Now using

$$P_l(z) \geqslant P_{l'}(z) \quad \text{if } l \geqslant l' \quad \text{and} \quad z \geqslant 1,$$

we see that the second term on the left-hand side of (B9) is positive. Let us therefore concentrate on the first term on the left-hand side of (B9). Now

$$\sum_{l=0}^{[K']} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_K(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$= \sum_{l=0}^{[K']} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_K(z)-P_{K'}(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$+ \sum_{l=0}^{[K']} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_{K'}(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$= [P_K(z)-P_{K'}(z)] \sum_{l=0}^{[K']} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_m{}^2(z)-P_l(z)P_m(z)],$$

and this is easily seen to be positive by using $P_K(z) \geqslant P_{K'}(z)$ and Schwartz's inequality. This proves Lemma 2

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Applications of Current Commutation Relations to Muon Capture and Neutrino (Antineutrino) Reactions in Nuclei

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Relations among total muon-capture rates in nuclei and the equal-time commutators of the space and time components of the strangeness-conserving weak hadron current are derived. Using the quark field algebra and the closure approximation, this relation yields a total muon-capture rate in He^3 of $\Gamma(He^3) \approx 2.36 \times 10^3$ sec^{-1}, in very good agreement with experiment. We demonstrate that application of the gauge field algebra to our relations does not yield a result that can be compared with experiment, since we cannot justify the use of the closure approximation in the context of this algebra. Using the quark field algebra and the closure approximation, similar relations are also derived for the total "elastic" differential cross section for forward scattering of neutrinos off nuclei.

I. INTRODUCTION

In a recent series of Letters,[1–3] we have discussed the application of the Gell-Mann algebra of currents[4] to the calculation of the total muon-capture rate in complex nuclei as well as the derivation of relations between cross sections for elastic neutrino and antineutrino scattering by nuclei.

The purpose of the present paper is twofold:

(i) To reproduce in all detail the derivation of the results quoted in Refs. 1–3. This was not done in our previous brief communications and is essential for the proper understanding of our results.

(ii) To investigate how results are modified when the quark field algebra[4] is replaced by the gauge field algebra.[5]

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[1] C. W. Kim and M. Ram, Phys. Rev. Letters **18**, 508 (1967).

[2] C. W. Kim and M. Ram, Phys. Rev. Letters **20**, 35 (1968).

[3] C. W. Kim and M. Ram, Phys. Letters **27B**, 306 (1968).

[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967); T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

Section II is a brief review of the current-current formalism[6] commonly used in weak-interaction calculations. In Sec. III we reproduce in detail the muon-capture calculation[1,7] in which differences of *total* muon-capture rates within nuclear isospin multiplets are related to current commutators. The quark[4] and gauge field[5] algebras are very briefly reviewed in Sec. IV. In Sec. V both algebras are separately applied to the relation for muon capture derived in Sec. III and the results compared with experiment. The relations between cross sections for elastic scattering of neutrinos and antineutrinos from nuclei are reviewed in Sec. VI.

Throughout this paper we use natural units ($\hbar=c=1$) and a Minkowski metric $x=(\mathbf{x},it)$ for space-time coordinates and $p=(\mathbf{p},iE)$ for four-momenta. Greek subscripts can assume the values 1, 2, 3, and 4 while Latin ones are restricted to 1, 2, and 3. Summation over repeated indices is always implied.

II. CURRENT-CURRENT WEAK-INTERACTION HAMILTONIAN FOR STRANGENESS-CONSERVING PROCESSES

It is well known[6] that leptonic and semileptonic strangeness-conserving weak interactions can be described by the following "phenomenological" interaction Hamiltonian:

$$\mathcal{H}_w(x)=(-G/\sqrt{2})\mathcal{J}_\alpha^\star(x)\mathcal{J}_\alpha(x)\,, \tag{1}$$

where

$$\begin{aligned}\mathcal{J}_\alpha^\star(x)&=\mathcal{J}_\alpha^\dagger(x)\,, \qquad \alpha=1,2,3\\ &=-\mathcal{J}_\alpha^\dagger(x)\,, \quad \alpha=4\end{aligned} \tag{2}$$

and $G=(1.02/m_p^2)\times10^{-5}$ is the μ-decay coupling constant (m_p is the proton mass). Within the framework of this theory it is generally conjectured that

$$\mathcal{J}_\alpha(x)=J_\alpha^{(-)}(x)+j_\alpha(x)\,, \tag{3}$$

where $J_\alpha^{(-)}(x)$ is the strangeness-conserving weak hadronic current and[8]

$$j_\alpha(x)=i\sum_{l=e,\mu}\bar{\psi}_l\gamma_\alpha(1+\gamma_5)\psi_{\nu l} \tag{4}$$

is the weak leptonic current. The hadron current $J_\alpha^{(-)}(x)$ is usually separated into vector and axial-vector parts as follows[9]:

$$J_\alpha^{(-)}=\cos\theta_C\,[V_\alpha^{(-)}(x)+A_\alpha^{(-)}(x)]\,, \tag{5}$$

where

$$V_\alpha^{(-)}(x)=V_\alpha^{(1)}(x)-iV_\alpha^{(2)}(x)\,, \tag{6a}$$

$$A_\alpha^{(-)}(x)=A_\alpha^{(1)}(x)-iA_\alpha^{(2)}(x)\,, \tag{6b}$$

and the $V_\alpha^{(i)}(x)$, $i=1$, 2, 3, are identified with the components of the isotopic-spin current (conserved-vector-current hypothesis[10]). The $A_\alpha^{(i)}(x)$ are assumed to transform like the components of the isotopic-spin current. The interaction Hamiltonian (1) is "phenomenological" in the sense that it is nonrenormalizable and therefore only useful in calculating transition amplitudes to order G.

III. RELATION FOR TOTAL MUON-CAPTURE RATES

Consider the muon-capture reaction

$$\mu^-+N_a\to N_b+\nu_\mu\,, \tag{7}$$

where N_a and N_b represent initial and final nuclear states, respectively. Neglecting the muon momentum and binding energy, it is easy to show that in the rest frame of nucleus N_a, the neutrino energy is given by

$$\nu=|\mathbf{v}|\cong\frac{(m_\mu+m_a)^2-m_b^2}{2(m_\mu+m_a)}\,, \tag{8}$$

where $\mathbf{v}$ is the neutrino momentum and m_μ, m_a, and m_b are the masses of μ, N_a, and N_b, respectively. In practice, reaction (7) is dominated by low-lying nuclear levels N_b for which

$$m_b-m_a\ll m_\mu\,. \tag{9}$$

Since also m_a, $m_b\gg m_\mu$, Eqs. (8) and (9) imply that for the dominant transitions

$$\nu\cong m_\mu\,, \tag{10}$$

i.e., the neutrino absorbs most of the energy released by the disappearing muon. From Eq. (1), the muon-capture rate is readily found to be

$$\begin{aligned}\Gamma(N_a\to N_b)\cong&\frac{G^2}{2\pi^2}\left[\frac{Z(N_a)}{a_B}\right]^3\frac{C(N_a)}{2J_a+1}\\ &\times\sum_{M_a=-J_a}^{J_a}\sum_{M_b=-J_b}^{J_b}\nu^2\left(1-\frac{\nu}{m_\mu+m_a}\right)\\ &\times\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\mathfrak{N}_{\sigma\lambda}(N_a\to N_b)\,,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathcal{L}_{\lambda\sigma}=(1/m_\mu\nu)[&(p_\mu)_\lambda(p_\nu)_\sigma+(p_\mu)_\sigma(p_\nu)_\lambda-(p_\mu\cdot p_\nu)\delta_{\lambda\sigma}\\ &+\epsilon_{\lambda\sigma\alpha\beta}(p_\nu)_\alpha(p_\mu)_\beta]\,,\end{aligned} \tag{12}$$

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[6] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

[7] H. Primakoff, in *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland, Amsterdam, 1967), pp. 409–445.

