PHYSICAL REVIEW D

PARTICLES AND FIELDS

THIRD SERIES, VOL. 1, NO. 10 15 MAY 1970

Mach's Principle, the Kerr Metric, and Black-Hole Physics*†

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(Received 2 October 1969)

The generalized Kerr solution is tentatively accepted as the description for the state of a charged rotating mass which has undergone complete gravitational collapse. Mass accretion by such a system is demonstrated to damp out the rotation. Even the most favorable selective capture of matter leaves the angular momentum bounded by m^2, precisely the upper limit that the Kerr geometry can accommodate without change of character. Mach's principle, exemplified by the rotation of inertial frames, is employed to obtain approximate expressions for perihelion precession of satellites, deflection of light trajectories, and the rotation of polarization of light. Results are compared with exact expressions, when available.

I. INTRODUCTION

A MAJOR problem in the investigation of gravitational collapse is in determining an adequate model for the final state of a completely collapsed star. The usual model, the "black hole"[1] of the Schwarzschild geometry, is balanced in its mathematical simplicity by its lack of generality. Static and spherically symmetric, it has but one free parameter, mass, while stars possess additionally both angular momentum and multipole moments of varying magnitudes.

In treating, then, the Schwarzschild solution as *the* correct form for a gravitationally collapsed body, one implicitly makes one of two assumptions. Either it is possible to so perturb the solution that it may accommodate the moments of the collapsed object it represents without destroying the essential characteristics of the Schwarzschild metric, or the object, while collapsing, must in some manner invariably discard these moments. The former possibility appears to be effectively ruled out by numerous recent computations,[2] while the latter may well violate the conservation laws of Newman and Penrose[3] for quadrupole moments, although the situation here is far from clear. In any case, there is today no compelling reason to suppose that either of the above assumptions *should* be true.

Hence, it is not at all inappropriate to seek more general models for the final state of collapse. The most general one to date is the Kerr solution[4] as extended by Newman *et al.*[5] It possesses a freely disposable mass m, an angular momentum per unit mass a, and an electric charge e. But it must be remembered that, since m and a uniquely determine all of the multipole moments, the possible objections voiced above for the Schwarzschild model are in part applicable here as well.

Crucial to the utility of the Kerr metric in the study of gravitational collapse is the range of values its parameters may meaningfully assume. For two reasons, it appears that

$$a^2+e^2\leq m^2. \tag{1}$$

Violation of this inequality has such unappealing direct consequences as the absence of horizons and the breakdown of causality.[6] There is, moreover, the theoretical objection that, since $|e|$ of the Reissner-Nordstrøm

* Part of this work was reported at the April 1969 Washington meeting of the American Physical Society, Bull. Am. Phys. Soc. **14**, 615 (1969).

† This work has been assisted in part by NSF Grant No. GP7669.

‡ National Science Foundation Predoctoral Fellow, 1967–1970.

[1] B. Harrison, K. Thorne, M. Wakano, and J. Wheeler, *Gravitation Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965), Chap. 11.

[2] See, for example, L. Myšak and G. Szekeres, Can. J. Phys. **44**, 617 (1966); W. Israel and K. Kahn, Nuovo Cimento **33**, 331 (1964).

[3] E. Newman and R. Penrose, Phys. Rev. Letters **15**, 231 (1965).

[4] R. Kerr, Phys. Rev. Letters **11**, 237 (1963).

[5] E. Newman, E. Couch, R. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. **6**, 918 (1965).

[6] B. Carter, Phys. Rev. **174**, 1559 (1968).

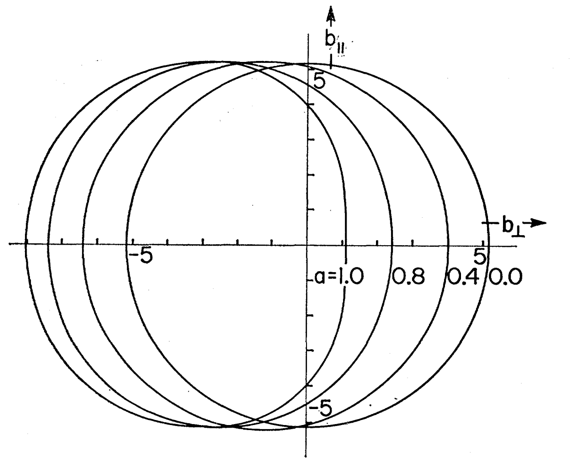

FIG. 1. Photon capture cross sections for black holes of various angular momenta. Photons approach perpendicularly to the rotation axis with impact parameter components $b_{\|}$ (along rotation axis) and $b_\perp$ (perpendicular to rotation axis). The Schwarzschild case with $b_{\|}=b_\perp=3\sqrt{3}$ corresponds to $a=0$. All quantities are in units of m.

