Axial-Vector Sum Rules for He^3 †

THOMAS F. WALSH* AND EARL A. PETERSON
School of Physics, University of Minnesota, Minneapolis, Minnesota 55455
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This paper presents and discusses two sum rules for the axial-vector form factor for the He^3-H^3 isodoublet, the conventional Weisberger-Adler sum rule, and a new plane-wave sum rule. Both sum rules are well satisfied. A method of improving the impulse-approximation estimate of pion-cross-section differences is presented, and the various disintegration cuts and anomalous thresholds inherent in a nuclear dispersion relation are discussed.

I. INTRODUCTION

THE success of the Weisberger-Adler (WA) sum rule for the nucleon axial-vector form factor f_A leads immediately to questions about its applicability to other systems, including nuclei, which one might hope to treat as elementary in some sense. One might use the sum rule either for information about pion-nucleus scattering or axial-vector form factors.

Our interest in the particularly simple case of the WA sum rule for He^3 is motivated by the latter point—that one might be able to calculate the $H^3 \to He^3 + e + \nu_e$ axial-vector form factor without having to consider the nuclear physics of the problem in any great detail, in analogy to the Thomas-Reiche-Kuhn sum rule for atomic systems. The elementary-particle treatment of light nuclei has had some success in nuclear β-decay calculations (particularly those of Kim and Primakoff)[1] and originated in the muon-capture calculations of Dreschsler and Stech,[2] and Fujii and Yamagouchi.[3]

The experimental value of the H^3-He^3 axial-vector constant F_A can be obtained directly from the (ft) value of H^3, or from a ratio of the (ft) values for H^3 and the neutron. Since the theory of radiative corrections to nuclear β decay is not well developed, we choose the second course and use (ft) values determined at a consistent level of approximation,[4]

$$\frac{(ft)_n}{(ft)_{H^3}} = \frac{1+3F_A{}^2}{1+3f_A{}^2} = \frac{1187\pm35}{1132\pm40},$$

for $|f_A| = 1.18\pm0.02$, $|F_A| = 1.22\pm0.07$.

In direct nuclear-physics calculations (i.e., outside the elementary-particle model) there is no apparent structure independence of F_A. The problems of determining the internal nuclear wave function and of expressing the exchange effects are both unsolved and linked. This is an enormous problem for heavy nuclei and is already acute in He^3, where the Q value of the β-transition is small (18 keV) and the nucleons are lightly bound (8 MeV).

As the Goldberger-Treiman (GT) relation for He^3 is marred by the appearance of anomalous thresholds in the axial-vector form factor, it is possible that the anomalous thresholds in the WA sum rule itself are just as serious. It is not obvious that the WA sum rule can be of practical utility in nuclei unless these structure effects are insignificant (or at least as minor as for the nucleon).

The price that must be paid for the simplicities of the WA sum rule is the covert appearance of the nuclear structure effects as analytic structure in the sum rule—a nuclear disintegration cut, and the appearance of anomalous thresholds in the extrapolation to zero pion mass of physical π-He^3 scattering. Also, the experimental input to the sum rule, $\pi^{\pm}$-He^3 scattering data, does not now exist. We choose to estimate the sum rule by using the impulse approximation in place of the scattering data.

We also find a new sum rule, involving only the nuclear disintegration pieces evaluated in a basis of plane-wave three-particle states (rather than the exact three-particle scattering states of the WA sum-rule disintegration contribution).

An alternative approach to WA sum rules for nuclei has been presented by Kim and Primakoff,[5] who use closure (via the infinite-momentum frame) to eliminate the explicit disintegration contributions. The following presentation is more closely allied to particle physics and, while in an incomplete state (the various admixtures in He^3 are ignored), indicates a direction that could lead to further applications of nuclear dispersion relations.

The structure of the paper is as follows: We derive the sum rule for He^3 and the extrapolation prescription which accompanies it. We then calculate the nuclear disintegration contribution and present the new sum rule which arises naturally at this point. Finally, we attempt to evaluate the sum rule in the absence of experimental π-He^3 scattering data, and discuss very briefly the question of forward π-He^3 dispersion relations.

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* Present address: Deutsches Elektronen-Sychrotron DESY, Hamburg, Germany.
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[2] W. Drechsler and B. Stech, Z. Physik **178**, 1 (1964).
[3] A. Fujii and Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto) **31**, 107 (1964).
[4] C. S. Wu and S. A. Moszkowski, *Beta Decay* (Wiley-Interscience, Inc., New York, 1966), p. 66.
[5] C. W. Kim and H. Primakoff, Phys. Rev. **147**, 1034 (1966).

II. SUM RULE

We consider the identity of Weisberger[6]

$$q_\mu q_\nu T_{\mu\nu}=iT$$

$$-\int d^4x\, e^{iqx}\tfrac{1}{2}\sum_{s=\pm}\langle p,s|\delta(x_0)[A_0^+(x),A_\nu^-(0)]|p,s\rangle$$

$$+i\int d^4x\, e^{iqx}\tfrac{1}{2}\sum_{s=\pm}\langle p,s|\delta(x_0)[A_0^-(x),D^+(0)]|p,s\rangle \quad (1)$$

for states $|p,s\rangle$ of He^3 with spin and isospin $\frac{1}{2}$. Further,

$$T_{\mu\nu}=\tfrac{1}{2}\sum_{s=\pm}\int d^4x\, e^{iqx}\langle p,s|(A_\mu^+(x)A_\nu^-(0))_+|p,s\rangle, \quad (2a)$$

$$T=\tfrac{1}{2}\sum_{s=\pm}\int d^4x\, e^{iqx}\langle p,s|(D^+(x)D^-(0))_+|p,s\rangle, \quad (2b)$$

utilizing the isotopic raising/lowering components of the axial-vector current $A_\mu^\pm(x)$, its divergence $D^\pm(x)$, and the time-ordering symbol $(\cdots)_+$. Neither He^3 nor the other member of the isodoublet H^3 has excited states, and He^3 satisfies all the field-theoretic conditions for (1) to hold.

