\mathcal{G}}_4 & \subset & \tilde{\mathcal{G}}_5. & & \end{array}$$

(The symbols stand for the respective Lie algebras. The arrows indicate contraction, and $\subset$ means inclusion.) These relations are quite instructive. For example, we see that nonrelativistic kinematics can be obtained from $\tilde{\mathcal{G}}_5$ either by going first to the nonrelativistic dynamics $\tilde{\mathcal{G}}_4$ and then to the corresponding Euclidean kinematics, or by going first to the Poincaré framework and then to the Euclidean system. This illustrates our point that the Poincaré framework should be considered more of a kinematical (rather than dynamical) symmetry.

Note added in proof. While this paper was in print, L. Castell kindly called our attention to his work in Nuovo Cimento **49,** 285 (1967), in which he constructed Lie algebras that contain a relativistic position operator. One of his algebras is isomorphic to the Lie algebra of our group $\tilde{\mathcal{G}}_5$. We also note that the recent paper by J. E. Johnson, Phys. Rev. **181,** 1755 (1969), contains an interesting discussion of position operators and proper time.

APPENDIX A: GROUP EXTENSIONS

In order to make certain subtle mathematical points (that were referred to in the text) more acessible to a wider set of readers, we give here some concepts concerning *group extensions.*

Let $\mathcal{G}$ and $\mathcal{K}$ be two groups and let Γ be an invariant subgroup of $\mathcal{G}$. If we have the isomorphism

$$\mathcal{K}\approx\mathcal{G}/\Gamma, \tag{A1}$$

then we say that $\mathcal{G}$ is an *extension* of $\mathcal{K}$. It then follows that there is a one-to-one correspondence $h:k\to c$ between the elements k of $\mathcal{K}$ and the elements c of the coset space $\mathcal{G}/\Gamma$. Thus, an element $c\in\mathcal{G}/\Gamma$ can be written as $c=h(k)$, and the composition law is

$$h(k_1)h(k_2)=\omega(k_1,k_2)h(k_1\cdot k_2). \tag{A2}$$

Here $\omega(k_1,k_2)\in\Gamma$ and is called a *factor system.* Furthermore, any element g of $\mathcal{G}$ can be uniquely decomposed as

$$g=\gamma\cdot h(k) \quad \text{with} \quad \gamma\in\Gamma \quad \text{and} \quad h(k)\in\mathcal{G}/\Gamma. \tag{A3}$$

The composition law of $\mathcal{G}$ is then given by

$$g_1g_2=\gamma_1\cdot h(k_1)\cdot\gamma_2\cdot[h(k_1)]^{-1}\cdot\omega(k_1,k_2)\cdot h(k_1\cdot k_2). \tag{A4}$$

We note that if Γ is a one-dimensional (Abelian) group $T_1{}^\theta$, then $\omega(k_1,k_2)$ is just a phase factor. We then speak of a *scalar extension.*

Suppose now that there exists for any $h(k)\in\mathcal{G}/\Gamma$ (hence, for any $k\in\mathcal{K}$) a certain element $\gamma_k\in\Gamma$ such that

$$h(k)\cdot\gamma\cdot[h(k)]^{-1}=\gamma_k\cdot\gamma\cdot\gamma_k{}^{-1} \quad \text{for all } \gamma\in\Gamma. \tag{A5}$$

We then call the group $\mathcal{G}$ a *central extension* of $\mathcal{K}$. In particular, if Γ is a one-dimensional (Abelian) group $T_1{}^\theta$ which belongs to the center of $\mathcal{G}$, then we have, obviously,

$$h(k)\cdot\gamma\cdot[h(k)]^{-1}=\gamma,$$

i.e., (A5) is fulfilled, and so we have a central extension $\mathcal{G}$ of $\mathcal{K}$ by the phase group $T_1{}^\theta$. Equation (A4) then reveals that

$$\mathcal{G}=T_1{}^\theta\otimes\mathcal{K}. \tag{A6}$$

If, in particular, the factor system $\omega(k_1,k_2)$ is not only a phase, but actually $\omega(k_1,k_2)=+1$ for every k_1, k_2, then the central extension becomes trivial and we have the direct product

$$\mathcal{G}=T_1{}^\theta\times\mathcal{K}. \tag{A7}$$

APPENDIX B: NONRELATIVISTIC QUANTUM-MECHANICAL GALILEI GROUP

For convenience and easy reference, we list here some basic facts about the *nonrelativistic Galilei group.*

