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Scattering Theory of Resonance "Mixtures"

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The scattering amplitude for scattering through several resonances is derived and shown to be always a sum of simple Breit-Wigner terms even for nontrival "particle mixtures." The formalism affords a particularly simple interpretation of the nonexponential decay or "dipole" phenomenon.

THE intriguing problem of particle mixtures continues to be of physical interest, in the original K^0 meson problem[1] as well as in various analogies inspired by the K^0 system.[2] For certain of these problems, such as $e^+e^-\to\rho^0$, ω^0, where ρ and ω are mixed by electromagnetic interactions or excitation of atomic levels mixed by external fields,[3] it would seem more natural to have a version of the theory treated as a scattering problem, rather than as, in the usual approach, an unstable particle problem. Thus a K^0 meson is thought of as a resonance (a rather narrow one it must be admitted) in $\pi\pi$, $\bar{\nu}\mu$, 3π, ... scattering.

It has been stated[4] that the natural result one would expect, namely, that the scattering matrix T is simply a sum of Breit-Wigner resonance forms, is not generally correct. It is claimed that in the case (such as seems to apply for the K mesons) of nonorthogonal eigenvectors of the "mass matrix," the corresponding T matrix would have to violate unitarity if it were just a sum of Breit-Wigner forms. We will show that this is not correct and that there exists a very simple, general formulation for the T matrix for the resonance-mixture problem which covers all cases and which *always* corresponds to a simple sum of Breit-Wigner forms.

To clarify the discussion, we start with the simple single-resonance problem and then add on the complications.

SIMPLE RESONANCE

We proceed by solving the Schrödinger equation for a Hamiltonian H_0+V. The H_0 operator has a multichannel continuous spectrum $|i,k\rangle, \ldots, |f,k\rangle$ corresponding to initial and final plane-wave states of momentum k, and a discrete state $|\alpha\rangle$, $H_0|\alpha\rangle=E_\alpha|\alpha\rangle$. This is the assumption that there is a particle or level which will give rise to a resonance when we "turn on" V. For the moment, V only connects the continuum states with $|\alpha\rangle$ but not the continuum states with each other.

We now look for a solution to $(H_0+V)\psi_{i,ki}{}^+=E\psi_{i,ki}{}^+$ with ψ^+ in the form

$$\psi_{i,ki}{}^+=\varphi_i\,|\alpha\rangle+\rho_{ij}(k)\,|j,k\rangle \tag{1}$$

(summed over all repeated indices).

Application of the Schrödinger equation and comparison of the coefficients of $|i\rangle$ and $|\alpha\rangle$ lead to

$$\varphi_i(E-E_\alpha)=\rho_{ij}(k)\,\langle\alpha|V|j,k\rangle, \tag{2a}$$

$$\rho_{ij}(k)[E-E(k)]=\varphi_i\langle k,j|V|\alpha\rangle. \tag{2b}$$

The solution to (2b), $\rho\sim\varphi V/[E-E(k)]$, is ambiguous at $E=E(k)$; to have outgoing waves in (1) we use the $i\epsilon$ prescription, and to have an incoming plane wave in channel i we add $\delta_{ij}\delta(k-k_i)$, giving

$$\rho_{ij}(k)=\delta_{ij}\delta(k-k_i)+\varphi_i\frac{\langle k,j|V|\alpha\rangle}{E-E(k)+i\epsilon}. \tag{3}$$

Putting this ρ in (2a) then gives, for φ,

$$\varphi_i=\frac{\langle\alpha|V|i,k_i\rangle}{E-E_\alpha-\langle\alpha|V|k,j\rangle\langle k,j|V|\alpha\rangle/[E-E(k)+i\epsilon]}. \tag{4}$$

Now, having found ψ^+, we use $T=\langle f,k_f|V|\psi^+\rangle$ to get the scattering amplitude. Since V only connects with $|\alpha\rangle$ in ψ^+, we have

$$T_{fi}(E)=\frac{\langle f,k_f|V|\alpha\rangle\langle\alpha|V|i,k_i\rangle}{E-E_\alpha-\langle\alpha|V|k,j\rangle\langle k,j|V|\alpha\rangle/[E-E(k)+i\epsilon]}. \tag{5}$$

The integral in the denominator is

$$\int dk\,\frac{\sum_j|\langle\alpha|V|j,k\rangle|^2}{E-E(k)+i\epsilon}\quad\text{(using V Hermitian),}$$

the real part being the level shift $\Delta(E)$ and the imaginary part one-half the total width, $\frac{1}{2}\Gamma(E)$, giving the usual Breit-Wigner form with the slight generalization that Δ and ϕ can depend on the energy.