The leading singularities of $T_{\mu\nu}$ (or T) in the variables we consider (q^2 and $\nu=p\cdot q/M_{He}$) are the Born term, arising from the H^3 intermediate state at $\nu_B=(M_H^2-M_{He}^2-q^2)/2M_{He}$ for the direct term and the double (and single) pion poles at $q^2=m_\pi^2$. At fixed q^2 there are also disintegration cuts beginning at $\nu_{nnp}=[(2m_n+m_p)^2-M^2-q^2]/2M$ and $\nu_{nd}=[(m_n+m_d)^2-M^2-q^2]/2M$ for the direct term ($M=M_{He}$ and m_d is the deuteron mass) and at $-\nu_{ppp}=[(3m_p)^2-M^2-q^2]/2M$ for the crossed term as well as the π-He^3 scattering cuts beginning at $\pm\nu\approx 140$ MeV for $q^2=0$. The disintegration cuts begin at $|\nu|\approx 8$ MeV.

For $q^2=0$ and $0<|\nu|\ll\nu_B$, the left-hand side of (1) is $O(\nu^2/\nu_B^2)$, and thus we may write the familiar sum rules of Weisberger and Adler,[7] and Adler[8] by equating to zero the terms of order ν, and of order unity, on the right-hand side of (1). We will be concerned only with the first (crossing-odd) sum rule.

Alternatively, we might choose to separate out the Born term on the two sides of (1) and show that the residues of the poles in ν match on both sides for $q^2=0$ as well as at $q^2=m_\pi^2$ (as they must). The nonsingular remainder is just F_A^2 and the left-hand side is then of order ν^2/ν_c^2 ($\nu_c\approx\nu_{nnp}$), where ν_c is the position of the new nearest singularity in $T_{\mu\nu}-T_{\mu\nu}^{\text{Born}}$. We choose the former prescription for the symmetry it provides between the Born term and the disintegration cuts.

Our procedure is first to extract the Born term in T at $q^2=0$ up to $O(\nu)$, use the algebra of currents for the equal-time commutators in (1), and then equate to zero the crossing-odd term of $O(\nu)$. Since the Born term is just $-F_A^2(q^2=0)$, the sum rule becomes

$$0=1-F_A^2-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\nu'(\nu')^{-2}\,\text{Im}T_D(\nu',q^2=0)$$

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\nu'(\nu')^{-2}\,\text{Im}T_c(\nu',q^2=0), \quad (3)$$

where the vector-current matrix element resulting from the axial-charge-axial-vector current commutator provides the factor of unity, and

$$\text{Im}T_D=\tfrac{1}{2}(2\pi)^4\sum_{l'}|\langle l'|D^-|He^3\rangle|^2\delta(p+q-p_{l'}) +\text{crossed term}, \quad (4)$$

$$\text{Im}T_c=\tfrac{1}{2}(2\pi)^4\sum_{n}|\langle n|D^-(0)|He^3\rangle|^2\delta(p+q-p_n) +\text{crossed term}. \quad (5)$$

The states $|l'\rangle$ include $|nd\rangle$ and $|nnp\rangle$ for the direct term ($|H^3\rangle$ is excluded) and $|ppp\rangle$ for the crossed term.

The states $|n\rangle$ for the direct and crossed terms include all states *except* the three-nucleon states; $|n\rangle$ begins with $|He^3\pi\rangle$, $|nnp\pi\rangle$, etc.

We rewrite the sum rule as

$$F_A^2=1-\Delta-\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu\,\nu^{-2}\,\text{Im}T_c(\nu,q^2=0), \quad (6)$$

where it is understood that Δ (the disintegration contribution) is to be evaluated from theory at $q^2=0$. We might have chosen to estimate it from experimental data at $q^2=m_\pi^2$ if such data existed. Our problem now is to relate $T_c(\nu,q^2=0)$, which we require, to $T_\pi(\nu,q^2=m_\pi^2)$, the physical pion-He^3 scattering amplitude. To this end, consider the related matrix element $\langle n\bar{p}|D|0\rangle$, where the state $|n\rangle$ has invariant mass $p_n^2=(M+m_\pi)^2$ and $\bar{p}=-p$ is the momentum of the anti-He^3 in the final state. Disregarding the complications of spin and isospin and suppressing the dependence on the internal variables of the state $|n\rangle$, we write

$$\langle n\bar{p}|D^-|0\rangle=(m_\pi^2-q^2)^{-1}[m_\pi^2F_\pi\langle n\bar{p}|j^-(0)|0\rangle]_{q^2=m_\pi^2}$$

$$+\int_{m_0^2}^{\infty}(m^2-q^2)^{-1}d(m^2)dm^2, \quad (7)$$

where the residue d of the one-pion pole has been explicitly displayed. The continuum begins at m_0^2, which equals $9m_\pi^2$ in the absence of anomalous thresholds. We thus expect the leading singularities of $\text{Im}T_c(\nu,q^2)$ in q^2 at fixed ν to be a pion pole of order 2 at $q^2=m_\pi^2$. Picking out the most singular term in the putative

[6] W. I. Weisberger, Phys. Rev. **143**, 1302 (1966).
[7] Reference 6 and S. L. Adler, Phys. Rev. **140**, B736 (1965).
[8] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

Laurent expansion at $m_\pi^2=q^2$, we find

$$\mathrm{Im}T_c(\nu,\,q^2=0)\approx F_\pi^2\,\mathrm{Im}T_\pi(\nu,\,q^2=m_\pi^2)\,.$$

Loosely, we say that the double pion pole at $q^2=m_\pi^2$ dominates $\mathrm{Im}T_c$ at $q^2=0$. This "extrapolation to $q^2=0$" can be carried out along any line in the (ν,q^2) plane and we choose, with Weisberger,[6] to pick $\nu=$ constant. The location of some singularities in this (ν,q^2) plane are shown in Fig. 1 and the various one- and two-dimensional reduced graphs giving the exhibited thresholds are shown in Fig. 2.