mass is due to its electric charge,[7] one should expect that $(a^2+e^2)^{1/2}$ of the Kerr mass is of rotational and electrical origin. In this argument, $a^2+e^2>m^2$ becomes an impossibility.

Many stars, including the sun,[8] have an a/m ratio exceeding unity in absolute value. Can they not collapse, will they break up upon collapsing, or must the Kerr model be discarded? The full answer probably will only emerge from detailed machine calculations of collapse.

This paper demonstrates the stability of condition (1), showing that an already collapsed object with $|a/m|<1$ even by the accretion of matter of the highest capturable angular momentum cannot increase the ratio above unity. Moreover, accretion of randomly incident particles is shown to damp out rotation.

The second part of the paper considers certain observables related to a rotating black hole, tying them together within the framework of Mach's principle.

II. MASS ACCRETION AND DAMPING OF ANGULAR MOMENTUM

The notion that mass accretion should damp out the rotation of the capturing body was first examined by Doroshkevich.[9] For a not as yet collapsed body of radius (Schwarzschild coordinate) $R>3m$ capturing nonrelativistic particles arriving with random impact parameters, he obtained on the basis of the well-known Lense-Thirring line element[10] the first-order formula for change of angular momentum L with mass m,

$$dL/dm=-(L/m)[2m/(R-2m)]. \quad (2)$$

An exact formula for the same quantity may be obtained by finding for what impact parameter the turninging point of a particle coincides with the radius[11] of the body. This is done most conveniently by setting $\dot{r}=0$ in Carter's turning-point equation [his Eq. (63)].[6] We consider only capture for a flux incident perpendicular to the body's axis of rotation, noting that such a flux contributes most to altering the angular momentum of the central body. The cross section is characterized by the components $b_{\|}$ and $b_\perp$ of the impact parameter, parallel and perpendicular to the axis of rotation, for a particle that can just barely make its way in to the coordinate value R. No point with coordinate r is accessible unless the reciprocal parameter $u=1/r$ satisfies the inequality

$$[E-\mathcal{E}ea-(pb_\perp+\beta Ea-Ea)au^2]^2 -(1-2mu+a^2u^2+e^2u^2) \times[(\mu^2+p^2b_{\|}^2u^2)+(pb_\perp+\beta Ea-Ea)^2u^2]\geq 0. \quad (3)$$

Here μ is the rest mass of the incident particle, $\mathcal{E}$ its charge, and E and p its energy and momentum at infinity;

$$E=\mu(1-\beta^2)^{-1/2}, \qquad p=\beta E. \quad (4)$$

At $R=r$ the inequality becomes an equality. It follows that the locus of the critical impact parameter is an ellipse,

$$[1-2m/R+(a^2+e^2)/R^2]b_{\|}^2 +[1-2m/R+e^2/R^2](b_\perp-b_0)^2=f^2, \quad (5)$$

where

$$f^2=R^2+e^2+b_0^2-\mu^2a^2/p^2+(2m\mu^2-2E\mathcal{E}e)R/p^2 \quad (6)$$

and the shift of the center of the ellipse from the origin is

$$b_0=\frac{1+\beta}{\beta}-\frac{aR(R/\beta-\mathcal{E}e/p)}{R^2+e^2-2mR}. \quad (7)$$

The average angular momentum transferred per particle is thus b_0p, and so

$$\frac{dL}{dm}=-\frac{L}{m}\left[\frac{R^2-\mathcal{E}eR/E}{R^2+e^2-2mR}+(\beta-1)\right]. \quad (8)$$

For $e=0$ and $\beta\to 0$, this gives precisely (2). The limit for extreme relativistic velocities is

$$dL/dm=-(L/m)[R^2/(R^2+e^2-2mR)]. \quad (9)$$

A similar equation may be written for the rate of

[7] A. Papapetrou, Proc. Roy. Irish Acad. **51A**, 191 (1947).