The carrier space is $E_3\times E_1$, with $x=(x_1,x_2,x_3)\in E_3$ and $t\in E_1$. The transformations are written, in a condensed form, as

$$\begin{aligned} x\to x'&=Rx+vt+a,\\ t\to t'&=t+\tau, \end{aligned} \tag{B1}$$

where R is a rotation matrix of $SO(3)$, a is a translation in E_3, τ is a translation in E_1, and v is the boost in $E_3\times E_1$. Here τ is a scalar and a and v are three-vectors under $SO(3)$. The group so defined is denoted by $\mathcal{G}_4$.

We shall be interested only in the central extension of the covering group of $\mathcal{G}_4$ by a phase group. The structure of this quantum-mechanical Galilei group $\tilde{\mathcal{G}}_4$ is then

$$\tilde{\mathcal{G}}_4=\{T_1{}^\alpha\times(T_3{}^a\times T_1{}^\tau)\}\otimes\{T_3{}^v\otimes SU(2)\}, \tag{B2}$$

as was discussed in Sec. I. Denoting the generators of $SU(2)$, $T_3{}^a$, $T_3{}^v$, and $T_1{}^\tau$ by J_k, P_k, G_k, and H, respectively, we have the Lie algebra

$$[J_i,J_k]=i\epsilon_{ikl}J_l, \tag{B3a}$$

$$[P_i,J_k]=-i\epsilon_{ikl}P_l, \tag{B3b}$$

$$[P_k,P_l]=[G_k,G_l]=[J_k,H]=[P_k,H]=0, \tag{B3c}$$

$$[P_k,G_l]=-i\delta_{kl}M, \tag{B3d}$$

$$[J_k,G_l]=i\epsilon_{klm}G_m, \tag{B3e}$$

$$[H,G_k]=-iP_k. \tag{B3f}$$

The Casimir operators are

$$\mathcal{B}=\tfrac{1}{2}M^{-1}\mathbf{P}^2-H \tag{B4}$$

and

$$\mathfrak{N}=\mathbf{F}^2, \tag{B5a}$$

where

$$F_k=J_k+M^{-1}\epsilon_{klj}P_lG_j. \tag{B5b}$$

Setting $X_k=M^{-1}G_k$, we may rewrite (B5b) as

$$\mathbf{F}=\mathbf{J}+\mathbf{P}\times\mathbf{X}, \tag{B5c}$$

which reveals that $\mathbf{F}$ is the spin angular momentum.

As suggested by (B4) and (B5), the irreducible unitary projective representations[14] of $\tilde{\mathcal{G}}_4$ are labeled by (a) an arbitrary real number M, (b) an arbitrary real number $\mathcal{B}$, and (c) an integer or half-integer number s. [The latter is the index of the familiar finite-dimensional unitary representation D^s of $SU(2)$. Alternatively, the eigenvalue of $\mathfrak{N}$ could be used too.] A representation is denoted by the symbol $(M\,|\,\mathcal{B}, s)$.

A realization of the $\tilde{\mathcal{G}}_4$ algebra in the Hilbert space over $E_3\times E_1$ is given by

$$J_k=-i\epsilon_{klj}x_l\partial_j-i\Sigma_k, \tag{B6a}$$

$$P_k=-i\partial_k, \tag{B6b}$$

$$G_k=-it\partial_k+Mx_k, \tag{B6c}$$

$$H=i\partial_t. \tag{B6d}$$

Here Σ_k denotes the familiar finite-dimensional representation matrices of the $SU(2)$ generators.

In the realization (B6), the angular momentum operator F_k becomes simply $F_k=-i\Sigma_k$, i.e., the spin. Hence, we can identify the representation label s with the particle spin. The label M is identified with the mass, and the label $\mathcal{B}$ with internal energy. However,[8] the representations with different values of $\mathcal{B}$ are equivalent in the sense that

$$\mathcal{U}_{M,\mathcal{B},s}(g)=\exp(i\tau\mathcal{B})A^{-1}\mathcal{U}_{M,0,s}(g)A, \tag{B7}$$

where $\mathcal{U}_{M,\mathcal{B},s}(g)$ is the operator corresponding to an arbitrary group element $g\in\tilde{\mathcal{G}}_4$ in the representation $(M\,|\,\mathcal{B}, s)$, and A is an isometric operator. Hence, in the free-particle realization, $\mathcal{B}$ has no significance, and can be taken to be zero. (See, however, Ref. 16.)