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[1] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957); R. P. Feynman, R. B. Leighton, M. Sinds, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1965), Vol. III, pp. 11–15.

[2] For just three in elementary particle physics, we mention J. Bernstein and G. Feinberg, Phys. Rev. **132**, 1227 (1962); M. Ross and L. Stodolsky, Phys. Rev. Letters **17**, 563 (1966); J. Harte and R. G. Sachs, Phys. Rev. **135**, B459 (1964). For an analysis of the recently observed ω-ρ mixing effect, see M. Gourdin, F. M. Renard, and L. Stodolsky, Phys. Letters **30B**, 347 (1969).

[3] See C. Cohen-Tannoudji, *Cargèse Lectures*, edited by M. Lévy (Gordon and Breach, New York, 1968), Vol. 2.

[4] W. D. McGlinn and D. Polis, Phys. Rev. Letters **22**, 908 (1969).

FINAL-STATE INTERACTIONS

We can drop the restriction that the potential does not interconnect continuum states by splitting a completely general potential into two terms, one (V) connecting only to $|\alpha\rangle$, the other (U) connecting continuum states. If we now call the scattering solutions of the ψ^+ type for H_0+U, $|i, k+\rangle$, etc., we see that the above manipulations go through until Eq. (5) just as well for H_0+U+V, by replacing $|i, k\rangle \rightarrow |ik+\rangle$. Equation (1) for ψ^+ will again have the correct boundary conditions since the term $|ik+\rangle$ coming from the δ function in ρ contains the plane wave. A detour is necessary in the step before Eq. (5) since $T=\langle f, k|U+V|\psi^+\rangle$. Using the two-potential theorem,[5] the U dependence can be brought into the wave functions, giving

$$T_{fi}(E)=T_{fi}{}^0(E)+\frac{\langle -f, k_f|V|\alpha\rangle\langle\alpha|V|i, k_i+\rangle}{E-E_\alpha-\langle\alpha|V|k, j+\rangle\langle +k, j|V|\alpha\rangle/[E-E(k)+i\epsilon]}. \tag{6}$$

T^0 is the scattering due to U as if V were not present at all, $T_0=\langle f, k_f|U|k_i i+\rangle$. We then have a completely rigorous form for the unstable particle resonance in the presence of other interactions; it is again a simple Breit-Wigner formula if the effects of U are smoothly varying near E_α.

SEVERAL RESONANCES

In this case, instead of one discrete state in the spectrum of H_0+U, we have several: $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, ...; these are the ordinary orthogonal states, e.g., $K^0\bar{K}^0$, or the levels of an atom in the absence of coupling to the radiation field. The calculations above can be repeated, keeping track of the additional index. Thus, finally we arrive at $T=T^0+T^R$, where T^0 is the "U-only" scattering and the resonant scattering is

$$T_{fi}{}^R(E)=\langle -f, k_f|V|\alpha\rangle[1/(\mathfrak{M}-E)]_{\alpha\beta}\langle\beta|V|k_i i+\rangle, \tag{7}$$

where $\mathfrak{M}$ is a matrix in the discrete state subspace

$$\mathfrak{M}_{\alpha\beta}(E)=M_{\alpha\beta}{}^0(E)+\frac{\langle\alpha|V|k, j+\rangle\langle +k, j|V|\beta\rangle}{E-E(k)+i\epsilon} \tag{8}$$

or

$$\mathfrak{M}_{\alpha\beta}(E)=M_{\alpha\beta}(E)-\tfrac{1}{2}\Gamma_{\alpha\beta}(E).$$

We have put the real level shift together with the M^0 which comes from the original resonance energies and from V connecting $|\alpha\rangle$, $|\beta\rangle$ directly. Note that since V is Hermitian, M and Γ are Hermitian. We shall assume that the number of eigenvectors (with nonzero eigenvalue) is equal to the dimension of the discrete subspace. This implies that M has an inverse. Aside from this, it is clear that $\mathfrak{M}$ is a completely arbitrary matrix and that its eigenvectors are not necessarily orthogonal. In particular, the expansion of the identity in terms of eigenstates $|n\rangle$ of $\mathfrak{M}$ is not $|n\rangle\langle n|$, but rather $I=|n\rangle g_{nm}\langle m|$, where $g_{nm}\langle m|p\rangle=\delta_{np}$. The vector $g_{nm}\langle m|$ is like the "dual" $\langle n^d|$ used by Sachs.[6] Inserting this in (7) we get, with E_n the complex eigenvalue for $|n\rangle$,

$$T_{fi}{}^R(E)=\frac{\langle -f, k_f|V|n\rangle\langle n^d|V|k_i, i+\rangle}{E_n-E}$$

$$=\frac{\langle -f, k_f|V|n\rangle g_{nm}\langle m|V|k_i, i+\rangle}{E_n-E}, \tag{9}$$

which is the general solution to the problem and is evidently just a sum of Breit-Wigner terms. Note, however, that the numerator does not have exactly the form supposed in Ref. 4.