[8] For the sun $a/m=1.26$, as may be obtained from $a/m=Lc/GM^2$. L is the angular momentum of the sun, M its mass, c the speed of light, and G the gravitational constant, all expressed in some standard system of units.

[9] A. Doroshkevich, Astron. Zh. **43**, 105 (1966) [Soviet Astron. AJ **10**, 83 (1966)]; see also Ya. Zel'dovich and I. Novikov, Astron. Zh. **43**, 758 (1966) [Soviet Astron. AJ **10**, 602 (1967)].

[10] J. Lense and H. Thirring, Physik. Z. **19**, 156 (1918).

[11] The surface of the body is taken as a constant $r-t$ surface in the standard Schwarzschild-like coordinates. See, e.g., Ref. 6. R must be greater than the $1/u$ corresponding to the double zero of Eq. (3).

change of charge with accretion. The explicit results are, however, involved and not very informative. They are best summarized by noting that nonrelativistic particles of charge opposite to that of the core are preferentially captured, much as in the Newtonian case. For particles found in most plasmas, $\mathcal{E}/\mu\approx 10^{20}$. Therefore the approach of the body to equilibrium is very rapid. If the captured particles are, on the other hand, highly relativistic, more nearly even numbers of oppositely charged particles are captured, and the approach to electrical equilibrium is comparatively slower.

When we turn to the case of a completely collapsed body, a black hole, calculations become more difficult. It is not sufficient merely to replace R in the preceding formulas by the radius of the black hole's outer horizon, for we soon find that Eq. (3) can be satisfied at the horizon while at the same time not be satisfied for slightly larger values of r. It is as though an effective potential barrier surrounds the horizon. Anything surmounting the barrier is captured.

For a given set of parameters, the barrier maximum occurs where the derivative of (3) vanishes. We therefore differentiate (3) with respect to u and annul the derivative, along with (3) itself. In this way we arrive at two expressions quadratic in the components of the impact parameter, to be solved simultaneously. Selected results for the case of photon capture, found numerically, appear in Fig. 1. Here too it is evident that mass accretion damps out angular momentum. Note that cross sections are relatively independent of e.

For $e=0$ the $b_\perp$ intercepts ($b_\perp^*$) are given by

$$(b_\perp^*+2a)^3-27m^2b_\perp^*=0. \qquad (10)$$

By means of the convenient parametric representation[12]

$$a=-m\cos 3\sigma, \qquad b_\perp^*=8m\cos^3\sigma, \qquad (11)$$

this curve may be plotted as in Fig. 2.[13] The shaded region represents values of $b_\perp$ and a for which a photon with equatorial trajectory is captured.

From Eq. (10), it follows that even the most favorable selective capture of photons cannot increase the absolute value of $\alpha=a/m$ above $\frac{3}{8}\sqrt{6}=0.926$. A short computation gives the change of α per photon as

$$d\alpha/dm=(b_\perp-2a)/m^2. \qquad (12)$$

Hence, for any $b_\perp$ the equilibrium value of α corresponds to $2a=b_\perp$. Inserting this into Eq. (10) gives the asserted upper bound.

The corresponding calculation for material particles, again with $e=0$, is considerably more difficult, and so is omitted here. From it we find that the limiting value of α assumes a maximum of 1 for nonrelativistic particles, an encouraging result in view of the proposed limit of $a^2<m^2$ given in the Introduction.

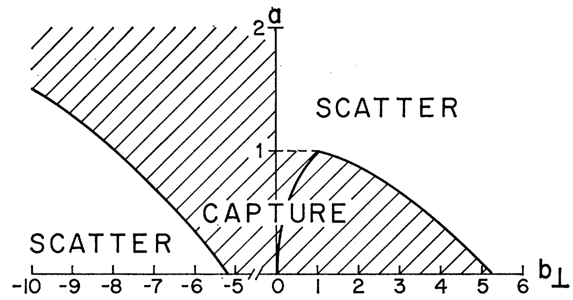

FIG. 2. Plot of a versus $b_\perp^*$ (solid curve). Both are in units of m. The plot is physically meaningless for $0\le b_\perp^*<1$, as these values would lie within the horizon. The shaded areas indicate values of a and $b_\perp$ for which equatorial photons are captured.

