Compton Interaction of a Photon Gas with a Plasma*

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The rate of energy transfer due to Compton collisions from a photon gas to a Maxwell electron gas is calculated to fourth order in the expansion parameters kT_e/mc^2 and kT_R/mc^2. The photon distribution is taken to be Planckian at a temperature T_R not equal to the electron temperature T_e. It is found that the nonrelativistic expression is accurate only for $kT_R \leq 2.5$ keV and $kT_e \leq 20$ keV, while the fourth-order expression is accurate for $kT_R \leq 10$ keV and $kT_e \leq 80$ keV.

INTRODUCTION

THE inverse Compton effect is a major energy-loss mechanism for energetic electrons interacting with lower-energy photons. This process has been recently discussed in the literature both in the context of a plasma interacting with Planck radiation[1] and in the astrophysical situation of a cosmic-ray electron interacting with an isotropic thermal radiation field.[2]

In this work, we include relativistic corrections to the Compton rate of energy exchange from a Maxwellian electron gas to a Planckian photon gas. In Sec. I, the correct relativistic formula for the energy exchange rate is written down and the integrand is expanded in powers of v/c and $\hbar\omega/mc^2$. After integration over the appropriate distributions, the rate is proportional to the energy density in the radiation field multiplied by a series expansion in the parameters kT_e/mc^2 and kT_R/mc^2.

In Sec. II, the results of the expansion are given and compared with "exact" calculations which have been completed recently. There is agreement to within 3% for $0 \leq kT_e \leq 92$ keV and $0 \leq kT_R \leq 7.2$ keV.

In Sec. III, the convergence of the series expansion is discussed, and the results of the calculation are summarized in the concluding section.

I. RATE OF ENERGY TRANSFER

We wish to calculate the rate of energy transfer due to Compton scattering from a gas of electrons at a temperature T_e to a gas of photons at a temperature T_R. We will assume that the electrons are in an isotropic Maxwell-Boltzmann distribution $n_e(p)$ and that the photons are in an isotropic Planck distribution $n_R(\omega)$. It is convenient to define

$$\theta_e \equiv kT_e, \qquad \theta_R \equiv kT_R, \qquad x \equiv \hbar\omega/mc^2, \qquad x' \equiv \hbar\omega'/mc^2,$$

where ω and ω' are the initial and final photon frequencies in the laboratory frame for a single scattering. Using the Klein-Nishina formula to obtain the energy transferred in a single collision, we get the following expression for the rate of energy transfer:

$$\begin{aligned} dE/dt = mc^2 \int d\Omega_0 d^3p d\Omega_k dx \{ (1-\boldsymbol{\beta}\cdot\hat{k}) n_R(x) c \\ \times (\tfrac{1}{2} r_0^2 [1+(1-\cos\theta_0) x_0]^{-2} \\ \times [(x_0'-x_0)^2/x_0'x_0 + 1 + \cos^2\theta_0]) \\ \times n_e(p) [1+\tfrac{1}{2}(2\pi\hbar/mc)^3 n_R(x')] \\ \times [1-\tfrac{1}{2}(2\pi\hbar)^3 n_e(p')] x^2 (x'-x) \}. \end{aligned} \quad (1)$$

In the above expression we have

$$n_R(x) = 2(mc/2\pi\hbar)^3 [\exp(xmc^2/\theta_R) - 1]^{-1} \quad (2)$$

and

$$n_e(p) = A\exp(-\epsilon/\theta_e), \qquad \epsilon = \text{electron energy} \quad (3)$$

with the normalization condition

$$\int d^3p\, n_e(p) = \rho_e, \qquad \text{the electron density.} \quad (4)$$

The subscript 0 refers to variables in the frame of reference in which the electron is initially at rest. In this frame the Klein-Nishina formula can be used to give for the collision cross section the term in the boldface parentheses. The factors $(1-\boldsymbol{\beta}\cdot\hat{k}) n_R(x) c$ give the incident photon flux in this frame, where a factor of $\gamma = (1-\boldsymbol{\beta}^2)^{-1/2}$ has been canceled by the time-dilatation factor γ^{-1} relating reaction rates in the two frames. $\boldsymbol{\beta}$ is the incident electron velocity in the lab divided by c, and $\hat{k}$ is a unit vector in the direction of the incident photon momentum in the lab. The density-of-final-states factors for the electron and photon, respectively, are

$$[1-\tfrac{1}{2}(2\pi\hbar)^3 n_e(p')] \quad \text{and} \quad [1+\tfrac{1}{2}(2\pi\hbar/mc)^3 n_R(x')].$$

In this paper we will ignore electron degeneracy effects and replace the first factor by unity.