Whether we wish to call the nonorthogonal eigenstates $|n\rangle$, $|m\rangle$, ... of $\mathfrak{M}$ "particles" or not is somewhat a matter of taste, but their physical significance is clear; either as poles in T or from the time-dependent point of view, they are those "directions" in the discrete state subspace which do not get "rotated" as time progresses. Since our S matrix is unitary, it of course leaves orthogonal states orthogonal.

UNITARITY

Since our T comes from solving the Schrödinger equation with a Hermitian Hamiltonian, it must necessarily have unitarity, that is, $S=1+iT$ is unitary. Forgetting again for a moment the potential U and thus the $+$, $-$ signs in the wave functions, it is easy to show by matrix manipulations that the form (7) gives unitarity automatically if

$$\tfrac{1}{2}[\mathfrak{M}-\mathfrak{M}^\dagger]_{\alpha\beta}=-i\pi\langle\alpha|V|j, k\rangle\langle j, k|V|\beta\rangle\delta(E-E(k)). \tag{10}$$

That this condition holds is seen from (8). Now with the "final-state interactions" U included, $(1+iT^R)$ alone is not unitary, since (7) has a $\langle -|$ state on the left and a $|+\rangle$ on the right, while the same states on both ends are needed to get (10). Using, however, $S^0=\langle -|+\rangle$, we can write for our total scattering matrix that

$$S=1+iT^0+iT^R=1+iT^0+iT'^R S^0=(1+iT'^R)S^0. \tag{11}$$

[5] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 202.
[6] R. G. Sachs, Ann. Phys. (N.Y.) **22**, 239 (1963).

T'^R is (7) with $|-\rangle$ states on both sides, the $|+\rangle \to |-\rangle$ resulting from the application of S^0. Now $1+iT'^R$ is unitary so S, the product of two unitary operators, is also unitary.

BELL-STEINBERGER CONDITION

This condition[7] is firstly a statement about the relation between the anti-Hermitian part of a general matrix and its eigenvectors and, as such, has nothing to do with unitarity. If $|n\rangle$ and $|m\rangle$ are eigenvectors of matrix $\mathfrak{M}$ with eigenvalues E_n and E_m, it follows that

$$\langle n | \mathfrak{M}-\mathfrak{M}^\dagger | m\rangle = (E_m - E_n^*)\langle n | m\rangle. \qquad (12)$$

Unitarity now comes in since it has to do with the anti-Hermitian part of our mass matrix, as explained above. Now substituting

$$\tfrac{1}{2}[\mathfrak{M}-\mathfrak{M}^\dagger]_{\alpha\beta} = -i\pi\delta(E-E(k)) \times \langle\alpha | V | k, j+\rangle\langle +k, j | V | \beta\rangle \qquad (13)$$

for the left-hand side of Eq. (12), we get the Bell-Steinberger conditions. Note that for $n=m$, (12) gives $\mathrm{Im}E_n$ in terms of the matrix elements of V.

$\mathfrak{M}$ SYMMETRIC, T INVARIANCE

So far we have used no symmetry properties of H (except, of course, that it is Hermitian). In general, there may be further simplifications; in the K^0 problem with CPT invariance, for example, the two diagonal elements of $\mathfrak{M}$ are equal. An important case is when we have T (time-reversal) invariance. Then $\langle\alpha | V | k, j+\rangle$ has the phase of the U-induced scattering in the channel j (taken to be an eigenchannel of the U scattering), and $M_{\alpha\beta}$ and $\Gamma_{\alpha\beta}$ are real and symmetric, and so $\mathfrak{M}$ is symmetric. Of course, in problems with atoms in *external* magnetic fields this may not be true, even though T is not violated. If $\mathfrak{M}$ is symmetric, the general formalism with the g_{nm} is too complicated, since it can be shown directly that the eigenvectors of a symmetric matrix are "orthogonal" in the sense $\tilde{U}(n)U(m)=\delta_{nm}$ (normalizing appropriately), where tilde means transpose.