III. MACHIAN EFFECTS

Within the framework of Mach's principle it is possible to provide simple explanations for a number of phenomena resulting from the rotation of the central body in the Kerr metric. Essentially, we consider the inertial frames to be dragged along with the central body, with objects passing through the frames consequently feeling a Coriolis force. Viewed in this way, the resulting deflections are more readily understandable, and, additionally, can be calculated to first order with comparative ease. The general question of Machian effects in the gravitational field of a rotating object has been considered in a series of articles by Brill and Cohen.[14] Here we specialize and consider three such effects, the precession of satellite orbits, the bending of light trajectories, and the rotation of the polarization of light, computing their approximate magnitudes on the basis of the rotation of inertial frames.

Before proceeding, we cite exact expressions for the first two effects, considered in the equatorial plane, and with $e=0$. Using the technique described by Boyer and Lindquist,[15] one obtains for the perihelion precession, by a straightforward computation,[16]

$$\Delta\phi=4[2mb_\perp^2(s_1-s_3)]^{-1/2}\times\left[\frac{A_+}{u_+-s_3}\Pi\left(\frac{s_2-s_3}{u_+-s_3},\left(\frac{s_2-s_3}{s_1-s_3}\right)^{1/2}\right)+\frac{A_-}{u_--s_3}\Pi\left(\frac{s_2-s_3}{u_--s_3},\left(\frac{s_2-s_3}{s_1-s_3}\right)^{1/2}\right)\right], \qquad (13)$$

where

$$\frac{A_+}{u_+-u}+\frac{A_-}{u_--u}=\frac{1-2mb_\perp u}{1-2m+a^2u^2}. \qquad (14)$$

[12] Note the invariance under the simultaneous change of sign of a and $b_\perp^*$.

[13] Similar diagrams are to be found in F. de Filice, Nuovo Cimento **57B**, 351 (1968).

[14] J. Cohen and D. Brill, Nuovo Cimento **56B**, 209 (1968).

[15] R. Boyer and R. Lindquist, J. Math. Phys. **8**, 265 (1967).

[16] The integrals and special functions involved are dealt with in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).

$s_1 \geq s_2 \geq s_3 \geq 0$ are the roots of expression (3). $\Pi(\alpha, k)$ is a complete elliptic integral of the third kind. For a circular orbit set $s_2 = s_3 = s$, with s the orbit reciprocal radius,

$$\Delta\phi = 2\pi[2mb_\perp^2(s_1 - s)]^{-1/2}. \tag{15}$$

One may obtain Darwin's Schwarzschild metric result[17] by setting $a=0$ and $s=1/d$:

$$\Delta\phi = 2\pi(1-6m/d)^{-1/2}. \tag{16}$$

To second order in s, Eq. (15) becomes

$$\Delta\phi \approx 2\pi(1+3ms-4am^{1/2}s^{3/2}+35m^2s^2/2+3a^2s^2/2). \tag{17}$$

With $h^2 = m/s$, the first three terms reproduce the result of Boyer and Price.[18] For the precession of the perihelion of Mercury, only the effect of the second term, $3m/d$, is measurable.

The expression for the deflection of light, derived explicitly by Boyer and Lindquist, is given here in slightly altered form:

$$\sin^2\delta = \frac{s_1 - s_3}{s_2 - s_3}\frac{s_2}{s_1}, \qquad q^2 = \frac{s_2 - s_3}{s_1 - s_3},$$

$$\Delta\phi = 4[2mb_\perp^2(s_1 - s_3)]^{-1/2}\left\{\frac{A_+}{(u_+ - s_1)(u_+ - s_2)} \times \left[(s_2 - s_1)\Pi\left(\delta, q^2\frac{u_+ - s_1}{u_+ - s_2}, q\right) + (u_+ - s_2)F(\delta, q)\right] + \frac{A_-}{(u_- - s_1)(u_- - s_2)}\left[(s_2 - s_1)\Pi\left(\delta, q^2\frac{u_- - s_1}{u_- - s_2}, q\right) + (u_- - s_2)F(\delta, q)\right]\right\}, \tag{18}$$

where $F(\alpha, k)$ and $\Pi(\alpha, n, k)$ are incomplete elliptic integrals of the first and third kinds. Note that now $s_1 \geq s_2 \geq 0 \geq s_3$. To first order in the impact parameter $s_1 \approx 1/d$, this yields the well-known result[14,15,19,20]