We will concern ourselves only with temperatures which satisfy the conditions

$$\theta_e/mc^2 \ll 1 \quad \text{and} \quad \theta_R/mc^2 \ll 1. \quad (5)$$

By expanding the various factors in the integrand of the expression for dE/dt in powers of β and $\hbar\omega/mc^2 = x$ and integrating over the distributions (2) and (3), we obtain an expansion of dE/dt in powers of θ_e/mc^2 and θ_R/mc^2. We handle the $n_R(x')$ appearing in the photon final-state factor by making a Taylor expansion about the point x. We handle $n_e(p)$ as follows. First we write

$$n_e(p) p^2 dp = A\exp(-\gamma mc^2/\theta_e)[\gamma^2 m^2 c^2 \beta^2 d\beta\, dp/d\beta].$$

* Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] R. Weymann, Phys. Fluids **8**, 2112 (1965).
[2] F. C. Jones, Phys. Rev. **137**, B1306 (1965).

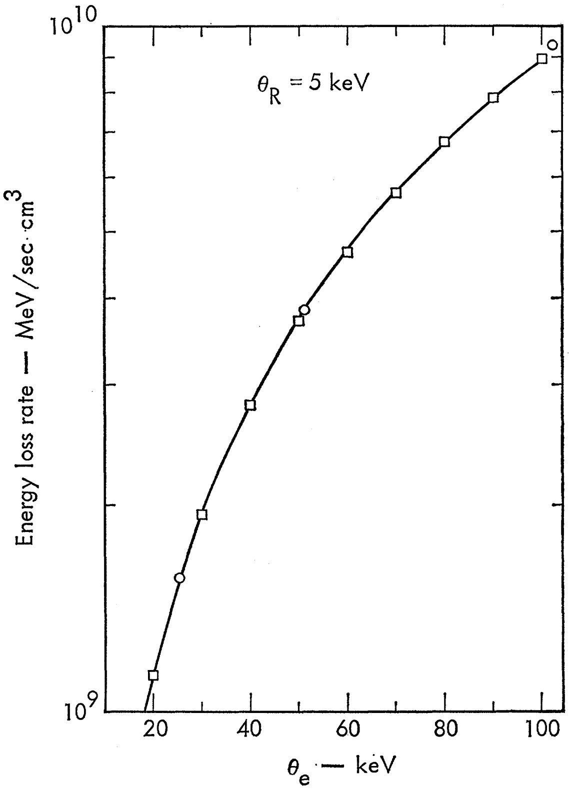

FIG. 1. $|dE/dt|$ in MeV/sec cm³ for $\rho_e=1$ cm^{-3} and $\theta_R=5$ keV as calculated by Warham (circles), Corman (squares), and Eq. (6) (solid line).

The right-hand side can then be expressed as a product of $A\exp(-mc^2/\theta_e-\beta^2mc^2/2\theta_e)d\beta$ and a power series in the two parameters β^2 and mc^2/θ_e. Integrating both sides and using the normalization condition (4), we therefore produce an expansion for $A\exp(-mc^2/\theta_e)$ in the parameter θ_e/mc^2. We need to evaluate integrals

$$F_{2j}\equiv\int\beta^{2j}n_e(p)p^2dp. \qquad (6)$$

Using the expansions for $n_e(p)p^2dp$ and $A\exp(-mc^2/\theta_e)$, we then obtain expansions for the F_{2j} integrals in the parameter θ_e/mc^2.

The calculation is extremely tedious and yields the result

$$\begin{aligned}dE/dt=&(32/45)\pi^3[r_0^2c/(\hbar c)^3]\rho_e\theta_R^4[(\theta_e-\theta_R)/mc^2]\\&\times[1.000+2.500(\theta_e/mc^2)-19.74(\theta_R/mc^2)\\&+1.875(\theta_e/mc^2)^2-122.2(\theta_e/mc^2)(\theta_R/mc^2)\\&+410.7(\theta_R/mc^2)^2-1.875(\theta_e/mc^2)^3\\&-292.0(\theta_e/mc^2)^2(\theta_R/mc^2)\\&+4357(\theta_e/mc^2)(\theta_R/mc^2)^2-8995(\theta_R/mc^2)^3], \qquad (7)\end{aligned}$$

where we have retained corrections to third order to the nonrelativistic result, which stands outside the bracket. The first-order correction terms are in agreement with those obtained earlier by Sessler and Riddell.[3]

To evaluate F_{2j} in (6), the variable of integration was changed to β. The upper limit of integration was then taken to be ∞ rather than 1, in order to obtain a simple series expansion for the integral in powers of θ_e/mc^2. By numerical integration, the error was estimated to be negligible with respect to the highest order of θ_e/mc^2 which was retained in the approximation.