The two-dimensional reduced graphs give anomalous thresholds which disappear into the thresholds of the one-dimensional graphs at the points of tangency. We show only the right-hand cut in $T_c(\nu,q^2)$ for clarity, and we note that the dangerous anomalous thresholds are, for $\nu\geq m_\pi$, sufficiently distant so as to justify taking the first term in (7). Note that the anomalous threshold for the three-pion state is also present in the nucleon WA sum rule (and, in fact, is nearly the same) and contributes only for ν well below the $\Delta(1236)$. The nearby anomalous nuclear structure threshold should have a minimal effect, since they approach $q^2\approx 5m_\pi^2$ only in the vicinity of one value of ν rather than over a range. Note that for $8\leq\nu\leq 140$ MeV the effect of the anomalous thresholds is expected to be more critical.

We find that the compositeness of He^3 has, apart from the appearance of Δ, a small effect on the WA sum rule (i.e., there is little likelihood that the axial-vector current will be absorbed before turning into a pion).

This is perhaps a good point at which to contrast the WA sum rule for F_A^2 to the GT relation. In the latter, we would extrapolate the axial-vector current matrix element $\langle H^3|A_\mu^-|He^3\rangle$ to $q^2=0$ assuming dominance of the one-pion pole. This extrapolation is valid if the nearest continuum threshold is at $m_0^2\gg m_\pi^2$ or if the discontinuities across the corresponding cuts are small. For He^3 we would extrapolate along the line $s=M^2$ in the (ν,q^2) plane and this intercepts the anomalous cut (see Fig. 1) at $q^2=4.6m_\pi^2$. For heavier nuclei this anomalous cut is closer to $q^2=0$ and therefore is even more influential. The somewhat surprising conclusion is that we can expect nuclei to be more "elementary" in the WA sum rule than they appear in the GT relation (or are in fact). This may be of some importance in heavier nuclei.

We might also note that in the field-theory version of PCAC (partially conserved axial-vector current) we define the extrapolation of the pion field by $\Phi^-(x)=\partial\mu A_\mu^-(x)/m_\pi^2F_\pi$ and thus the anomalous thresholds appear directly in the extrapolation of $\langle n\bar{p}|D|0\rangle=m_\pi^2F_\pi(m_\pi^2-q^2)^{-1}\langle n\bar{p}|j|0\rangle$, where the matrix element of the pion source is taken at q^2 (rather than m_π^2).

Using the crossing relation and unitarity ($\mathrm{Im}T_\pi$

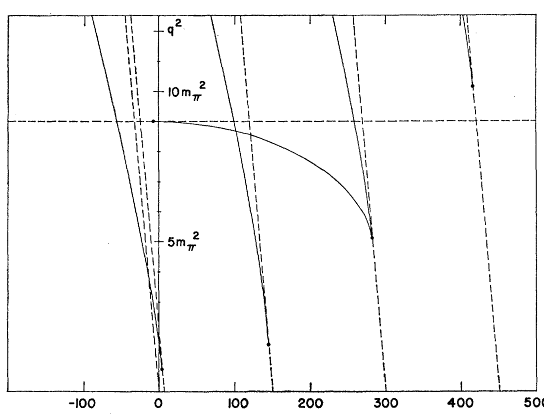

FIG. 1. Some of the normal and anomalous thresholds in the ν-q^2 plane. Normal thresholds are shown as dashed lines, anomalous thresholds as solid curves. The normal threshold at $q^2=m_\pi^2$ corresponds to the second one-dimensional reduced graph in Fig. 2(a). The anomalous threshold [from the two-dimensional reduced graph in Fig. 2(a)] goes from this first normal threshold into the normal threshold given by the last one-dimensional reduced graph of Fig. 2(a) (the slanted lines intersecting $q^2=0$ at $\nu\approx 300$ MeV). The other normal and anomalous thresholds correspond to, from left to right, Figs. 2(b)–2(d), the normal thresholds shown intersecting $q^2=0$ at $\nu\approx 8$ MeV, $\nu\approx 140$ MeV, and $\nu\approx 300$ MeV, respectively. The normal thresholds in q^2 for these reduced graphs are not shown, being off the graph. The dashed line intersecting $q^2=\nu=0$ is $s=M^2$ (the pole).

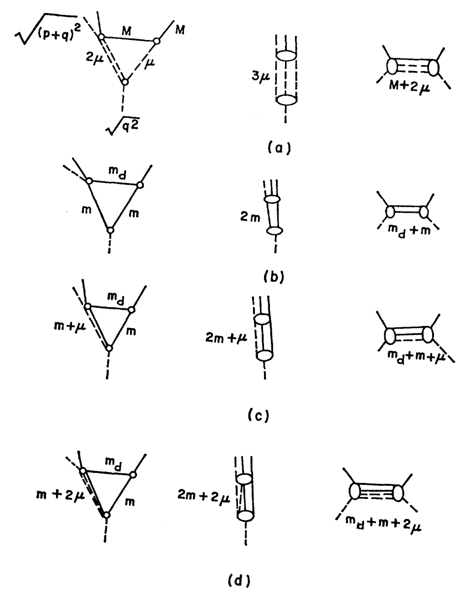

FIG. 2. Two- and one-dimensional reduced graphs corresponding to the normal and anomalous thresholds in Fig. 1. From left to right in each line are shown the two-dimensional reduced graph, the one-dimensional reduced graph giving the normal q^2 threshold, and the graph giving the ν threshold (whose location is q^2-dependent). μ is the pion mass, M is the He^3 mass, and m_d and m are the deuteron and nucleon masses. The masses of the external legs are shown only once.