[8] $\psi_l(x)$ ($l=e$ or μ) is the lepton (electron or muon) field and $\psi_{\nu l}(x)$ is the corresponding neutrino field. We adopt the following Hermitian representation for the Dirac matrices:

$$\gamma_j=-i\begin{pmatrix}0&\sigma_j\\-\sigma_j&0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma_5=-\begin{pmatrix}0&I\\I&0\end{pmatrix},$$

where σ_j are the 2×2 Pauli matrices and I is the 2×2 unit matrix.

[9] The angle θ_C is the Cabibbo angle (see N. Cabibbo, Ref. 6). We assume that $\cos\theta_C\simeq0.98$.

[10] R. P. Feynman and M. Gell-Mann, Ref. 6; S. S. Gershtein and J. B. Zeldovich, Zh. Eksperim. i Teor. Fiz. **29**, 698 (1955) [Soviet Phys. JETP **2**, 576 (1955)].

$$\mathfrak{N}_{\sigma\lambda}(N_a\rightarrow N_b) =\Omega^2\langle N_a;\mathbf{k}_a=0,M_a|J_\sigma^{(+)}(0)|N_b;\mathbf{k}_b=-\boldsymbol{\nu},M_b\rangle \times\langle N_b;\mathbf{k}_b=-\boldsymbol{\nu},M_b|J_\lambda^{(-)}(0)|N_a;\mathbf{k}_a=0,M_a\rangle, \quad (13)$$

$p_\mu=(\mathbf{p}_\mu=0,\,im_\mu)=$four-momentum of muon,

$p_\nu=(\boldsymbol{\nu},i\nu)=$four-momentum of neutrino,

$k_a=(\mathbf{k}_a=0,\,im_a)=$four-momentum of N_a,

$k_b=(\mathbf{k}_b,iE_b)=$four-momentum of N_b.

The nuclei N_a and N_b have spins J_a and J_b, and the summations in Eq. (11) extend over the third components of the nuclear spins M_a and M_b (since $\mathbf{k}_b\neq 0$, M_b refers really to the helicity of N_b). The number of protons in the N_a nucleus is $Z(N_a)$, and

$$a_B=\frac{1}{\alpha}\left(\frac{m_\mu+m_a}{m_\mu m_a}\right)$$

is the Bohr radius of the muon. $\alpha\sim 1/137$ is the fine-structure constant. Also we have

$$J_\sigma^{(+)}(x)=J_\sigma^{(-)\star}(x)=\cos\theta_C\left[V_\sigma^{(+)}(x)+A_\sigma^{(+)}(x)\right], \quad (14)$$

where

$$V_\sigma^{(+)}(x)=V_\sigma^{(1)}(x)+iV_\sigma^{(2)}(x), \quad (15a)$$

$$A_\sigma^{(+)}(x)=A_\sigma^{(1)}(x)+iA_\sigma^{(2)}(x). \quad (15b)$$

The integration in Eq. (11) is over the neutrino solid angle, and $C(N_a)$ is a correction factor arising from the nonpoint character of the charge distribution of N_a. We have chosen our wave functions to be normalized in a box of volume Ω. Note that since we are neglecting the muon momentum and binding energy and working in a reference frame in which nucleus N_a is at rest,

$$\mathbf{k}_b=-\boldsymbol{\nu} \quad (16a)$$

and

$$E_b+\nu=m_a+m_\mu. \quad (16b)$$

The total capture rate $\Gamma(N_a)$ is obtained by summing Eq. (11) over all possible final hadronic states N_b, subject to the constraint $E_b\leq m_\mu+m_a$, i.e.,

$$\Gamma(N_a)=\sum_{N_b(E_b\leqslant m_\mu+m_a)}\Gamma(N_a\rightarrow N_b). \quad (17)$$

The summation in Eq. (17) is over all independent quantum numbers distinguishing N_b (aside from $\mathbf{k}_b$ and M_b). Introducing

$$Q_\alpha^{(\pm)}(\boldsymbol{\nu})=\int J_\alpha^{(\pm)}(\mathbf{x},t=0)e^{\pm i\boldsymbol{\nu}\cdot\mathbf{x}}d\mathbf{x}, \quad (18)$$

one can readily show that

$$\Gamma(N_a)\cong\frac{G^2}{2\pi^2}\left[\frac{Z(N_a)}{a_B}\right]^3\frac{C(N_a)}{2J_a+1}\sum_{M_a}\int\frac{d\Omega_\nu}{4\pi}\sum_{N_b(E_b\leqslant m_\mu+m_a)}\sum_{M_b}\sum_{\mathbf{k}_b}\nu^2\left(1-\frac{\nu}{m_\mu+m_a}\right) \times\mathfrak{L}_{\lambda\sigma}\langle N_a;\mathbf{k}_a=0,M_a|Q_\sigma^{(+)}(\boldsymbol{\nu})|N_b;\mathbf{k}_b,M_b\rangle\langle N_b;\mathbf{k}_b,M_b|Q_\lambda^{(-)}(\boldsymbol{\nu})|N_a;\mathbf{k}_a=0,M_a\rangle, \quad (19)$$

where $\mathbf{k}_b$ is no longer restricted by Eq. (16a), and use has been made of the periodic boundary condition

$$\frac{1}{\Omega}\int e^{-i(\mathbf{k}_a-\mathbf{k}_b-\nu)\cdot\mathbf{x}}d\mathbf{x}=\delta_{\mathbf{k}_a,\mathbf{k}_b+\nu}. \quad (20)$$