APPENDIX C: RELATIVISTIC GALILEI GROUP

In this appendix we summarize the mathematical properties of our proposed new *relativistic Galilei group*. For completeness, we shall repeat a few items and formulas that were already presented in the main text.

The carrier space is $E_{3,1}\times E_1$, with $x=(x^0, x^1, x^2, x^3)\in E_{3,1}$ and $u\in E_1$. The transformations are written, in a condensed form, as

$$x\to x'=\Lambda x+bu+a, \qquad u\to u'=u+\sigma, \tag{C1}$$

where Λ is a Lorentz matrix of the connected component $SO_0(3, 1)$ of the Lorentz group,[1] a is a translation in $E_{3,1}$, σ is a translation in E_1, and b is the RG-boost in $E_{3,1}\times E_1$. Here σ is a scalar and a and b are four-vectors under $SO_0(3, 1)$. The group so defined is denoted by $\mathcal{G}_5$. An arbitrary element will by symbolized by $g=(\sigma, a, b, \Lambda)$. The unit element is $(0, 0, 0, 1)$. The composition law is

$$(\sigma_2, a_2, b_2, \Lambda_2)(\sigma_1, a_1, b_1, \Lambda_1) =(\sigma_2+\sigma_1,\ a_2+\Lambda_2a_1+\sigma_1b_2,\ b_2+\Lambda_2b_1,\ \Lambda_2\Lambda_1). \tag{C2}$$

The inverse element is

$$g^{-1}=(-\sigma,\ -\Lambda^{-1}(a-\sigma b),\ -\Lambda^{-1}b,\ \Lambda^{-1}). \tag{C3}$$

The group structure is[28]

$$\mathcal{G}_5=\{T_4{}^a\times T_1{}^\sigma\}\otimes\{T_4{}^b\otimes SO_0(3, 1)\}. \tag{C4}$$

Even though our interest lies in the corresponding quantum-mechanical group $\tilde{\mathcal{G}}_5$, we first give here some details about the algebra of $\mathcal{G}_5$ itself.

Denoting the generators of $SO_0(3, 1)$, $T_4{}^a$, $T_4{}^b$, and $T_1{}^\sigma$ by $J_{\mu\nu}$, P_μ, Q_μ, and S, respectively, we find

$$[J_{\mu\nu}, J_{\rho\sigma}]=i(g_{\nu\rho}J_{\mu\sigma}-g_{\mu\rho}J_{\nu\sigma}-g_{\mu\sigma}J_{\rho\nu}+g_{\nu\sigma}J_{\rho\mu}), \tag{C5a}$$

$$[P_\mu, J_{\rho\sigma}]=i(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho), \tag{C5b}$$

$$[P_\mu, P_\nu]=[Q_\mu, Q_\nu]=[J_{\mu\nu}, S]=[P_\mu, S]=0, \tag{C5c}$$

$$[P_\mu, Q_\nu]=0, \tag{C5d}$$

$$[J_{\mu\nu}, Q_\rho]=i(g_{\nu\rho}Q_\mu-g_{\mu\rho}Q_\nu), \tag{C5e}$$

$$[S, Q_\mu]=iP_\mu. \tag{C5f}$$

The Casimir operators are

$$I_1=P_\mu P^\mu, \tag{C6a}$$

$$I_2=W_\mu W^\mu. \tag{C6b}$$

Here we used the following notation:

$$W_\mu=\epsilon_{\mu\rho\sigma\nu}J^{\rho\sigma}P^\nu. \tag{C7}$$

It is interesting to note that I_1 and I_2 are precisely the familiar invariants of the Poincaré group.