$=k_\pi\sigma$), we find

$$F_A{}^2=1-\Delta-\frac{F_\pi{}^2}{\pi}\int d\nu\, \nu^{-2}k_\pi \times[\sigma'(\pi^-\,\mathrm{He}^3)-\sigma'(\pi^+\,\mathrm{He}^3)], \quad (8)$$

where the prime is a reminder that $\mathrm{Im}T_\pi$ excludes the processes $\pi^-\mathrm{He}\to nnp$, nd; $\pi^+\mathrm{He}^3\to ppp$ which are included explicitly at $q^2=0$ in Δ. Except for the omission of these processes, the σ' are total cross sections.

The set of states $|l'\rangle$ in Eq. (4), plus the state $|\mathrm{H}^3\rangle$, constitutes a complete set in the subspace of three-nucleon states where the states $|nnp\rangle$, $|nd\rangle$, and $|ppp\rangle$ are exact scattering states orthogonal to the H^3 state. We may, however, choose to span this space with a complete set of noninteracting (plane wave) three-nucleon states which we denote $|\mathrm{pw}\rangle$. We may then recast (6) as follows: The definition of $\mathrm{Im}T_D$ is extended so as to contain the state $|\mathrm{H}^3\rangle$; then we simply substitute for the complete subset $|l\rangle=|l'\rangle$, $|\mathrm{H}^3\rangle$ the alternative subset $|l;\mathrm{pw}\rangle$. The sum rule then becomes

$$0=1-\Delta_{\mathrm{pw}}-\frac{1}{\pi}\int_{-\infty}^{+\infty} d\nu\, \nu^{-2}\,\mathrm{Im}T_c \quad (9)$$

with $\mathrm{Im}T_c$ unchanged and

$$\Delta_{\mathrm{pw}}=\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu\, \nu^{-2}\,\mathrm{Im}T_D{}^{\mathrm{pw}}, \quad (10)$$

$$\mathrm{Im}T_D{}^{\mathrm{pw}}=\tfrac{1}{2}(2\pi)^4\sum_{l;\,\mathrm{pw}}|\langle l;\mathrm{pw}|D^-|\mathrm{He}^3\rangle|^2 \times\delta(p+q-p_l)+\text{crossed term}, \quad (10')$$

where $|l;\mathrm{pw}\rangle=|nnp;\mathrm{pw}\rangle$ for the direct term and $|ppp;\mathrm{pw}\rangle$ for the crossed term.

Subtracting (9) from (6), we have a new "plane-wave" sum rule

$$F_A{}^2=\Delta_{\mathrm{pw}}-\Delta. \quad (11)$$

III. CALCULATION OF Δ_{pw} AND Δ

We turn now to the calculation of Δ_{pw} and Δ at $q^2=0$. For both we use the impulse approximation and a simple symmetric He^3 wave function. The calculation of Δ_{pw} is a prelude to the calculation of Δ.

Since the outgoing nucleons in Δ_{pw} are in plane-wave states, it is natural to use the impulse approximation represented by

$$D^\pm(0)=\sum_{\alpha,\beta} a_\alpha{}^\dagger\langle\alpha|D^\pm(0)|\beta\rangle a_\beta \quad (12)$$

with α and β free one-nucleon plane-wave states and

$$\langle\alpha|D^\pm(0)|\beta\rangle =if_A(2\pi)^{-3}(m_\alpha m_\beta/E_\alpha E_\beta)^{1/2}(m_\alpha+m_\beta)\bar{u}_\alpha\gamma_5\tau^\pm u_\beta \approx if_A(2\pi)^3\chi_\alpha{}^\dagger\boldsymbol{\sigma}\cdot(\mathbf{p}_\alpha-\mathbf{p}_\beta)\chi_\beta. \quad (13)$$

$f_A=$ nucleon axial-vector decay constant; $|f_A|=1.18$. We also use the symmetric He^3 wave function[9]

$$|\mathrm{He}^3;\mathbf{p},s_z=\tfrac{1}{2}\rangle=(2\pi)^{-6}\left(\frac{3\sqrt{3}\alpha^6}{\pi^3}\right)^{1/2}\int\prod_{i=1}^3 d^3\mathbf{r}_i\prod_{j=1}^3 d^3\mathbf{p}_j\exp[-i\sum_j^3\mathbf{p}_j\cdot\mathbf{r}_j-\tfrac{1}{2}\alpha^2(\mathbf{r}_{12}{}^2+\mathbf{r}_{13}{}^2+\mathbf{r}_{23}{}^2)] \times a_p{}^\dagger(\mathbf{p}_1;s_z=\tfrac{1}{2})a_p{}^\dagger(\mathbf{p}_2;s_z=-\tfrac{1}{2})a_n{}^\dagger(\mathbf{p}_3;s_z=\tfrac{1}{2})|0\rangle, \quad (14)$$

$$i,j=1,2,3;\quad \mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j;\quad \alpha=0.384\ \mathrm{F}^{-1}\approx 75\ \mathrm{MeV}.$$

The final plane-wave states are

$$|nnp;\mathrm{pw}\rangle=(2)^{-1/2}a_p{}^\dagger(\mathbf{p}_1',s_1')a_n{}^\dagger(\mathbf{p}_2',s_2')a_n{}^\dagger(\mathbf{p}_3',s_3')|0\rangle$$

and

$$|ppp;\mathrm{pw}\rangle=(6)^{-1/2}a_p{}^\dagger(\mathbf{p}_1',s_1')a_p{}^\dagger(\mathbf{p}_2',s_2')a_p{}^\dagger(\mathbf{p}_3';s_3')|0\rangle.$$

We may now carry out the spin sums in (10) and (10′), as well as perform the spatial integrals from the He wave function.