In the spirit of the closure approximation,[11] we can rewrite Eq. (19) as follows:

$$\Gamma(N_a)=\frac{G^2}{2\pi^2}\left[\frac{Z(N_a)}{a_B}\right]^3\frac{C(N_a)}{2J_a+1}\sum_{M_a}\int\frac{d\Omega_\nu}{4\pi}\mathfrak{L}_{\lambda\sigma}\langle\nu^2[1-\nu/(m_\mu+m_a)]\rangle_a \times\sum_{N_b(E_b\leqslant m_\mu+m_a)}\sum_{\mathbf{k}_b}\sum_{M_b}\langle N_a;\mathbf{k}_a=0,M_a|Q_\sigma^{(+)}(\langle\boldsymbol{\nu}\rangle_a)|N_b;\mathbf{k}_b,M_b\rangle\langle N_b;\mathbf{k}_b,M_b|Q_\lambda^{(-)}(\langle\boldsymbol{\nu}\rangle_a)|N_a;\mathbf{k}_a=0,M_a\rangle, \quad (21)$$

where $\langle\cdots\rangle_a$ denotes an appropriate weighted average over possible final states N_b. Defining

$$D(N_a)=\frac{G^2}{2\pi^2}\left[\frac{Z(N_a)}{a_B}\right]^3 C(N_a)\langle\nu^2[1-\nu/(m_\mu+m_a)]\rangle_a, \quad (22)$$

Eq. (21) becomes

$$\frac{\Gamma(N_a)}{D(N_a)}\cong\frac{1}{2J_a+1}\sum_{M_a}\int\frac{d\Omega_\nu}{4\pi}\mathfrak{L}_{\lambda\sigma}\sum_{N_b(E_b\leqslant m_\mu+m_a)}\sum_{\mathbf{k}_b}\sum_{M_b}\langle N_a;\mathbf{k}_a=0,M_a|Q_\sigma^{(+)}(\langle\boldsymbol{\nu}\rangle_a)|N_b;\mathbf{k}_b,M_b\rangle \times\langle N_b;\mathbf{k}_b,M_b|Q_\lambda^{(-)}(\langle\boldsymbol{\nu}\rangle_a)|N_a;\mathbf{k}_a=0,M_a\rangle. \quad (23)$$

[11] H. Primakoff, Rev. Mod. Phys. **31**, 802 (1959).

Consider now the nuclear isotopic-spin doublet (He^3,H^3). Applying Eq. (23) to these nuclei, we have

$$\frac{\Gamma(He^3)}{D(He^3)} \cong \frac{1}{2}\sum_{M_{He^3}}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\sum_{N_b(E_b\leqslant m_\mu+m_{He^3})}\sum_{\mathbf{k}_b}\sum_{M_b}\langle He^3;\mathbf{k}_{He^3}=0,M_{He^3}|Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{He^3})|N_b;\mathbf{k}_b,M_b\rangle \times\langle N_b;\mathbf{k}_b,M_b|Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{He^3})|He^3;\mathbf{k}_{He^3}=0,M_{He^3}\rangle \quad (24)$$

and

$$\frac{\Gamma(H^3)}{D(H^3)} \cong \frac{1}{2}\sum_{M_{H^3}}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\sum_{N_c(E_c\leqslant m_\mu+m_{H^3})}\sum_{\mathbf{k}_c}\sum_{M_c}\langle H^3;\mathbf{k}_{H^3}=0,M_{H^3}|Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{H^3})|N_c;\mathbf{k}_c,M_c\rangle \times\langle N_c;\mathbf{k}_c,M_c|Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{H^3})|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle$$

$$=\frac{1}{2}\sum_{M_{H^3}}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\{[\sum_{N_c(E_c\leqslant m_\mu+m_{H^3})}\sum_{\mathbf{k}_c}\sum_{M_c}\langle H^3;\mathbf{k}_{H^3}=0,M_{H^3}|Q_\sigma^{(+)}(-\langle\mathbf{v}\rangle_{He^3})|N_c;\mathbf{k}_c,M_c\rangle \times\langle N_c;\mathbf{k}_c,M_c|Q_\lambda^{(-)}(-\langle\mathbf{v}\rangle_{He^3})|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle]+\Delta_{\sigma\lambda}^{(1)}\}, \quad (25)$$

where

$$\Delta_{\sigma\lambda}^{(1)}=\sum_{N_c}\langle H^3;\mathbf{k}_{H^3}=0,M_{H^3}|Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{H^3})|N_c;\mathbf{k}_c,M_c\rangle\langle N_c;\mathbf{k}_c,M_c|Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{H^3})|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle -\langle H^3;\mathbf{k}_{H^3}=0,M_{H^3}|Q_\sigma^{(+)}(-\langle\mathbf{v}\rangle_{He^3})|N_c;\mathbf{k}_c,M_c\rangle\langle N_c;\mathbf{k}_c,M_c|Q_\lambda^{(-)}(-\langle\mathbf{v}\rangle_{He^3})|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle. \quad (26)$$

Note that $\langle\mathbf{v}\rangle_{He^3}$ and $\langle\mathbf{v}\rangle_{H^3}$ are slightly different since the accessible final states in μ capture by He^3 and H^3 are not the same.

Using the closure relation

$$\sum_{N_b}\sum_{\mathbf{k}_b}\sum_{M_b}|N_b;\mathbf{k}_b,M_b\rangle\langle N_b;\mathbf{k}_b,M_b|=1, \quad (27)$$

we can write Eqs. (24) and (25) as follows:

$$\frac{\Gamma(He^3)}{D(He^3)} \cong \frac{1}{2}\sum_{M_{He^3}}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}[\langle He^3;\mathbf{k}_{He^3}=0,M_{He^3}|Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{He^3}) \times Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{He^3})|He^3;\mathbf{k}_{He^3}=0,M_{He^3}\rangle-\Delta_{\sigma\lambda}^{(2)}(He^3)] \quad (28)$$

and

$$\frac{\Gamma(H^3)}{D(H^3)} \cong \frac{1}{2}\sum_{M_{H^3}}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}[\langle H^3;\mathbf{k}_{H^3}=0,M_{H^3}|Q_\sigma^{(+)}(-\langle\mathbf{v}\rangle_{He^3})Q_\lambda^{(-)}(-\langle\mathbf{v}\rangle_{He^3})|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle +\Delta_{\sigma\lambda}^{(1)}-\Delta_{\sigma\lambda}^{(2)}(H^3)], \quad (29)$$

where

$$\Delta_{\sigma\lambda}^{(2)}(N_a)=\sum_{N_b(E_b\leqslant m_\mu+m_a)}\sum_{\mathbf{k}_b}\sum_{M_b}\langle N_a;\mathbf{k}_a=0,M_a|Q_\sigma^{(+)}(\pm\langle\mathbf{v}\rangle_{He^3})|N_b;\mathbf{k}_b,M_b\rangle \times\langle N_b;\mathbf{k}_b,M_b|Q_\lambda^{(-)}(\pm\langle\mathbf{v}\rangle_{He^3})|N_a;\mathbf{k}_a=0,M_a\rangle \quad (N_a=He^3 \text{ or } H^3). \quad (30)$$

In Eq. (30), $+\langle\mathbf{v}\rangle_{He^3}$ and $-\langle\mathbf{v}\rangle_{He^3}$ correspond to $N_a=He^3$ and H^3, respectively.