As was explained, from the physical point of view, in Sec. III, the next step is to go to the covering group of $\mathcal{G}_5$ [by replacing $SO_0(3, 1)$ with $SL(2, C)$] and then to perform a central extension by a phase group $T_1{}^\theta$. We thus obtain the quantum-mechanical relativistic Galilei group, which has the structure[30]

$$\tilde{\mathcal{G}}_5=\{T_1{}^\theta\times(T_4{}^a\times T_1{}^\sigma)\}\otimes\{T_4{}^b\otimes SL(2, C)\}. \tag{C8}$$

We shall symbolize an element $g\in\tilde{\mathcal{G}}_5$ by $g=(\exp(i\theta);\ \sigma, a, b, \Lambda)$. For the factor system (cf. Appendix A) we conveniently use the form[41]

$$\omega(g_1, g_2)\equiv\exp\{il^{-1}f(\sigma_1, a_1, b_1, \Lambda_1; \sigma_2, a_2, b_2, \Lambda_2)\}. \tag{C9}$$

[41] The function f has the explicit form $f=(b_2\Lambda_2a_1+\frac{1}{2}b_2{}^2\sigma_1)$, but alternative forms are possible. The constant l^{-1} with the dimension of inverse length appears in the exponent of ω because f has the dimension of length.

The composition law of $\tilde{\mathcal{G}}_5$ can then be written as

$$(\exp(i\theta_2);\sigma_2,a_2,b_2,\Lambda_2)(\exp(i\theta_1);\sigma_1,a_1,b_1,\Lambda_1)$$
$$=(\exp i(\theta_2+\theta_1+l^{-1}f);\sigma_2+\sigma_1,a_2+\Lambda_2a_1+\sigma_1b_2,$$
$$b_2+\Lambda_2b_1,\Lambda_2\Lambda_1). \quad \text{(C10)}$$

The unit element is $(1;0,0,0,1)$. The inverse element of g becomes

$$g^{-1}=(\exp i(-\theta-l^{-1}\hat{f});-\sigma,-\Lambda^{-1}(a-\sigma b),-\Lambda^{-1}b,\Lambda^{-1}),$$

where $\hat{f}$ is the function in (C9) which corresponds to $\omega(g,g^{-1})$.

The Lie algebra of $\tilde{\mathcal{G}}_5$ is now easily found. One obtains the same relations (C5) as were found for $\mathcal{G}_5$ itself, *with the exception of* Eq. (C5d), *which is replaced by*[42]

$$[P_\mu,Q_\mu]=-il^{-1}g_{\mu\nu}. \quad \text{(C11)}$$

Furthermore, the meaning of the operators $J_{\mu\nu}$ changes: They are to be looked upon as the generators of $SL(2,C)$ rather than of $SO_0(3,1)$. We note that if l^{-1} is set equal to zero, by (C9), the factor system ω reduces to the constant value 1, so that the central extension becomes trivial, a direct product of T_1^θ and the covering of $\mathcal{G}_5$.

An explicit realization of the Lie algebra (3.7)–(3.12) of $\tilde{\mathcal{G}}_5$ in the Hilbert space $\mathcal{H}(E_{3,1}\times E_1)$ built upon the carrier space $E_{3,1}\times E_1$ is easily constructed and has the following form[43]:

$$J_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu)+i\Sigma_{\mu\nu}, \quad \text{(C12a)}$$

$$P_\nu=i\partial_\nu, \quad \text{(C12b)}$$

$$Q_\nu=iu\partial_\nu-l^{-1}x_\nu, \quad \text{(C12c)}$$

$$S=i\partial_u. \quad \text{(C12d)}$$

In (C12a) the matrix $i\Sigma_{\mu\nu}$ is what the physicist usually calls "the intrinsic spin part" of $J_{\mu\nu}$. One may, of course, choose these operators to be the familiar finite-dimensional representatives of the $SL(2,C)$ group. [Thus, for example, for "spin $\frac{1}{2}$" one has $\Sigma_{\mu\nu}=\frac{1}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, with γ_μ being the Dirac matrices.] When so doing, Eq. (4.13) tells us that the realization (C12) describes a particle with unique spin, whose value is determined by the eigenvalues of $i\Sigma_{ik}$. However, as will be discussed below, the use of finite-dimensional representations for $\Sigma_{\mu\nu}$ would imply that the realization of the algebra of $\tilde{\mathcal{G}}_5$ by (C12) is not Hermitian. Conversely, in order that (C12) represent a Hermitian realization, it is necessary to interpret the $\Sigma_{\mu\nu}$ as the infinite-dimensional matrices associated with the irreducible unitary representations of $SL(2,C)$.