$$\Delta\phi - \pi = \frac{4m(1-a/d)}{d}. \tag{19}$$

Mach's principle asserts that inertial frames near the origin of the generalized Kerr metric should be dragged along in the direction of rotation of the central body. Since the rate of dragging varies with position, a differential rotation between adjacent frames results.[21] This differential rotation is expressed by the Landau-Lifshitz formula[22]

$$\mathbf{\Omega} = -\tfrac{1}{2}h^{1/2}\nabla \times \boldsymbol{g}, \tag{20}$$

where

$$h = -g_{00}, \qquad g_i = -g_{0i}/g_{00}. \tag{21}$$

After lengthy but straightforward calculations, one obtains

$$\begin{aligned} \Omega_r &= a\Delta(2mr-e^2)[1+(2mr-e^2)/\rho^2]\cos\theta/x, \\ \Omega_\theta &= a[(2mr-e^2)r - m\rho^2]\rho\sin\theta/x, \\ \Omega_\phi &= a^2(2mr-e^2)[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]^{1/2}\sin 2\theta/2x\rho, \\ x &= \rho^2[\rho^2-(2mr-e^2)][\rho^4-(2mr-e^2)^2]^{1/2}, \\ \rho &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2+a^2+e^2-2mr. \end{aligned} \tag{22}$$

We note that the expressions for angular velocity bebecome infinite precisely on the infinite-red-shift surface, where the timelike Killing vector becomes null.[23] Within this surface the formula is no longer valid, since the metric is not stationary.

In the limit of large r, $\mathbf{\Omega}$ takes the form

$$\begin{aligned} \Omega_r &= 2ma\cos\theta/r^3, \\ \Omega_\theta &= ma\sin\theta/r^3, \\ \Omega_\phi &= 0. \end{aligned} \tag{23}$$

Near the equatorial plane the unit $\boldsymbol{\theta}$ vector is antiparallel to the angular momentum vector $\mathbf{L}$ ($\mathbf{L}$ points out of the paper for $a>0$ in Fig. 3, while $\boldsymbol{\theta}$ points in) and therefore, interestingly enough, the rotation of inertial frames is retrograde[24]:

$$\mathbf{\Omega} \approx -\mathbf{L}/r^3 \qquad (\theta \approx \tfrac{1}{2}\pi). \tag{24}$$

With these results we can reproduce expressions (17) and (19). Because of the differential rotation of inertial frames, photons experience a transverse Coriolis acceleration given by[25]

$$\dot{\mathbf{v}} = 2\mathbf{\Omega} \times \mathbf{v}, \tag{25}$$

or, for large r and in the equatorial plane,

$$\dot{v} = 2ma/r^3 \tag{26}$$

and to the right in the sense of Fig. 3. Incidentally, Fig. 3 shows well the dramatic effect of rotation upon photon trajectories. At small distances the whole pattern is swept around to the right, accounting for the asymmetric capture cross sections already mentioned.

Equation (26) may be used directly to obtain the

[17] C. Darwin, Proc. Roy. Soc. (London) **A249**, 180 (1969).
[18] R. Boyer and T. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965).
[19] J. Plebanski, Phys. Rev. **118**, 1396 (1960).
[20] G. Skrotskii, Dokl. Akad. Nauk SSSR **114**, 73 (1958) [Soviet Phys. Doklady **2**, 226 (1957)].
[21] I an indebted to Professor James Bardeen for pointing up and explaining this situation.

[22] L. Landau and E. Lifshitz, *Theoria Polia*, 5th ed., (Nauka, Moscow, 1965).
[23] C. Vishveshwara, J. Math. Phys. **9**, 1319 (1968).
[24] First pointed out by J. Cohen, Phys. Rev. **173**, 1258 (1968).
[25] A minus sign is inserted into the standard formula since $\mathbf{\Omega}$ measures rotation of the inertial frames relative to the global metric, rather than the reverse.