Applying the relations

$$e^{i\pi I^{(1)}}|He^3;\mathbf{k}_{He^3}=0,M_{He^3}\rangle=|H^3;\mathbf{k}_{H^3}=0,M_{H^3}\rangle, \quad (31a)$$

$$e^{i\pi I^{(1)}}\mathcal{J}_\alpha^{(\pm)}(x)e^{-i\pi I^{(1)}}=\mathcal{J}_\alpha^{(\mp)}(x), \quad (31b)$$

and

$$e^{i\pi I^{(1)}}Q_\alpha^{(\pm)}(\mathbf{v})e^{-i\pi I^{(1)}}=Q_\alpha^{(\mp)}(-\mathbf{v}), \quad (31c)$$

where $\mathbf{I}=(I^{(1)},I^{(2)},I^{(3)})$ is the total isotopic-spin operator, one can readily show that

$$\frac{\Gamma(H^3)}{D(H^3)} \cong \frac{1}{2}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\{\sum_{M_{He^3}}\langle He^3;\mathbf{k}_{He^3}=0,M_{He^3}|Q_\sigma^{(-)}(\langle\mathbf{v}\rangle_{He^3})Q_\lambda^{(+)}(\langle\mathbf{v}\rangle_{He^3})|He^3;\mathbf{k}_{He^3}=0,M_{He^3}\rangle +\sum_{M_{H^3}}[\Delta_{\sigma\lambda}^{(1)}-\Delta_{\sigma\lambda}^{(2)}(H^3)]\}. \quad (32)$$

Since we have set $\mathbf{p}_\mu=0$ and are summing over both He^3 polarizations, it is easy to see that only the symmetric part of $\mathcal{L}_{\lambda\sigma}$ contributes to the first term in the curly brackets of Eq. (32). We can therefore write

$$\frac{\Gamma(\mathrm{H}^3)}{D(\mathrm{H}^3)}\cong\tfrac{1}{2}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\Big\{\sum_{M_{\mathrm{He}^3}}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|Q_\lambda^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})Q_\sigma^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle+\sum_{M_{\mathrm{H}^3}}[\Delta_{\sigma\lambda}^{(1)}-\Delta_{\sigma\lambda}^{(2)}(\mathrm{H}^3)]\Big\}. \quad (33)$$

Combining Eqs. (28) and (33), we find

$$\frac{\Gamma(\mathrm{He}^3)}{D(\mathrm{He}^3)}-\frac{\Gamma(\mathrm{H}^3)}{D(\mathrm{H}^3)}\cong\tfrac{1}{2}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\Big\{\sum_{M_{\mathrm{He}^3}}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|[Q_\sigma^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}),Q_\lambda^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})]\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle-\Delta_{\sigma\lambda}-\sum_{M_{\mathrm{H}^3}}\Delta_{\sigma\lambda}^{(1)}\Big\}, \quad (34)$$

where

$$\Delta_{\sigma\lambda}=\sum_{M_{\mathrm{He}^3}}\Delta_{\sigma\lambda}^{(2)}(\mathrm{He}^3)-\sum_{M_{\mathrm{H}^3}}\Delta_{\sigma\lambda}^{(2)}(\mathrm{H}^3). \quad (35)$$

The quantity $\Delta_{\sigma\lambda}^{(1)}$ depends on the difference of the weak form factors evaluated at values of the average four-momentum transfer squared

$$\langle q^2\rangle_{\mathrm{H}^3}=\{-(m_c-m_a)^2+2m_a[m_c^2+\langle\nu\rangle_{\mathrm{H}^3}{}^2]^{1/2}\}$$

and

$$\langle q^2\rangle_{\mathrm{He}^3}=\{-(m_c-m_a)^2+2m_a[m_c^2+\langle\nu\rangle_{\mathrm{He}^3}{}^2]^{1/2}\}.$$

Since

$$|\langle q^2\rangle_{\mathrm{He}^3}-\langle q^2\rangle_{\mathrm{H}^3}|\cong 2m_\mu|\bar{m}_b-\bar{m}_c|\ll m_\mu^2,$$

$\Delta_{\sigma\lambda}^{(1)}$ can be estimated as being no more than 1%,[7] and we shall therefore neglect it. After this approximation and substitution of Eq. (12) into Eq. (34), we finally obtain

$$\frac{\Gamma(\mathrm{He}^3)}{D(\mathrm{He}^3)}-\frac{\Gamma(\mathrm{H}^3)}{D(H^3)}\cong\tfrac{1}{2}\sum_{M_{\mathrm{He}^3}}\int\frac{d\Omega_\nu}{4\pi}\{g_{\sigma\lambda}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|[Q_\lambda^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}),Q_\sigma^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})]|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle$$
$$+i(\hat{\nu})_l\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|([Q_l^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}),Q_4^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})]+[Q_4^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}),Q_l^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})])\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle-\Delta\}, \quad (36)$$

where

$$\Delta=\tfrac{1}{2}\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\Delta_{\sigma\lambda} \quad (37)$$

and $\hat{\nu}=\boldsymbol{\nu}/\nu$.

Relation (36) can be readily generalized to the case of an arbitrary nucleus $N(A,I,I^{(3)})$ of mass number A, isotopic spin I, and third component of isotopic spin $I^{(3)}>0$. The result is

$$\frac{\Gamma(N(A,I,I^{(3)}))}{D(N(A,I,I^{(3)}))}-\frac{\Gamma(N(A,I,-I^{(3)}))}{D(N(A,I,-I^{(3)}))}\cong\frac{1}{2J+1}\sum_M\int\frac{d\Omega_\nu}{4\pi}\{g_{\lambda\sigma}\langle N(A,I,I^{(3)});\mathbf{k}=0,M|[Q_\lambda^{(+)}(\langle\boldsymbol{\nu}\rangle),Q_\sigma^{(-)}(\langle\boldsymbol{\nu}\rangle)]$$
$$\times|N(A,I,I^{(3)});\mathbf{k}=0,M\rangle+i(\hat{\nu})_l\langle N(A,I,I^{(3)});\mathbf{k}=0,M|([Q_l^{(+)}(\langle\boldsymbol{\nu}\rangle),Q_4^{(-)}(\langle\boldsymbol{\nu}\rangle)]$$
$$+[Q_4^{(+)}(\langle\boldsymbol{\nu}\rangle),Q_l^{(-)}(\langle\boldsymbol{\nu}\rangle)])|N(A,I,I^{(3)});\mathbf{k}=0,M\rangle\}-\Delta, \quad (38)$$

where $D(N(A,I,I^{(3)}))$ is given by Eq. (22) with N_a replaced by $N(A,I,I^{(3)})$, and $\mathbf{k}$, J, and M are the momentum, spin, and third component of spin, respectively, of $N(A,I,I^{(3)})$. Also, Δ in Eq. (38) is given by Eqs. (30), (35), and (37) with He^3 replaced by $N(A,I,I^{(3)})$ and H^3 replaced by $N(A,I,-I^{(3)})$.