II. RESULTS

The range of validity of the approximate series in (7) can be estimated by a comparison with other calculations of dE/dt which involve approximations of a different character. Warham[4] has calculated L, the ratio of dE/dt to the nonrelativistic approximation to dE/dt [obtained by replacing the quantity in brackets in (7) by unity], for several values of θ_e/mc^2 and θ_R/mc^2 by performing a multiple numerical integration using the exact integrand. Also Corman[5] has calculated dE/dt by a Monte Carlo simulation of the Compton interaction of a Maxwellian gas of electrons with a Planckian

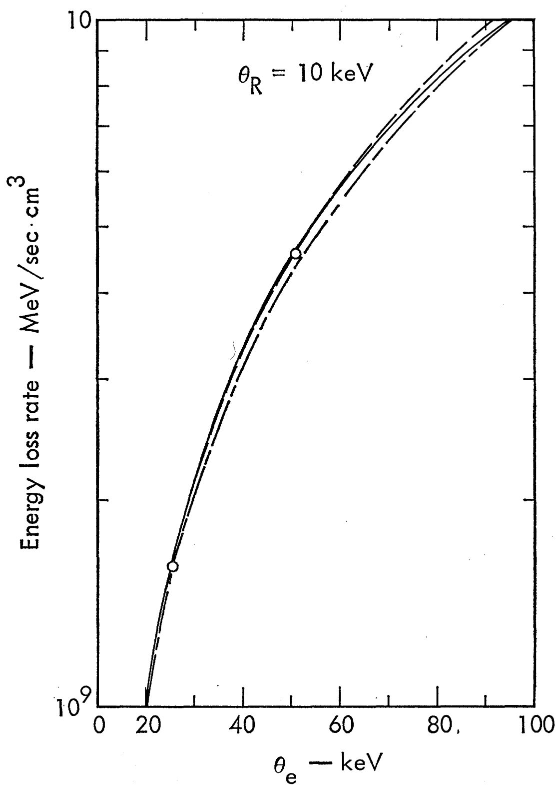

FIG. 2. $|dE/dt|$ in MeV/sec cm³ for $\rho_e=1$ cm^{-3} and $\theta_R=10$ keV as calculated by Warham (dashed), Corman (dashed and dotted), and Eq. (6) (solid).

[3] A. M. Sessler and R. J. Riddell, Jr. (private communication).
[4] A. Warham, Atomic Weapons Research Establishment Report No. 003/68 (unpublished).
[5] E. G. Corman, following paper, Phys. Rev. D 1, 2734 (1970).

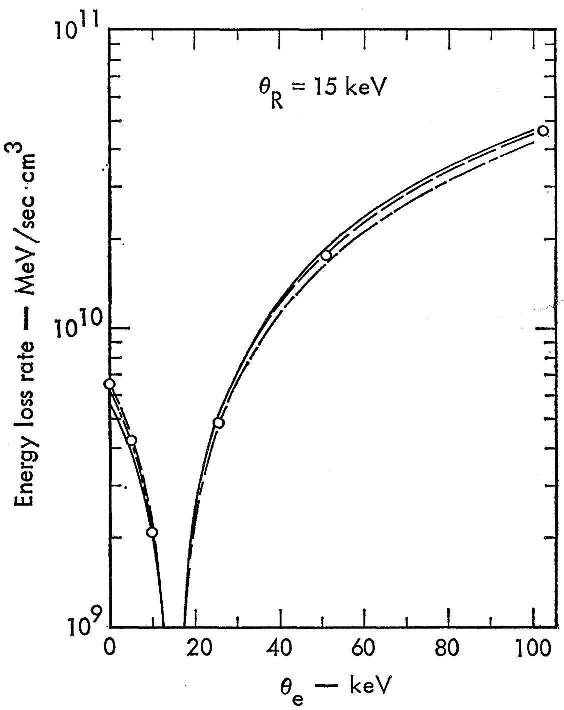

FIG. 3. $|dE/dt|$ in MeV/sec cm^3 for $\rho_e = 1$ cm^{-3} and $\theta_R = 15$ keV as calculated by Warham (dashed), Corman (dashed and dotted), and Eq. (6) (solid).