The result for $\Delta_{\mathrm{pw}}{}^{nnp}$ is

$$\Delta_{\mathrm{pw}}{}^{nnp}=f_A{}^2(3\sqrt{3}\alpha^6\pi^3)^{-1}\int d\nu\int\prod_{i=1}^3 d^3\mathbf{p}_i \times\left\{2\exp\left[-\frac{\mathbf{p}_2{}^2}{2\alpha^2}-\frac{2(\mathbf{p}_1+\mathbf{p}_2/2)^2}{3\alpha^2}\right] -\exp\left[-\frac{(\mathbf{p}_2{}^2+\mathbf{p}_3{}^2)}{4\alpha^2}-\frac{(\mathbf{p}_1+\mathbf{p}_2/2)^2}{3\alpha^2}-\frac{(\mathbf{p}_1+\mathbf{p}_3/2)^2}{3\alpha^2}\right]\right\}\delta^4(p+q-\sum_{i=1}^3 p_i), \quad (15)$$

where we choose $q=(\nu\hat{n},\nu)$ and the frame $p=(\mathbf{0},m)$. There are three convenient levels of approximation to (15). First, we may use a pole approximation by writing for the argument of the energy δ function

$$(M-\sum_{i=1}^3 E_i+\nu)\approx(M-3m+\nu).$$

Next, we can in addition keep the first-order corrections to the pole approximation coming solely from the inverse of the derivative of the argument of the delta function which is implicitly set to unity in the zeroth approximation, but not corrections from the momentum dependence of ν. Third, we can include these latter corrections as well. There are no corrections of this third type to the first term in (15), and the corrections to the second term of this type are of order $(\nu_{nnp}{}^2/\alpha^2)(\alpha^2/m^2)=(\nu_{nnp}/m)^2$, where we denote the threshold value of ν by $\nu_{nnp}=9.0$ MeV. Thus the only important corrections to the pole approximation are of the second type.

[9] L. I. Schiff, Phys. Rev. 133, B802 (1964).

Including these corrections of the second type,

$$\Delta_{\mathrm{pw}}{}^{nnp}=f_A{}^2[1+11\alpha^2/m^2+\nu_{nnp}{}^2/6\alpha^2+\nu_{nnp}/m], \quad (16)$$

and, similarly,

$$\Delta_{\mathrm{pw}}{}^{ppp}=f_A{}^2[\nu_{ppp}{}^2/6\alpha^2+\nu_{ppp}/m]. \quad (17)$$

Note that the pole-approximation contribution to $\Delta_{\mathrm{pw}}{}^{ppp}$ vanishes.

The calculation of Δ itself is difficult because of the appearance of exact three-particle scattering wave functions for the state $|n\rangle$. A typical term leading to Eq. (15) contains the square of the matrix element

$$T=\int \prod_{i=1}^{3} d^3\mathbf{r}_i \Phi_n{}^* e^{i\mathbf{q}\cdot\mathbf{r}_i}\Phi_{\mathrm{He}}\chi_n{}^\dagger \boldsymbol{\sigma}\cdot\mathbf{q}\chi_{\mathrm{He}} \quad (18)$$

[the q^2 factor in the squared matrix element will be cancelled by the factor ν^{-2} in (10)]. As $|q|\to 0$ the overlap integral $\int\Phi_n{}^*\Phi_{\mathrm{He}}\to 0$ if Φ_n and Φ_{He} are eigenstates of an exactly charge-independent Hamiltonian (which we assume). The matrix element is then of order $|\mathbf{q}|\cdot|(\mathbf{r})_{n,\mathrm{He}}|$, and thus Δ^{nnp} is of order $(\nu_{nnp}/\alpha)^2\approx 1\%$. Another such situation arose in the calculation of $\Delta_{\mathrm{pw}}{}^{ppp}$ where the radial three-proton wave function was constrained to be asymmetric by the Pauli principle, and thus orthogonal to the (symmetric) He^3 radial wave function. Thus the leading term in $\Delta_{\mathrm{pw}}{}^{ppp}$ is $O((\nu_{ppp}/\alpha)^2)$.

Rather than attempt a calculation with the exact wave function Φ_n, we expand the exponentials $e^{i\mathbf{q}\cdot\mathbf{r}_i}$ and use the orthogonality assumption to eliminate the first term. The next term can then be estimated by replacing Φ_n *in this term* by a plane-wave three-particle wave function, i.e.,

$$\int \Phi_n{}^*(\mathrm{exact})(i\mathbf{q}\cdot\mathbf{r}_i)\Phi_{\mathrm{He}}\approx \int \Phi_n{}^*(\mathrm{pw})(i\mathbf{q}\cdot\mathbf{r}_i)\Phi_{\mathrm{He}}. \quad (19)$$

We have compared this approximation with the exact result in a simple two-particle model and find that the approximation overestimates Δ by 30%. Since Δ is small, this level of accuracy is acceptable. The two-particle model calculation is outlined in the Appendix.

It is now a straightforward matter to adapt the calculation of Δ_{pw} to this approximation. If we write (15) in terms of the T's, eliminate the contribution at $q=0$ and keep only the remainder, square, and perform the integrals as before, we find

$$\Delta^{nnp}=f_A{}^2(\nu_{nnp}{}^2/3\alpha^2). \quad (20)$$

Since for Δ^{ppp} we are already assured that there is no $q=0$ component in the matrix element, our approximation is just

$$\Delta^{ppp}=\Delta_{\mathrm{pw}}{}^{ppp}=f_A{}^2[\nu_{ppp}{}^2/6\alpha^2+|\nu_{ppp}|/m].$$

The additional contribution from the $|nd\rangle$ final state in Δ can be calculated by using a two-term Gaussian wave function,[10] similar to that used for He^3, and applying the same procedure. Then

$$\Delta^{nd}=f_A{}^2(\nu_{nd}{}^2/6\alpha^2), \quad (21)$$

where the thresholds are as follows: $\nu_{nnp}=9.0$ MeV, $-\nu_{ppp}=6.4$ MeV, and $\nu_{nd}=6.8$ MeV. Numerically, we find

$$\Delta=\Delta^{nd}+\Delta^{nnp}-\Delta^{ppp}=-0.0014f_A{}^2, \quad (22a)$$

$$\Delta_{\mathrm{pw}}=\Delta_{\mathrm{pw}}{}^{nnp}-\Delta_{\mathrm{pw}}{}^{ppp}=1.035f_A{}^2. \quad (22b)$$

Note that the magnitude of Δ does not depend strongly on cancellations.