IV. WEAK HADRON-CURRENT EQUAL-TIME COMMUTATORS

To evaluate the right-hand side of Eq. (38), we have to know the equal-time commutators of the Fourier transforms of the weak hadron currents. Two field-theory models for determining these commutators have been widely discussed in the literature. They are (i) the quark field algebra,[4] and (ii) the gauge field algebra.[5] In this section we simply quote the well-known values of these commutators in the two models. Introducing the following Fourier transforms of the vector and axial-vector $SU(3)$ currents[12]:

[12] In the quark field model, $V_\alpha^{(j)}(\mathbf{x},t)=\frac{1}{2}iq^\dagger(\mathbf{x},t)\gamma_4\gamma_\alpha\lambda^{(j)}q(\mathbf{x},t)$ and $A_\alpha^{(j)}(\mathbf{x},t)=\frac{1}{2}iq^\dagger(\mathbf{x},t)\gamma_4\gamma_\alpha\gamma_5\lambda^{(j)}q(\mathbf{x},t)$ $(j=0,1,2,\ldots,8)$, where $q(\mathbf{x},t)$ is the quark field, $\lambda^{(0)}$ the 3×3 unit matrix, and $\lambda^{(j)}$ $(j=1,\ldots,8)$ the 3×3 $SU(3)$ matrices.

$$Q_{V,\mu}{}^{(j)}(\mathbf{k})=\int V_\mu{}^{(j)}(\mathbf{x},0)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\,, \tag{39a}$$

$$Q_{A,\mu}{}^{(j)}(\mathbf{k})=\int A_\mu{}^{(j)}(\mathbf{x},0)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \quad (j=0,1,2,\ldots,8)\,, \tag{39b}$$

$$Q_{V,\mu}{}^{(\pm)}(\mathbf{k})=\int V_\mu{}^{(\pm)}(\mathbf{x},0)e^{\pm i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\,, \tag{39c}$$

$$Q_{A,\mu}{}^{(\pm)}(\mathbf{k})=\int A_\mu{}^{(\pm)}(\mathbf{x},0)e^{\pm i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\,, \tag{39d}$$

then[4,5]

$$[Q_{V,l}{}^{(+)}(\mathbf{k}),Q_{V,m}{}^{(-)}(\mathbf{k}')]=[Q_{A,l}{}^{(+)}(\mathbf{k}),Q_{A,m}{}^{(-)}(\mathbf{k}')]$$
$$=-2i\eta\{g_{lm}Q_{V,4}{}^{(3)}(\mathbf{k}-\mathbf{k}')+\epsilon_{lmn4}[\tfrac{2}{3}Q_{A,n}{}^{(0)}(\mathbf{k}-\mathbf{k}')+(\surd\tfrac{1}{3})Q_{A,n}{}^{(8)}(\mathbf{k}-\mathbf{k}')]\}\,, \tag{40a}$$

$$[Q_{A,l}{}^{(+)}(\mathbf{k}),Q_{V,m}{}^{(-)}(\mathbf{k}')]=[Q_{V,l}{}^{(+)}(\mathbf{k}),Q_{A,m}{}^{(-)}(\mathbf{k}')]$$
$$=-2i\eta\{g_{lm}Q_{A,4}{}^{(3)}(\mathbf{k}-\mathbf{k}')+\epsilon_{lmn4}[\tfrac{2}{3}Q_{V,n}{}^{(0)}(\mathbf{k}-\mathbf{k}')+(\surd\tfrac{1}{3})Q_{V,n}{}^{(8)}(\mathbf{k}-\mathbf{k}')]\}\,, \tag{40b}$$

$$[Q_{V,4}{}^{(+)}(\mathbf{k}),Q_{V,4}{}^{(-)}(\mathbf{k}')]=[Q_{A,4}{}^{(+)}(\mathbf{k}),Q_{A,4}{}^{(-)}(\mathbf{k}')]=2iQ_{V,4}{}^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{40c}$$

$$[Q_{A,4}{}^{(+)}(\mathbf{k}),Q_{V,4}{}^{(-)}(\mathbf{k}')]=[Q_{V,4}{}^{(+)}(\mathbf{k}),Q_{A,4}{}^{(-)}(\mathbf{k}')]=2iQ_{A,4}{}^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{40d}$$

$$[Q_{V,4}{}^{(+)}(\mathbf{k}),Q_{V,l}{}^{(-)}(\mathbf{k}')]=[Q_{A,4}{}^{(+)}(\mathbf{k}),Q_{A,l}{}^{(-)}(\mathbf{k}')]$$
$$=2iQ_{V,l}{}^{(3)}(\mathbf{k}-\mathbf{k}')-2ik_l\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}S(\mathbf{x})d\mathbf{x}-2\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\partial_l S(\mathbf{x})d\mathbf{x}\,, \tag{40e}$$

$$[Q_{A,4}{}^{(+)}(\mathbf{k}),Q_{V,l}{}^{(-)}(\mathbf{k}')]=[Q_{V,4}{}^{(+)}(\mathbf{k}),Q_{A,l}{}^{(-)}(\mathbf{k}')]=2iQ_{A,l}{}^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{40f}$$

where

$$\eta=1,\quad \text{quark field model}$$
$$=0,\quad \text{gauge field model} \tag{41}$$

and the terms depending on $S(\mathbf{x})$ are the so-called Schwinger terms[13] in momentum space. In the quark filed model, $S(\mathbf{x})$ is really ambiguous and its value depends on the limiting procedure adopted in defining the currents. If the original definition adopted by Schwinger[13] is used, then $S(\mathbf{x})$ is an infinite c number. Källen[14] has demonstrated, however, that if the electromagnetic current is properly regularized, the Schwinger terms vanish in quantum electrodynamics. In the gauge field model, $S(\mathbf{x})$ is a well-defined constant, namely,[5]

$$S(\mathbf{x})=(m_\rho/g_\rho)^2\,, \tag{42}$$

where m_ρ is the physical ρ-meson mass and g_ρ the renormalized ρ-meson strong decay coupling constant.

V. APPLICATIONS

In this section we wish to apply the current commutation relations in the nonrelativistic impulse approximation and the two models of current commutators reviewed in Sec. IV to relation (36). Before proceeding, however, we wish to point out that it is to be expected that $\Gamma(\mathrm{H}^3)\ll\Gamma(\mathrm{He}^3)$ since there exist low-lying bound and partially bound states such as H^3 and H^2+n that contribute to muon capture by He^3, while only unbound states such as the three-neutron state contribute to muon capture by H^3. Muon capture by H^3 will therefore be suppressed considerably relative to capture by He^3 because of the very poor spatial overlap between initial and final hadron states. In fact, estimates by Primakoff[7,11] indicate that $\Gamma(\mathrm{H}^3)\approx0.004\Gamma(\mathrm{He}^3)$. It is therefore quite reasonable to neglect the $\Gamma(\mathrm{H}^3)$ term in relation (36) to write

$$\frac{\Gamma(\mathrm{He}^3)}{D(\mathrm{He}^3)}\cong\frac{1}{2}\sum_{M_{\mathrm{He}^3}}\int\frac{d\Omega_\nu}{4\pi}\{g_{\lambda\sigma}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|[Q_\lambda{}^{(+)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3}),Q_\sigma{}^{(-)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3})]|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle$$
$$+i(\hat{\nu})_l\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|([Q_l{}^{(+)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3}),Q_4{}^{(-)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3})]+[Q_4{}^{(+)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3}),Q_l{}^{(-)}(\langle\mathbf{v}\rangle_{\mathrm{He}^3})])$$
$$\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle\}-\Delta\,. \tag{43}$$

As a first application, we will first evaluate the right-hand side of Eq. (43) using the nonrelativistic impulse approximation.