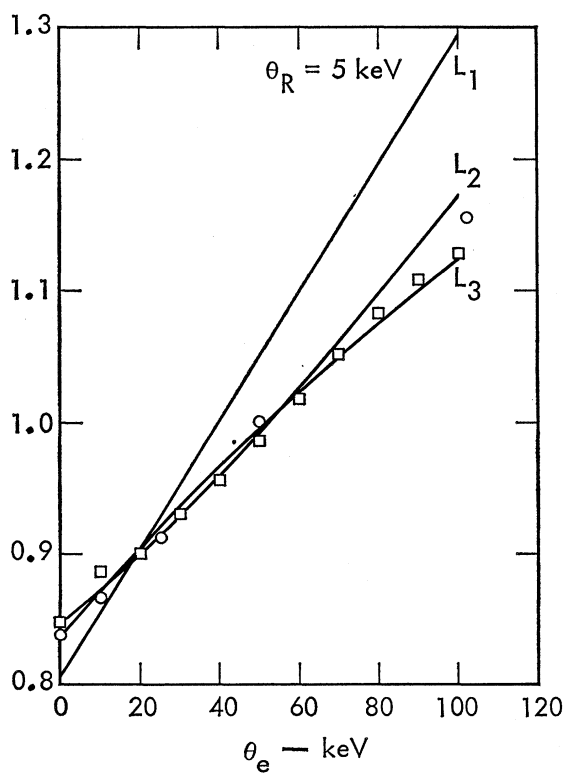

FIG. 4. L_1, L_2, L_3, L for $\theta_R = 5$ keV. L points as computed by Warham are circles and L points computed by Corman are squares.

photon gas. The values of dE/dt so obtained[6] are compared with values obtained from Eq. (7) in Figs. 1–3. When θ_R is 5, 10, and 15 keV, our series over the range $0 \leq \theta_e \leq 100$ keV agrees with Warham's results to 2.5, 4.5, and 14%, respectively, and with the Monte Carlo calculations of Corman to 1.1, 6.5, and 13%, respectively. When θ_R is 20 keV the series is of no use, since it disagrees with Warham's and Corman's calculations by as much as 40% in the range $0 \leq \theta_e \leq 100$ keV.

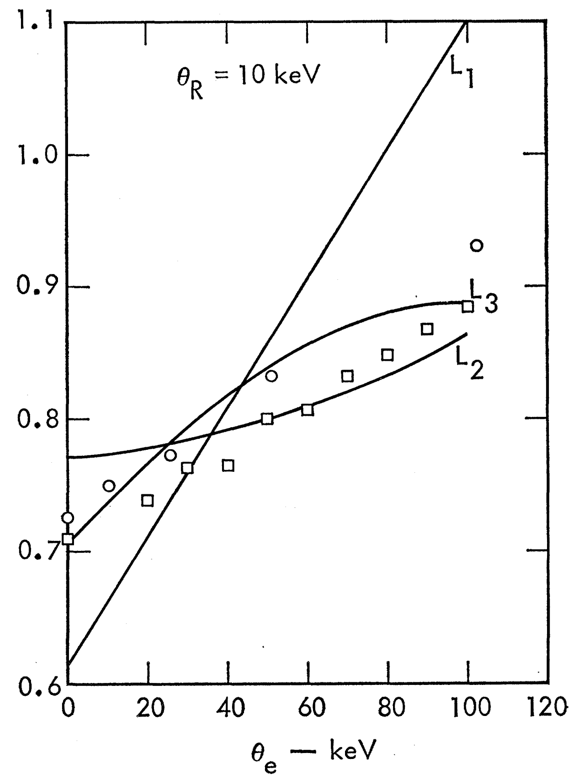

FIG. 5. L_1, L_2, L_3, L for $\theta_R = 10$ keV. L points as computed by Warham are circles and L points as computed by Corman are squares.