Now going back to quantum-mechanical notation, suppose that we have an eigenstate of $\mathfrak{M}$ expressed in terms of the discrete level subspace $|m\rangle = a|\alpha\rangle + b|\beta\rangle + \cdots$. The transpose vector $\tilde{U}(m)$ that we want is then $\langle\alpha | a + \langle\beta | b + \cdots$, that is, the vector $\langle Tm|$, where T is the usual time-reversal operator[8] and where we have chosen $|\alpha\rangle$, $|\beta\rangle$ to be eigenstates of T. Now we have $I = |m\rangle\langle Tm|$, so (7) becomes

$$T_{fi}^R(E) = \frac{\langle -f, k_f | V | m\rangle\langle Tm | V | k_i i, +\rangle}{E_m - E}.$$

Now using $|T+\rangle = |-\rangle$, the nature of T operation,[8] T invariance for V, and finally the Hermiticity of V, we get

$$T_{fi}^R(E) = \frac{\langle -f, k_f | V | m\rangle\langle -i, k_i | V | m\rangle}{E_m - E}. \qquad (14)$$

Here, time-reversal symmetry $T_{fi}=T_{if}$ is explicit. Note again that the numerator, even without final-state interactions, does not have the form suggested in Ref. 4. Because of the special character of the intermediate states $|m\rangle$, essentially the imaginary parts of the $\langle f | V | m\rangle$ are correlated, so that unitarity is fulfilled without the numerator being a projection operator. Attention should be paid to the fact that although with T invariance the $\langle f | V | \alpha\rangle$-type matrix elements are real, the $\langle f | V | m\rangle$ are not, because of the mixing effects.

TIME DEPENDENCE

To assure ourselves that these results correspond to nothing unusual from the time-dependent point of view, let us see briefly how to arrive at the analog of the conventional results. We start from a vector, $|\gamma\rangle$, lying entirely in the discrete "particle" subspace. This represents the state at $t=0$. After a time t, it becomes $\exp(-iHt)|\gamma\rangle$, and we want the probability amplitude for a component, say, $|\delta\rangle$, remaining in the discrete subspace, namely, $\langle\delta | \exp(-iHt) | \gamma\rangle$. Expanding in terms of the $\psi_i{}^\dagger$, we have

$$\langle\delta | \exp(-iHt) | \gamma\rangle = \langle\delta | \psi_i{}^\dagger\rangle \exp(-iE_it)\langle\psi_i{}^\dagger | \gamma\rangle, \qquad (15)$$

where E is the energy of the state i.

Now from the generalization of Eq. (1) to the several-resonance case, we have

$$\langle\delta | \psi_i{}^\dagger\rangle = \varphi_{i,\delta},$$

and from the generalization of Eq. (4), we have

$$\varphi_{i,\delta} = [1/(\mathfrak{M}-E)]_{\delta\alpha}\langle\alpha | V | i\rangle. \qquad (16)$$

Equation (15) now becomes, using the Hermiticity of V,

$$\exp(-iE_it)\{[1/(\mathfrak{M}-E_i)]V | i\rangle\langle i | V[1/(\mathfrak{M}^\dagger - E)]\}_{\delta\gamma}. \qquad (17)$$

Now the continuum state $|i\rangle$ in the sum has the energy E_i, so that $V|i\rangle\langle i|V$ is just, from Eq. (10), the anti-Hermitian part of the mass matrix at energy E. Then with

$$\mathfrak{M}-\mathfrak{M}^\dagger = (\mathfrak{M}-E) - (\mathfrak{M}^\dagger - E),$$

we have

$$[e^{-iEt}]/2i\pi \times \{[1/(\mathfrak{M}-E)] - [1/(\mathfrak{M}\pm E)]\}_{\delta\gamma}. \qquad (18)$$

Equation (18) corresponds to the conventional result, since with the assumption that E can be taken to run from $-\infty$ to $+\infty$ and the assumption that $\mathfrak{M}(E)$

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[7] J. S. Bell and J. Steinberger, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford Laboratory, Chilton, Berkshire, England, 1966).