A. Nonrelativistic Impulse Approximation

By nonrelativistic impulse approximation we mean here the approximation in which the hadronic weak currents are replaced by sums over single-nucleon weak currents, and terms of order v/c are neglected (v is the nucleon velocity). This approximation serves as the basis of many successful calculations.[11] With this approximation and

[13] T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955); J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[14] G. Källen, in *Particles Currents Symmetries*, edited by P. Urban (Springer, Berlin, 1968), pp. 268–319.

neglecting the weak magnetism and induced pseudoscalar terms,[15] we can write

$$[Q_\mu^{(+)}(\mathbf{k}),Q_\mu^{(-)}(\mathbf{k})]=2[g_A^2(1-\delta_{\mu,4})-g_V^2\delta_{\mu,4}]\sum_{n=1}^{A}\tfrac{1}{2}\tau_n^{(3)}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_n}\quad(\mu\text{ not summed over})\tag{44a}$$

and

$$[Q_l^{(+)}(\mathbf{k}),Q_4^{(-)}(\mathbf{k}')]=-2g_Vg_A\sum_{n=1}^{A}\tfrac{1}{2}i(\sigma_l)_ne^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_n}\quad(l=1,2,3),\tag{44b}$$

where $g_V=1$ and $g_A=1.23$ are the weak vector and axial-vector renormalization constants, respectively, A is the nuclear mass number, and $\mathbf{r}_n$ is the radius vector of the nth nucleon. The subscript n denotes the fact that the matrices only act on the nth nucleon. Substituting relations (44a) and (44b) into Eq. (43), we find that

$$\Gamma(\mathrm{He}^3)\cong D(\mathrm{He}^3)[2(3g_A^2+g_V^2)I^{(3)}(\mathrm{He}^3)-\Delta],\tag{45}$$

where $I^{(3)}(\mathrm{He}^3)=\frac{1}{2}$ is the third component of isospin of He^3. In the spirit of the nonrelativistic closure approximation,[11] we can neglect Δ. This approximation is entirely justified here since we are calculating rates for muon capture by nuclei and treating the weak hadronic current as a sum of single-nucleon currents (all meson exchange and baryon resonances, pair states,..., contributions are neglected in the nonrelativistic impulse approximation). As a result, contributions to Δ only come from free-nucleon intermediate states whose wave functions overlap very poorly with the wave function of the initial bound nucleon state. With this approximation, we finally obtain

$$\Gamma(\mathrm{He}^3)\cong 5.55D(\mathrm{He}^3).\tag{46}$$

Using the estimates[11]

$$C(\mathrm{He}^3)\cong 0.965,\quad |\langle\nu\rangle_{\mathrm{He}^3}|\cong 0.95m_\mu,\tag{47}$$

Eqs. (22) and (46) yield

$$\Gamma(\mathrm{He}^3)\cong 1.64\times10^3\ \mathrm{sec}^{-1}.\tag{48}$$

This result is somewhat lower than the experimental values

$$\Gamma(\mathrm{He}^3)_{\mathrm{expt}}=(2.14\pm0.18)\times10^3\ \mathrm{sec}^{-1}\quad(\text{Dubna}^{16})$$
$$=(2.17_{-0.43}^{+0.17})\times10^3\ \mathrm{sec}^{-1}\quad(\text{Berkeley}^{17}).\tag{49}$$

Better agreement with experiment is achieved if we introduce a factor[1] $R\approx1.08$ on the right-hand side of Eq. (46) to correct for neglect of the weak-magnetism and induced-pseudoscalar terms.

[15] It is well known that the weak-magnetism and induced-pseudoscalar contributions to *total* muon-capture rates are of opposite signs and practically cancel one another.

[16] J. V. Falomkin *et al.*, Phys. Letters **6**, 100 (1968).

[17] L. B. Auerbach *et al.*, Phys. Rev. **138**, B127 (1965).

B. Quark Field-Theory Model with c-Number Schwinger Terms

We shall assume that the Schwinger terms appearing in the quark field-theory current commutators are c-number functions. Substituting relations (40a)–(40f) with $\eta=1$ into Eq. (43) and using[18]

$$\sum_{M_{\mathrm{He}^3}}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|\int V_l^{(j)}(\mathbf{x},0)d\mathbf{x}\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle=0,\tag{50}$$

$$\sum_{M_{\mathrm{He}^3}}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|\int A_4^{(j)}(\mathbf{x},0)d\mathbf{x}\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle=0,\tag{51}$$

$$\sum_{M_{\mathrm{He}^3}}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|\int A_l^{(3)}(\mathbf{x},0)d\mathbf{x}\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle=0,\tag{52}$$

we find that[19]

$$\begin{aligned}\frac{\Gamma(\mathrm{He}^3)}{D(\mathrm{He}^3)}&\cong\tfrac{1}{2}\sum_{M_{\mathrm{He}^3}}\int\frac{d\Omega_\nu}{4\pi}g_{\lambda\lambda}\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|\\&\quad\times[Q_\lambda^{(+)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}),Q_\lambda^{(-)}(\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3})]\\&\quad\times|\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle\\&\quad+8|\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}|\int S(\mathbf{x})d\mathbf{x}-\Delta\\&=\tfrac{1}{2}\sum_{M_{\mathrm{He}^3}}\int\frac{d\Omega_\nu}{4\pi}16\langle\mathrm{He}^3;\mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}|\\&\quad\times I^{(3)}|\mathrm{He}^3,\ \mathbf{k}_{\mathrm{He}^3}=0,M_{\mathrm{He}^3}\rangle\\&\quad+8|\langle\boldsymbol{\nu}\rangle_{\mathrm{He}^3}|\int S(\mathbf{x})d\mathbf{x}-\Delta,\end{aligned}\tag{53}$$

[18] Equations (50) and (51) follow immediately from the transformation properties of the vector and axial-vector currents under the parity operation. The left-hand side of Eq. (52) vanishes, since no pseudovector can be constructed out of the only available vector $\mathbf{k}_{\mathrm{He}^3}=0$.