Equations (22) lead directly to the plane-wave sum rule (10), $F_A{}^2=1.036f_A{}^2$, $|F_A|=1.20$. This is in striking agreement with the experimental value, and particularly encouraging in that the ratio $|F_A/f_A|$ is larger than unity. This latter point is suggested by the experimental information, and is almost impossible to predict on the basis of an ordinary (as opposed to "elementary particle") theory.

IV. EVALUATION

In the absence of π-He^3 scattering data, we wish to attempt to *estimate* the integral in the WA sum rule (8). To this end we use the impulse approximation in the form

$$\int d\nu\ \nu^{-2}k_\pi[\sigma'(\pi^-\ \mathrm{He}^3)-\sigma'(\pi^+\ \mathrm{He}^2)] \approx \int d\nu\ \nu^{-2}k_\pi\langle\sigma(\pi^-p)-\sigma(\pi^+p)\rangle_F, \quad (23)$$

where the bracket $\langle\ \rangle_F$ means that the cross sections are to be averaged over the Fermi motion of the nucleons in He^3. We used the Gaussian wave function (14) to give a Gaussian "smearing" function. The results are shown in Fig. 3. With this approximation for the π-He^3 cross sections,

$$\int d\nu\ \nu^{-2}k_\pi\langle\sigma(\pi^-p)-\sigma(\pi^+p)\rangle_F\approx 33.1\ \mathrm{mb} \quad (24)$$

compared with

$$\int d\nu\ \nu^{-2}k_\pi[\sigma(\pi^-p)-\sigma(\pi^+p)]\approx 36.7\ \mathrm{mb},$$

without the Gaussian smearing function. The chief source of the 10% decrease in the integral is the smearing of the $\Delta(1236)$ into the region where $k_\pi(k_\pi{}^2+m_\pi{}^2)^{-1/2}$ is small.

[10] C. Ernst and S. Flügge, Z. Physik **162**, 448 (1961).

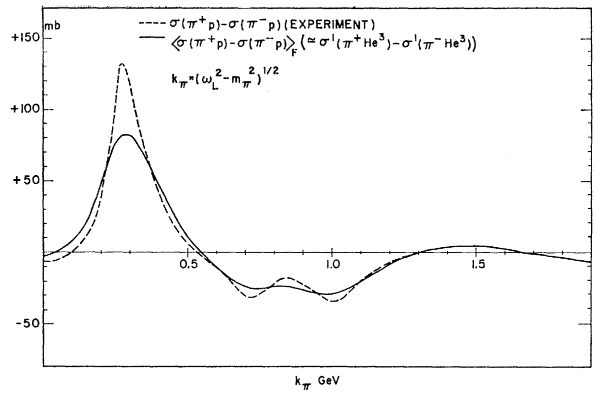

FIG. 3. Comparison of the "Fermi averaged" and impulse-approximation values for the difference of cross sections appearing in Eq. (23). k_π is the laboratory momentum of the incident pion.

If we use the GT relation (for the nucleon), we find

$$|F_A| = 1.16$$

for He^3 [from (8) and (24)] compared with $|f_A| = 1.17$ for the nucleon. Using the physical pion decay constant, which we regard as more reasonable, we find

$$|F_A| = 1.19,$$

compared with $|f_A| = 1.21$ ($|f_A|$ from the nucleon sum rule in both cases). The agreement with the experimental value of $|F_A|$ is well within the error limits stated in the Introduction, but not as striking as the result from the plane-wave sum rule. The reason for this is apparent: One further series of approximations has been introduced [in (23)]. It is clear, however, that the WA sum rule is valid for He^3,[11] and we turn to a discussion of the approximation (23).

We expect that the impulse approximation (23) is reasonable on several counts. First, the most obvious processes which violate the impulse approximation are $\pi^\pm He^3 \to 3N$ near threshold, corresponding to a two-nucleon capture process. These processes are explicitly excluded from σ', however. We can attempt to estimate what they would contribute if included in the integral (24), by using the S-orbital capture rates calculated by Cheon[12] to estimate the matrix elements. The result would be to increase (24) by $\approx 2\%$ if, rather than σ', we (wrongly) used σ, the *total* cross section. Possibly, then, other violations of the impulse approximation have a similarly small effect.

Secondly, we note that the impulse approximation (understood to include the smearing due to the nucleon's Fermi motion) is a fair approximation for pion-deuteron scattering, though resonances do seem somewhat more "smeared out" than our impulse approximation indicates. Also, Ericson, Formanek, and Locher[13] attempted to find $\sigma(\pi^- Be^9) - \sigma(\pi^+ Be^9)$ by an entirely different approximation technique, and the general decrease in the height of the $\Delta(1236)$ resonance is about the same in their work as in ours. Note that the Fermi motion effect is not too much different in Be^9 and He^3.