[8] G. C. Wick, Ann. Rev. Nucl. Sci. **8** (1958), Eq. 53.

is approximately constant, the $1/(\mathcal{M}-E)$ term gives an exponential in the Fourier transform while the $\mathcal{M}^+$ term gives zero. We have then the anticipated time dependence. In terms of the $|n\rangle$ eigenvectors of $\mathcal{M}$, so that $|\gamma\rangle = C_n |n\rangle$, we have, for the evolution of the state vector in the discrete "particle" subspace,

$$\exp(-iHt)\,|\gamma\rangle \rightarrow \exp(-iE_n t)\,C_n\,|n\rangle. \tag{19}$$

DEGENERACY OF EIGENVECTORS, NONEXPONENTIAL DECAY

So far we have assumed that eigenvectors of $\mathcal{M}$ span the space of $\mathcal{M}$, i.e., if $\mathcal{M}$ is $n\times n$ then there are n linearly independent eigenvectors. Now it can happen, as we vary some parameters of the system, that two eigenvectors come closer and closer together and in the limit merge into one or, more generally, the number of eigenvectors becomes less than the dimension of the space. This can occur because of the special kind of "orthogonality" used between different eigenvectors. Note that even for the case of $\mathcal{M}$ symmetric, the vectors

$$\begin{pmatrix} 1 \\ i+\delta \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1/(i+\delta) \end{pmatrix}$$

obey $\tilde{U}_1 U_2 = 0$, although in the limit $\delta \rightarrow 0$ they become identical.

This kind of behavior is not necessarily related to the invertability of $\mathcal{M}$, incidentally. Observe that even if $\mathcal{M}$ does become singular, meaning that there is a vector v such that $\mathcal{M}v=0$, $\mathcal{M}-E$ will generally have an inverse, which is sufficient for the operations used above. This corresponds then to a stable state with a purely real eigenvalue ($E=0$), but (12) and (13) with $n=m$ tells us that states with real eigenvalues are, naturally, not coupled to the continuum. Thus the singularity of $\mathcal{M}$ or, more generally, the existence of real eigenvalues (for then a simple change of energy scale make $\mathcal{M}$ singular) is related to the question of the existence of stable states.

The degeneration of *eigenvectors*, however, is related to the "dipole" or nonexponential decay phenomena[9] in which the scattering amplitude has a double-humped structure as a function of energy and, correspondingly, the system has a nonexponential decay in time.

Consider two eigenvectors which are very close together, but not exactly degenerate—that simply being a mathematical "point" without physical realizability. Being nearly "parallel," they have almost exactly equal eigenvalues also. Now consider the system placed in a state which has components outside of the limiting degenerate vector which the two vectors approach. These components must be resolved in terms of the two almost degenerate vectors, and the decay of the system in time will involve the difference of two almost equal exponentials, which for a finite time acts like a linear term in t times the exponential; as the limit is approached, this time becomes longer and longer. Thus physically, in effect, we always have two states with ordinary exponential decays but, as they become parallel, there is only one "direction" which decays as a amplitude as they have.

Note that the term dipole "state" is therefore something of a misnomer, since there is actually no state in a direction outside of that of the degenerate vector which remains invariant as t increases even in the limit. If we start the system with a vector in a general direction, it is continually rotated and projected on the degenerate vector as time goes on.

From the point of view of the scattering amplitude, our Eq. (14) with two resonances can be manipulated into the form of Eq. (1) of Lassila and Ruuskanen,[9] with appropriate changes in notation, so we arrive at the same description of the energy dependence of the amplitude as they have.

Finally, we note that if two eigenvalues were to become the same while the eigenvectors remained distinct, then any vector in the corresponding subspace would be an eigenvector and then $\mathcal{M}$ is a multiple of the identity there. Thus, in terms of the $\mathcal{M}$ matrix, a criterion for the effect is a degeneracy of eigenvalues while the corresponding part of $\mathcal{M}$ is not proportional to the identity matrix.

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[9] For a recent discussion and a report of experimental work on the effect in atoms, see J. Dupont-Roc, Thèse de 3e Cycle, Laboratorie de l'Ecole Normale Supérieure, Paris (unpublished); J. Dupont-Roc, N. Polonsky, and C. Cohen-Tannoudji, Compt. Rend. **266**, 613 (1968). A general formula for scattering through two atomic levels in an external field is given by K. E. Lassila and V. Ruuskanen, Phys. Rev. Letters **17**, 490 (1966); see also M. L. Goldberger and K. M. Watson, Phys. Rev. **136**, B1472 (1964); *Collision Theory* (Wiley, New York, 1964), Ch. 8; J. S. Bell and C. J. Goebel, Phys. Rev. **138**, B1198 (1965); H. Osborn, *ibid.* **145**, 1272 (1966); L. Mower, *ibid.* **142**, 799 (1966).