[19] We are discarding the term proportional to $\int\partial_lS(\mathbf{x})d^3x$. If $S(\mathbf{x})$ is a constant, then this term is obviously zero. If it is not a constant, we assume that $S(\mathbf{x})$ vanishes at infinity so that $\int\partial_lS(\mathbf{x})d^3x=0$ (this is trivially seen by transforming the volume integral to a surface integral using Green's theorem).

where $I^{(3)}$, the third component of the isotopic-spin operator, is given by

$$I^{(3)}=-iQ_{V,4}{}^{(3)}(0). \tag{54}$$

Therefore

$$\frac{\Gamma(\mathrm{He}^3)}{D(\mathrm{He}^3)}\cong 16I^{(3)}(\mathrm{He}^3)+8|\langle\mathbf{v}\rangle_{\mathrm{He}^3}|\int S(\mathbf{x})d\mathbf{x}-\Delta$$

$$=8+8|\langle\mathbf{v}\rangle_{\mathrm{He}^3}|\int S(\mathbf{x})d\mathbf{x}-\Delta. \tag{55}$$

Let us separate Δ into two parts as follows:

$$\Delta=\Delta_{\mathrm{conn}}+\Delta_{\mathrm{dis}}. \tag{56}$$

By definition, Δ_{dis} gets contributions only from disconnected diagrams, i.e., diagrams in which the initial nucleus (He^3 or H^3) in relation (30) does not "interact," while Δ_{conn} gets contributions only from connected diagrams. In the Appendix we show that

$$\Delta_{\mathrm{dis}}=8|\langle\mathbf{v}\rangle_{\mathrm{He}^3}|\int S(\mathbf{x})d\mathbf{x}. \tag{57}$$

Together with Eq. (55) this implies

$$\Gamma(\mathrm{He}^3)/D(\mathrm{He}^3)\cong 8-\Delta_{\mathrm{conn}}. \tag{58}$$

As defined previously, the quantity Δ_{conn} contains sums of squares of matrix elements of $Q_\alpha{}^{(\pm)}$ between the state $|\mathrm{He}^3\rangle$ and the states $|N_b\rangle$ which are inaccessible energetically in the actual muon-capture process, e.g., the nuclear states with one nucleon replaced by a nucleon isobar, the states with a particle-antiparticle pair (the so-called Z diagram), etc. As already mentioned, it seems that the neglect of Δ_{conn} is quite justified in the nonrelativistic approximation. Since the values of the commutators in the quark model are not very much different from those of the nonrelativistic impulse approximation [see Eqs. (40), (41), and (44)], it is not entirely unreasonable to neglect Δ_{conn} in the case of the quark model. In the absence of a more convincing proof, however, this to to be taken as an additional *ad hoc* assumption.[20] With this assumption, we finally obtain

$$\Gamma(\mathrm{He}^3)/D(\mathrm{He}^3)\cong 16I^{(3)}=8. \tag{59}$$

One of the remarkable points to note is that the right-hand side of relation (59) depends on only one nuclear parameter, namely, $I^{(3)}(\mathrm{He}^3)$. Substituting Eq. (22) and the estimates (47), Eq. (59) yields

$$\Gamma(\mathrm{He}^3)\cong 2.36\times 10^3\ \mathrm{sec}^{-1}, \tag{60}$$

in good agreement with the experimental values (49). In addition to using several well-established approximations, we have made the following two crucial assumptions in deriving this result:

(a) The weak hadronic currents satisfy the quark field algebra.

(b) The closure approximation (i.e., Δ_{conn} is negligible) is valid in the context of the quark field algebra.

It is quite clear that if one of these assumptions could be verified independently, our result would provide the necessary justification for the other assumption within the limits of the approximations introduced. In the absence of such independent verification of either of the assumptions, our method serves as a probe testing both assumptions together. Because of this, one cannot rule out the possibility that both assumptions are wrong in such a way that the mistakes they introduce cancel to yield the good result. To test assumptions (a) and (b) further, it is therefore imperative to look for applications of relation (38) to other nuclear isodoublets such as (Be^7, Li^7), (C^{11},B^{11}), etc., This, however, is not feasible in practice since one of the members of these isodoublets is always very unstable and the required muon-capture experiments cannot all be carried out. In the absence of further tests of assumptions (a) and (b) in muon capture, we shall focus our attention on elastic neutrino (antineutrino) scattering by nuclei in Sec. VI. Using assumptions (a) and (b), we will derive a relation between cross sections for elastic scattering of neutrinos and antineutrinos by nuclei. This relation will be not as restricted in its experimental applications as relation (38), in the sense that it involves one nucleus only and is therefore applicable to a variety of stable nuclear targets.

C. Gauge Field Algebra

If we now substitute relations (40a)–(40f) with $\eta=0$ into Eq. (43) and use Eqs. (50)–(52), (54), and (57), we find that

$$\Gamma(\mathrm{He}^3)/D(\mathrm{He}^3)\cong 4I^{(3)}(\mathrm{He}^3)-\Delta_{\mathrm{conn}}=2-\Delta_{\mathrm{conn}}. \tag{61}$$

Using the estimates (47), this gives

$$\Gamma(\mathrm{He}^3)\cong 0.59\times 10^3\ \mathrm{sec}^{-1}-\Delta_{\mathrm{conn}}. \tag{62}$$

If we also assume here that Δ_{conn} is negligible, we end up with a result in violent disagreement with experiment. However, since the values of the commutators in the gauge field algebra are, in contrast to the quark model, quite different from those of the nonrelativistic impulse approximation [see Eqs. (40), (41), and (44)], the neglect of Δ_{conn}, which has, after all, always been justified on the basis of the nonrelativistic approximation, is quite unjustified in the context of the gauge field algebra. Thus it is clear that we cannot reach any conclusions regarding the validity of the gauge field algebra from our calculations. At any rate, it can be concluded that the validity of the gauge field algebra necessarily implies a large contribution of Δ_{conn} to the capture rate, in contrast to the case of the quark model. This difference will enable us to distinguish between two algebras when and if the calculation of Δ_{conn} becomes possible.

[20] A. Wolsky (University of Pennsylvania) is currently investigating the magnitude of Δ_{conn} in the case of muon capture by proton (private communication).

VI. NEUTRINO (ANTINEUTRINO) REACTIONS IN NUCLEI

In this section, we shall investigate the possibility of applying assumptions (a) and (b) of Sec. V B to neutrino (antineutrino) elastic scattering by nuclei.

Consider the "elastic"[21] reaction

$$\nu_l+N_a\rightarrow l+N_b, \tag{63}$$

where N_a is the initial nucleus and N_b any allowed final state of hadrons excluding pions, baryon resonances, and strange particles. The lepton l can be either an electron or a muon. We work in the laboratory frame of reference in which the nucleus N_a is at rest, and adopt the same notation for the kinematical and dynamical variables associated with the neutrino and hadron states N_a and N_b as the one used in Sec. II. The charged lepton four-momentum is $p_l=(\mathbf{l},iE_l)$ and its mass is m_l. We restrict our attention to values of $l=|\mathbf{l}|$ such that $(m_l/l)^2\ll 1$, and terms of order $(m_l/l)^2$ are neglected so that $E_l\approx l$. From energy-momentum conservation, one can then show that

$$l\cong\frac{(m_a{}^2-m_b{}^2)+2\nu m_a}{2(E-\nu\cos\theta)}, \tag{64}$$

where $E=m_a+\nu$ is the total initial energy and θ is the angle of the final charged lepton relative to the original neutrino direction.