The presence of the average over the Fermi motion of the nucleons is important for reasons evident from a naive argument. A pion incident on a single nucleon at rest at an impact parameter b can excite a resonance of orbital angular momentum $l \approx k_\pi b$. If we suppose that another nucleon (also at rest) lies also within an impact parameter b, we might expect that both nucleons would be excited coherently into, for example, the $\Delta(1236)$ resonance. This would then lead to a violation of the unitarity relation for this particular configuration and partial wave. Of course this violation is spurious since the effects of rescattering are precisely such as to enforce unitarity. The ordinary impulse approximation (ignoring the relative motion and rescattering) then leads to violations of unitarity. In the presence of relatively narrow resonances which saturate the unitarity relation and the Fermi motion, the relative momenta of the nucleons will usually be such as to prevent such coherent excitation of the resonant state (as well as reduce the rescattering corrections). One might naively expect that the Fermi motion and the rescattering corrections might together lead to somewhat greater "smearing" than the Fermi motion alone, but we prefer to avoid further excursions into the thicket of pion-nucleus scattering. Note that the rms nucleon momentum in He^3 (75 MeV) is not very different from the half-width of the $\Delta(1236)$.

It is worth noting that the usual condition for the validity of the impulse approximation is not really valid here, since the $\Delta(1236)$ moves very little before it decays (≈ 0.1 F) and the average over the Fermi motion has been taken.

V. DISCUSSION

We now summarize our results and point out some significant features of the WA sum rule for He^3. The sum rule is useful chiefly because of the possibility of carrying out in a simple way the continuation in the variable q^2 from 0 to $m_\pi{}^2$ for at least a part of the absorptive amplitude (that part excluding the three-nucleon disintegration states with threshold at $\nu = 8$ MeV, i.e., beginning with the pion threshold near 140 MeV) while estimating the remaining disintegration contribution at $q^2 = 0$. This prescription depends critically on the location of the anomalous threshold singularities in the (ν, q^2) plane—particularly those anomalous thresholds due to the nuclear structure. We note that for the direct term these singularities are

[11] This conclusion is the opposite of that reported previously by one of us (EAP). We gratefully acknowledge comments on this earlier paper by C. W. Kim, H. Primakoff, G. Barton, and J. E. Paton.

[12] Il-Tong Cheon, Phys. Rev. **145**, 794 (1966).

[13] T. E. O. Ericson, J. Formanek, and M. P. Locher, Phys. Letters **26B**, 91 (1967).

generally to the left of the region of interest in the continuation ($0 \leq q^2 \leq m_\pi^2$, $\nu \geq 140$ MeV). In fact, these singularities seem to lie above and to the left of the parabola passing through $\nu = q^2 = 0$ and $\nu = m_\pi$, $q^2 = m_\pi^2$ which Fubini has conjectured does not cross any anomalous thresholds.[14] These singularities do, however, prevent a reliable extrapolation of the disintegration contribution from $q^2 = 0$ (where it is required for the sum rule) to $q^2 = m_\pi^2$, where it is either the absorptive part of the amplitude in the unphysical region ($8 \leq \nu \leq 140$ MeV), or the pion absorption cross section for ν above m_π. This can be most simply noted by remarking that energy-momentum conservation excludes the impulse approximation (one-nucleon capture) for $q^2 = m_\pi^2$, $\nu = m_\pi$, but not for $q^2 = 0$. Corresponding to this, one might expect that higher-order anomalous thresholds than the ones of Fig. 1 cover the region $q^2 = m_\pi^2$, $\nu = m_\pi$ and prevent one from extrapolating the impulse-approximation result at $q^2 = 0$ to $q^2 = m_\pi^2$ near $\nu = m_\pi$. We have found, in fact, that some three-dimensional reduced graphs contribute anomalous thresholds just in this region. This makes any extrapolation of Δ even more suspect than one might judge from Fig. 1. In this respect, the WA sum rule is even better than one might think from phenomenological estimates of $\Delta(q^2 = m_\pi^2)$.[15]

In connection with our use of the impulse approximation we may remark that, taking the difference of the cross sections $\sigma(\pi^+ \mathrm{He}^3) - \sigma(\pi^- \mathrm{He}^3)$ seems, from the work of Ericson *et al.*,[13] to lead to an absorptive amplitude for which the impulse approximation is reasonably good. In contrast, the impulse approximation for the sum $\sigma(\pi^+ \mathrm{He}^3) + \sigma(\pi^- \mathrm{He}^3)$ is subject to uncertainties coming, for example, from the scattering of pions by the potential which tends to cancel in the difference of cross sections. Likewise, Glauber shadowing corrections[16] ultimately affect only the already small asymptotic-cross-section difference. In the sum of the cross sections, shadowing effects lead to a violation of the linear A dependence following from the impulse approximation.

The WA sum rule is related to the problem of forward pion-nucleus scattering dispersion relations. The connection involves extrapolation of the pole term in the sum rule to $q^2 = m_\pi^2$ and a similar extrapolation of Δ, together with the extrapolation of $\mathrm{Im} T_c$ which we have discussed. The disintegration contribution below the physical threshold becomes just the absorptive contribution in the unphysical region to the dispersion relation, and above the physical threshold becomes the difference of the absorption cross sections for pions into three-nucleon states. The pole term becomes just the pole term in the dispersion relation. If the unphysical region in the dispersion relation contributes little, then the sum rule, the Goldberger-Treiman relation, and the forward dispersion relation give essentially the same result (in terms, for example, of a prediction of the π-He^3-H^3 coupling constant). We have seen, however, that the Goldberger-Treiman relation should be rather poor for nuclei, and that the extrapolation of Δ is nontrivial (though it might remain fairly small at $q^2 = m_\pi^2$). It is not clear to us whether or not one can use the WA sum rule to infer anything about the strength of the pole term (or terms) in the forward dispersion relation. This interesting question is outside the boundaries of this paper, however.