[42] To save space, we do not write down here the full Lie algebra of $\tilde{\mathcal{G}}_5$. In any case, it has been written down before; see Eqs. (3.7)–(3.12).

[43] It may be worth while to point out that the corresponding realization of the Lie algebra (C5) of $\mathcal{G}_5$ is of the same form as (C12) except for (C12c), which is replaced by $Q_\nu=iu\partial_\nu$.

We now turn to the discussion of the Casimir operators of $\tilde{\mathcal{G}}_5$. They are found to be[44,45]

$$\mathfrak{D}=P_\mu P^\mu+2l^{-1}S, \quad \text{(C13a)}$$

$$\mathcal{J}=\tfrac{1}{2}T_{\mu\nu}T^{\mu\nu}, \quad \text{(C13b)}$$

$$\mathcal{K}=\tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}. \quad \text{(C13c)}$$

Here we used the abbreviation

$$T_{\mu\nu}\equiv J_{\mu\nu}-lM_{\mu\nu}, \quad \text{(C14a)}$$

with

$$M_{\mu\nu}\equiv P_\mu Q_\nu-P_\nu Q_\mu. \quad \text{(C14b)}$$

We observe that *the operators* $\mathcal{J}$ *and* $\mathcal{K}$ *are the Casimir operators of an* $SL(2,C)$ *algebra.* It is well known[46] that in the unitary irreducible representations, the eigenvalues of these operators are

$$\mathcal{J}'=a_0^2+a_1^2-1, \qquad \mathcal{K}'=2ia_0a_1, \quad \text{(C15)}$$

where

$$a_0=0,\tfrac{1}{2},1,\tfrac{3}{2},\cdots, \qquad a_1=\text{arbitrary pure imaginary} \quad \text{(C16a)}$$

in the representations belonging to the "principal series," and

$$a_0=0, \qquad a_1=\text{arbitrary real}, \quad 0<a_1<1 \quad \text{(C16b)}$$

in the representations belonging to the "supplementary series."

Equations (C13a)–(C13c) thus suggest that the irreducible unitary projective representations of our $\tilde{\mathcal{G}}_5$ are labeled by (a) an arbitrary real number l, (b) an arbitrary real number $\mathfrak{D}$, and (c) two quantum numbers a_0 and a_1, as specified by (C16a) or (C16b). Such a representation will be denoted by the symbol $(l\,|\,\mathfrak{D},a_0,a_1)$.

In order to justify this result fully, as well as to explore the physical interpretation of the quantum numbers, it is necessary to construct explicitly the irreducible unitary projective representations of $\tilde{\mathcal{G}}_5$. The details of this procedure will be given elsewhere; here we only sketch the calculation.

The representations in question will be induced by the subgroup

$$\{T_4^a\times T_1^\sigma\times T_1^\theta\}\otimes\mathfrak{R}, \quad \text{(C17)}$$

where $\mathfrak{R}$ is the stabilizer of the orbits in $T_4^a\times T_1^\sigma$. Denoting the eigenvalues of P_μ by p_μ and those[47] of S by r, a point of an orbit is represented by (p_μ,r)

[44] Using the fact that, as stated in Sec. VI, $\tilde{\mathcal{G}}_5$ arises as the contraction of $ISO_0(3,2)$, one can check that there are no more invariants than the three listed below.

[45] It is interesting to compare these invariants of $\tilde{\mathcal{G}}_5$ with those of the original $\mathcal{G}_5$; cf. Eq. (C6).

[46] See, for example, I.M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Group* (Pergamon, New York, 1963), p. 200.

[47] In view of Eq. (4.3), the eigenvalues r of S are related to the mass-squared eigenvalues m^2 by $r=-\frac{1}{2}lm^2$.

and the orbits are given by the equation

$$p_\mu p^\mu + 2l^{-1}r = \mathfrak{D}. \tag{C18}$$

It is then found that the stabilizer $\mathfrak{R}$ in (C17) is precisely the group $SL(2,C)$.