first-order deflection of light due to rotation,

$$\Delta\phi = -\int_{-\infty}^{\infty} \dot{v}\, ds, \tag{27}$$

$$= -4ma/d^2, \tag{28}$$

in agreement with Eq. (19). Similarly, the lowest-order correction due to charge is

$$\Delta\phi = \pi a e^2/d^3. \tag{29}$$

A like calculation for the perihelion precession is not so simple. Merely inserting (26) into the standard formula[26]

$$d^2u/d\phi^2 + u = m/h^2 + 3mu^2, \tag{30}$$

with h the classical orbital angular momentum, to obtain

$$d^2u/d\phi^2 + u = m/h^2 + 3mu^2 - 2mau^2/h, \tag{31}$$

gives a result for circular orbits too small by a factor of 2. The difficulty is that not h but Carter's[6] Φ is the conserved quantity. A brief computation reveals that the following approximate substitution is therefore appropriate:

$$h^2 \to h^2(1+4mau/h). \tag{32}$$

The resultant equation

$$d^2u/d\phi^2 + u = (m/h^2)(1-4mau/h) + 3mu^2 - 2mau^2/h \tag{33}$$

then yields the result of Boyer and Price [Eq. (17) and subsequent comments].

The rotation of a photon's plane of polarization may also be computed to first order based on the rotation of inertial frames. For instance, consider a photon emitted from a body of radius R along its rotation axis. Assuming its polarization vector to be dragged along by the rotation of the inertial frames (i.e., rotated at a rate Ω_r), we obtain a net rotation, measured at a great distance, of

$$\Delta\Psi = \int_R^{\infty} \frac{2ma}{r^3}\, dr, \tag{34}$$

$$= L/R^2. \tag{35}$$

Other approximate results include $3L/R^2$ by Skrotskii[20] and 0 by Plebanski.[19] The latter, indeed, obtains no rotation for any trajectory in the vacuum outside a slowly rotating body.

In a similar manner, one may calculate the rotation for a photon traveling parallel to, and at some distance from, the rotation axis from minus to plus infinity. By an interesting cancellation, the result is zero. On the other hand, Skrotskii obtains $8L/d^2$, Balazs[27] finds $2L/d^2$, and, of course, Plebanski predicts no rotation.

[26] R. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford U. P., Oxford, 1934).

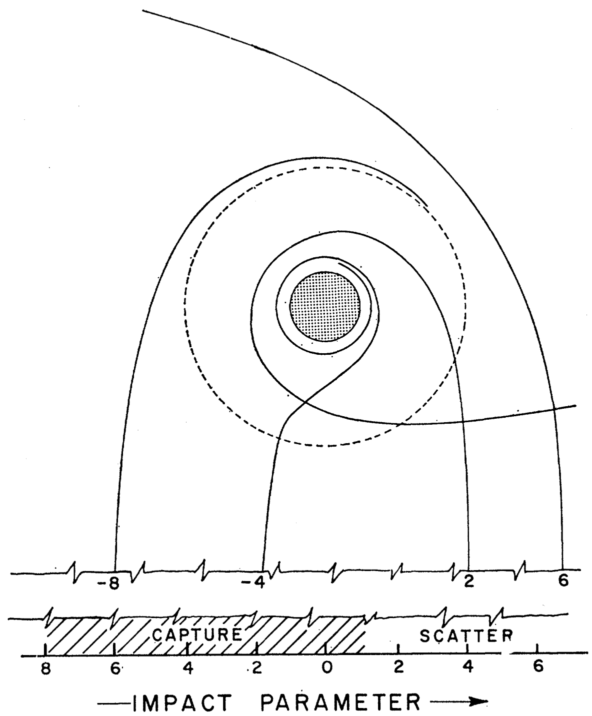

FIG. 3. Equatorial photon trajectories for the Kerr metric with $e=0$ and $a=1$. Photons enter from the bottom of the figure. A photon with impact parameter -8 is captured into an unstable orbit at $r=4$. A similar fate befalls a photon (not shown) with impact parameter $+1$. All quantities are given in units of mass m.

Since the polarization vector is parallel transported along the null geodesic,[28] one in principle may compute exactly the two rotations given above. However, only in the first case (a photon on the symmetry axis) is an analytic expression readily obtainable,

$$\Delta\Psi = L/(R^2+a^2), \tag{36}$$

which reduces to (35) in the limit of large R, thereby lending credence to the Machian approximation as opposed to the cited results of other methods.

ACKNOWLEDGMENTS

I am pleased to acknowledge the numerous suggestions and constant encouragement given by Professor John Wheeler. Also, Mr. F. Goldrich gave generously of his time to assist in numerical computations.

[27] N. L. Balazs, Phys. Rev. **110**, 236 (1958).

[28] Kindly pointed out to the author by Professor Charles Misner.