APPENDIX

In this appendix we consider the approximation in Eq. (19) for T. It is necessary for simplicity to study a two-body problem in the absence of tractable three-body models in our problem. We replace T by

$$4\pi k I(\mathbf{k},\mathbf{q}) = \int d^3r\, \Phi_B(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \Phi_k(\mathbf{r}). \tag{A1}$$

Then the fictitious Δ in this model is proportional to

$$J(\mathbf{q}) = \int d^3k\, |I(\mathbf{k},\mathbf{q})|^2, \tag{A2}$$

where we might choose to think of this as the disintegration term for a particle with $I = J = \frac{1}{2}$ bound to another particle with $I = J = 0$ via a simple (soluble) potential. Since the bound-state wave function Φ_B is orthogonal to $\Phi_k(r)$, and since we are interested in small $|q| \approx \epsilon =$ binding energy, we may make an expansion in I of $\exp(i\mathbf{q}\cdot\mathbf{r})$, the lowest term of which is $O(\epsilon/\alpha)$ and the next $O(\epsilon^2/\alpha^2)$. Consider Eq. (A1) for a zero-range potential with pure S-wave scattering and a bound state, where

$$\Phi_B = r^{-1} u_B(\mathbf{r}) = r^{-1} (\alpha/2\pi)^{1/2} e^{-\alpha r},$$

$$\Phi_k = r^{-1} u_0(\mathbf{r}) = r^{-1} (2/\pi)^{1/2} \sin(kr + \delta_0),$$

$$\tan(\delta_0) = -k/\alpha.$$

The exact results are

$$I_{\mathrm{ex}} = \frac{\alpha^{3/2}}{\pi} \left[-\frac{2}{3} \frac{\mathbf{q}^2}{(\mathbf{k}^2 + \alpha^2)^{5/2}} + O(\mathbf{q}^4) \right],$$

$$J_{\mathrm{ex}} = \frac{5\mathbf{q}^4}{144\alpha^4} + O(\mathbf{q}^6).$$

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[14] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 322 (1968).

[15] Such phenomenological estimates can be abstracted from R. Seki, Phys. Rev. Letters **21**, 1494 (1968); **21**, 1786(E) (1968). In this paper it is conjectured that the earlier calculation [Earl A. Peterson, Phys. Rev. Letters **20**, 776 (1968); **20**, 1134(E) (1968)] failed because of an inconsistent use of the impulse approximation. This proved not to be the case—the culprit was an incorrect use of the plane-wave approximation.

[16] R. J. Glauber, Phys. Rev. **100**, 242 (1955).

The approximation (19) corresponds here to $\delta_0=0$ and

$$I_{\rm apx}=I_{\rm apx}|_{\mathbf{q}=0}-\frac{\alpha^{1/2}}{\pi}\left[\frac{\mathbf{q}^2(3\alpha^2-\mathbf{k}^2)}{3(\alpha^2+\mathbf{k}^2)^3}+O(\mathbf{q}^4)\right],$$

$$J_{\rm apx}=\int d^3k\,|I_{\rm apx}-(I_{\rm apx}|_{\mathbf{q}=0})|^2$$

$$=\frac{6\mathbf{q}^4}{144\alpha^4}\approx 1.2J_{\rm ex}.$$

A more realistic example is a finite-range square well, where the leading term in I can be $O(\epsilon/\alpha)$. Here

$$I_{\rm ex}(\mathbf{k},\mathbf{q})=k^{-1}3\cos(\theta_{\mathbf{k},\mathbf{q}})e^{i\delta_1}\times\int d^3r\,\Psi_B(r)r^{-1}u_1(kr)j_1(qr)\,,\quad \text{(A3)}$$

since the leading term of the expansion of $e^{i\mathbf{q}\cdot\mathbf{r}}$ can combine with the p-wave scattering term. Note that the approximation is reasonable, since the linear dependence on $\mathbf{r}$ of the first moment of $e^{i\mathbf{q}\cdot\mathbf{r}}$ minimizes the contribution from small $\mathbf{r}$ where the potential strongly modifies the scattered wave. We choose for the numerical example a (deuteron) square well (a single bound state at -2.225 MeV) with a well depth of 38.5 MeV and a radius of 1.93 F. The approximate integral $I_{\rm apx}$ is simply obtained by replacing u_1 by the unscattered-wave Bessel function j_1. The resulting expressions are

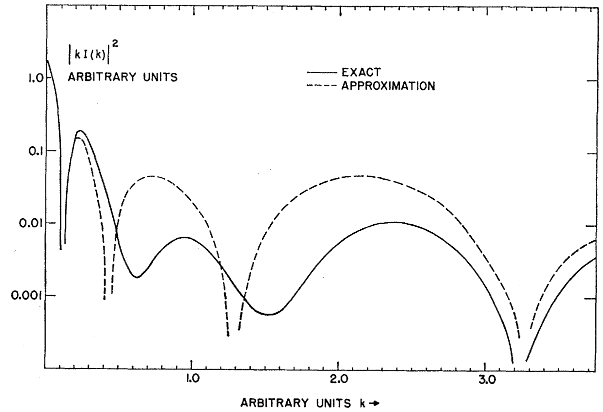

FIG. 4. Exact and approximate integrands leading to Δ in the two-particle square-well model discussed in the Appendix.

too involved to quote (see Fig. 4) but the numerical results for J are

$$J_{\rm ex}(\mathbf{q})=(93\mathrm{F}^2)\mathbf{q}^2+O(\mathbf{q}^4)\,,$$

$$J_{\rm apx}(\mathbf{q})=(120\mathrm{F}^2)\mathbf{q}^2\approx 1.29J_{\rm ex}.$$

We expect the approximation to be better for the higher moments but a comparison of the $\mathbf{q}^4$ term in the zero-range approximation with the corresponding $\mathbf{q}^2$ term indicates that it probably does not increase in accuracy very rapidly. All we require is the 30% accuracy of the model, since the Δ calculated is very small indeed.