We now choose a set of basis functions

$$|\,p_\mu, r, \xi, \eta\rangle,$$

where ξ, η serve to label the components of the representation space of $SL(2, C)$. As is well known,[48] in the unitary irreducible representations of $SL(2, C)$ (discussed above), the labels ξ and η take on the discrete values

$$\xi = a_0,\ a_0+1,\ a_0+2,\ \cdots,$$
$$\eta = -\xi,\ -\xi+1,\ \cdots,\ \xi-1,\ \xi. \tag{C19}$$

Eventually, with a suitable invariant measure being defined, the irreducible unitary projective representations of $\tilde{\mathcal{G}}_5$ are found to have the following form[49]:

$$U(\exp(i\theta);\sigma, a, b, \Lambda)\,|\,p_\mu, r, \xi, \eta\rangle$$
$$= \exp[i(r\sigma + p_\mu a^\mu + \beta\theta)]$$
$$\times [D^{a_0a_1}(G)]_{\xi'\eta'\xi\eta}\,|\,p_\mu', r', \xi', \eta'\rangle. \tag{C20a}$$

Here the RG-boosted p_μ' and r' are given by

$$p' = \Lambda^{-1}(p - l^{-1}b), \qquad r' = r + pb - \tfrac{1}{2}l^{-1}b^2. \tag{C20b}$$

The $D^{a_0a_1}(G)$ is an infinite-dimensional matrix belonging to some unitary irreducible representation of $SL(2, C)$ (see above), representing the group element G. The latter is given by

$$G = V_{p_\mu,r}^{-1}\Omega V_{p_\mu',r'}, \tag{C21a}$$

where Ω is a particular element of the factor group $T_4^b \otimes SL(2, C)$, viz.,

$$\Omega = (1; 0, 0, b, \Lambda), \tag{C21b}$$

and $V_{p_\mu,r}$ is that element of the same factor group which transforms an (arbitrary) fixed point $(\hat{p}_\mu, \hat{r})$ of the orbit (C18) into the given point (p_μ, r). Finally, in (C20) summation over ξ' and η' [with the ranges as given in (C19)] is understood.

Now we briefly discuss the physical interpretation of the quantum numbers associated with the above representation $(l\,|\,\mathfrak{D}, a_0, a_1)$. We have already pointed out at the beginning of Sec. IV that l can be looked upon as a universal length. Concerning $\mathfrak{D}$, then, we have the following comment to make. From (C20a), it can be shown that representations differing only in the value of $\mathfrak{D}$ are equivalent projective representations, in the sense that

$$\mathfrak{U}_{l,\mathfrak{D},a_0,a_1}(g) = \exp(-\tfrac{1}{2}il\sigma\mathfrak{D})A^{-1}\mathfrak{U}_{l,0,a_0,a_1}(g)A, \tag{C22}$$

where we used a notation analogous to the one employed in (B7). Thus, in the free-particle realization, $\mathfrak{D}$ is of no significance and can be taken to be zero.[50]

Finally, we consider the two remaining quantum numbers. From (4.13) and (C12a) it is evident that we wish to interpret the label a_0 of the representation as *spin*. Equation (C20) together with (C19) and (C16a) tells us that this indeed is possible. However, and this is a very interesting feature of our framework, (C19) shows that *our representations describe not a single spin value but rather, for each representation* $(l\,|\,\mathfrak{D}, a_0, a_1)$, *an infinite tower of spin states*, starting with the lowest value $s = a_0$, and going up in integral steps. [To each value $s = a_0 + n$ we, of course, have a $(2s+1)$-fold degeneracy, differing in spin component.]

The additional quantum number a_1, related to the eigenvalues of the noncompact part Σ_{0k} of the "spin operator" $\Sigma_{\mu\nu}$, does not lend itself to such a simple interpretation. At the present stage, we can only say that *both* a_0 and a_1 are needed to select a definite "tower."

[48] See, for example, p. 188 of Ref. 46.

[49] The constant β is arbitrary.

[50] For the interacting case, this is no longer true; but we may still renormalize the "internal" part of the mass squared of the composite system by the amount $\mathfrak{D}^1 + \mathfrak{D}^2$. The reduction of products of representations will be discussed elsewhere.