III. CONVERGENCE OF SERIES

We now investigate the convergence of the series in brackets in (7) by comparing the quantities L_n, which we define as the sum of terms up to nth order in the two parameters θ_e/mc^2 and θ_R/mc^2. A criterion for acceptable convergence of the series is that

$$|L_3 - L_2| \ll |L_2 - L_1| \ll |L_1 - 1|. \qquad (8)$$

When this condition is satisfied, we are justified in asserting that

$$|(L_3 - L)/L| \lesssim |(L_3 - L_2)/L_2|, \qquad (9)$$

[6] The values from Warham's report have been interpolated from his table.

where L is the exact value to which L_n is an approximation. In Figs. 4 and 5, L_1, L_2, and L_3 are plotted for $\theta_R=5$, 10 keV and $0\leq\theta_e\leq 100$ keV. The results of Warham's and Corman's calculations are also plotted.

Convergence of the series in (7) is satisfactory over the range $0\leq\theta_e\leq 100$ keV and $0\leq\theta_R\leq 10$ keV. When θ_R is as large as 15 keV the convergence of the series is not so strong, since the $\ll$ signs in (8) can no longer be taken so strictly. At $\theta_e=0$ and $\theta_e=100$ keV, $|L_3-L_2|$ is more than half $|L_2-L_1|$.

This analysis indicates, using Eq. (9), that Eq. (7) gives dE/dt to within about 3% of the exact value for $0\leq\theta_e\leq 92$ keV and for $0\leq\theta_R\leq 7.2$ keV. The limit on θ_R can be raised to 10 keV if we desire only about 10% accuracy.

IV. CONCLUSIONS

The rate of energy transfer from an isotropic Maxwell-Boltzmann electron gas with temperature $T_e=\theta_e/k$ to an isotropic Planckian photon gas with temperature $T_R=\theta_R/k$ can be accurately calculated using the series expansion in Eq. (7) when $\theta_e\lesssim 100$ keV and $\theta_R\lesssim 10$ keV. The size of the relativistic corrections incorporated in the higher-order terms in (7) indicates that the non-relativistic rate cannot be used when $\theta_e>20$ keV or when $\theta_R>2.5$ keV.

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Relativistic Compton Energy Exchange*

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A relativistically accurate table of values is calculated for the Compton energy exchange rate between photons in a Planckian distribution at temperature T_R varying from 5 to 50 keV and electrons in a nondegenerate Maxwell-Boltzmann distribution at temperatures T_e varying from 0 to 100 keV.

INTRODUCTION

The Compton energy exchange between electrons and photons (both direct and inverse) has been discussed in recent literature.[1–4] Such calculational methods have involved series expansions in v/c and $h\nu/mc^2$ or numerical integration procedures.

In this treatment Monte Carlo techniques are applied in lieu of series expansions or multiple integrations to obtain the Compton energy exchange rate. At any given temperature as much accuracy as desired can be achieved simply by using a large enough sample of photon "bundles", each bundle representing a large number of true photons of a given energy K. For these calculations a sampling size of 100 000 photon bundles was selected, which required approximately 10-sec computing time on the CDC 7600 computer, for each dE/dt value.

I. METHOD OF CALCULATION

The photons were selected from a Planckian distribution at temperature T_R. Each photon of energy K collided with an electron of velocity v approaching the point of collision at an angle whose cosine is γ_1 (see Fig. 1). The electron velocity is selected from a relativistic Maxwell-Boltzmann distribution $\exp(-\epsilon/kT_e)P^2dP$, where the electron kinetic energy

$$\epsilon=mc^2/(1-\beta^2)^{1/2}-mc^2,\qquad \beta=v/c,$$

and P is the electron momentum. The normalized distribution used for selecting β was a third-order expansion[5]

$$f(\eta)\,d\eta=(P_1f_1+P_2f_2+P_3f_3)\,d\eta,$$

where

$$\beta\equiv v/c=(1-[1+(kT_e/mc^2)\eta^2]^{-2})^{1/2},$$

and

$$f_1(\eta)=(4/\sqrt{\pi})\eta^2\exp(-\eta^2),$$
$$f_2=(8/3\sqrt{\pi})\eta^4\exp(-\eta^2),$$
$$f_3=(16/15\sqrt{\pi})\eta^6\exp(-\eta^2),$$
$$D=1+(15/8)(kT_e/mc^2)+(105/128)[(kT_e)^2/m^2c^4],$$
$$P_1=D^{-1},\qquad P_2=(15/8)(kT_e/mc^2)D^{-1},$$
$$P_3=(105/128)[(kT_e)^2/m^2c^4]D^{-1},$$

each f_i being separately normalized to unity. Note that f_1 is the nonrelativistic velocity distribution, f_2 and f_3 being relativistic corrections.

* Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] R. Weymann, Phys. Fluids **8**, 2112 (1965).
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