Using the interaction Hamiltonian given by Eq. (1) and the definition (18), one can readily show that the differential cross section for reaction (63) is

$$\begin{aligned}\frac{d\sigma^{(\nu)}(N_a\rightarrow N_b;\nu,\theta)}{d(\cos\theta)}&=\frac{G^2}{2\pi}\frac{1}{2J_a+1}\sum_{M_a}\sum_{M_b}\sum_{\mathbf{k}_b}\frac{l^2(E-l)}{E-\nu\cos\theta}\mathcal{L}_{\alpha\beta}{}^{(\nu)}\\&\times\langle N_a;\mathbf{k}_a=0,M_a|Q_\beta{}^{(-)}(\mathbf{q})|N_b;\mathbf{k}_b,M_b\rangle\\&\times\langle N_b;\mathbf{k}_b,M_b|Q_\alpha{}^{(+)}(\mathbf{q})|N_a;\mathbf{k}_a=0,M_a\rangle,\end{aligned} \tag{65}$$

where

$$\mathcal{L}_{\alpha\beta}{}^{(\nu)}=(1/\nu l)[(p_\nu)_\alpha(p_l)_\beta+(p_l)_\alpha(p_\nu)_\beta-(p_\nu\cdot p_l)\delta_{\alpha\beta}+\epsilon_{\alpha\beta\rho\sigma}(p_\nu)_\rho(p_l)_\sigma] \tag{66}$$

and

$$\mathbf{q}=\boldsymbol{\nu}-\mathbf{l}. \tag{67}$$

We shall be particularly interested in the *total* "elastic"[21] differential cross section

$$\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}=\sum_{N_b(E_b\leqslant E)}\frac{d\sigma^{(\nu)}(N_a\rightarrow N_b;\nu,\theta)}{d(\cos\theta)}. \tag{68}$$

If we restrict ourselves to the case when the charged lepton emerges in the forward direction ($\theta=0$), then

$$\begin{aligned}\left.\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}&=\frac{G^2}{2\pi}\frac{1}{2J_a+1}\\&\sum_{M_a}\Big\{\sum_{N_b(E_b\leqslant E)}\sum_{M_b}\sum_{\mathbf{k}_b}[l^2(E-l)/m_a]\mathcal{S}_{\alpha\beta}{}^{(\nu)}\\&\times\langle N_a;\mathbf{k}_a=0,M_a|Q_\beta{}^{(-)}(\mathbf{q})|N_b;\mathbf{k}_b,M_b\rangle\\&\times\langle N_b;\mathbf{k}_b,M_b|Q_\alpha{}^{(+)}(\mathbf{q})|N_a;\mathbf{k}_a=0,M_a\rangle\Big\},\end{aligned} \tag{69}$$

where

$$|\mathbf{q}|=\nu-l\cong(1/2m_a)(m_b{}^2-m_a{}^2) \tag{70}$$

and

$$\mathcal{S}_{\alpha\beta}{}^{(\nu)}=(1/l\nu)[(p_\nu)_\alpha(p_l)_\beta+(p_l)_\alpha(p_\nu)_\beta]. \tag{71}$$

We have dropped the $\delta_{\alpha\beta}$ and $\epsilon_{\alpha\beta\rho\sigma}$ terms in $\mathcal{L}_{\alpha\beta}{}^{(\nu)}$ since we are neglecting terms of order $(m_l/l)^2$, and they do not contribute for $\theta=0$.

In practice, the main contributions to relation (69) come from low-lying nuclear states N_b since the wave functions of such states overlap well with the wave function of the initial nuclear state. This is the assumption on which the closure approximation is essentially based. It implies that only those states N_b for which (m_b-m_a) is less than 20 MeV contribute appreciably to relation (69). It subsequently follows from Eq. (70) that the values of l for those states N_b that contribute substantially to relation (69) differ from ν by at most 20 MeV. Therefore, $\langle m_b\rangle_a{}^{(\nu)}-m_a\ll\nu$,[22] and one can write with sufficient accuracy that

$$\begin{aligned}\left.\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\cong\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}{}^{(\nu)}}{2J_a+1}\sum_{M_a}\Big[\sum_{N_b(E_b\leqslant E)}\sum_{\mathbf{k}_b}\sum_{M_b}\langle N_a;\mathbf{k}_a=0,M_a|Q_\beta{}^{(-)}(\langle\mathbf{q}\rangle_a{}^{(\nu)})|N_b;\mathbf{k}_b,M_b\rangle\\\times\langle N_b;\mathbf{k}_b,M_b|Q_\alpha{}^{(+)}(\langle\mathbf{q}\rangle_a{}^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle\Big].\end{aligned} \tag{72}$$

Similarly, the *total* elastic forward differential cross section for the corresponding antineutrino-induced reaction,

$$\bar{\nu}_l+N_a\rightarrow\bar{l}+N_c, \tag{73}$$

[21] By "elastic" scattering we mean reactions in which the three-momentum transfer to the nucleus (and therefore also the energy transfer, since we neglect terms of the order of the charged lepton mass squared and limit ourselves to forward charged lepton production) is less than 30–40 MeV/c. This automatically excludes reactions in which pions, baryon resonances, and strange particles are produced.

[22] $\langle\cdots\rangle_a{}^{(\nu)}$ denotes an appropriate weighted average over possible final states N_b for the neutrino-induced reaction (63).

is

$$\left.\frac{d\sigma^{(\bar{\nu})}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}=\sum_{N_c(E_c\leqslant E)}\left.\frac{d\sigma^{(\bar{\nu})}(N_a\rightarrow N_c;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}$$

$$\cong\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}^{(\nu)}}{2J_a+1}\sum_{M_a}\Big[\sum_{N_c(E_c\leqslant E)}\sum_{\mathbf{k}_c}\sum_{M_c}\langle N_a;\mathbf{k}_a=0,M_a|Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\bar{\nu})})|N_c;\mathbf{k}_c,M_c\rangle \times\langle N_c;\mathbf{k}_c,M_c|Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\bar{\nu})})|N_a;\mathbf{k}_a=0,M_a\rangle\Big],\quad(74)$$

where ν is the incident antineutrino energy and

$$\mathcal{S}_{\alpha\beta}^{(\bar{\nu})}=(1/\nu l)[(p_{\bar{\nu}})_\alpha(p_l)_\beta+(p_l)_\alpha(p_{\bar{\nu}})_\beta],\quad(75)$$

$$p_{\bar{\nu}}=(\mathbf{v},-i\nu),\quad p_l\cong(\mathbf{l},-il).\quad(76)$$

$\langle\cdots\rangle_a^{(\bar{\nu})}$ denotes an appropriate weighted average over possible final state N_c for the antineutrino-induced reaction (73). In general, $\langle\mathbf{q}\rangle_a^{(\nu)}$ and $\langle\mathbf{q}\rangle_a^{(\bar{\nu})}$ will be different since the final hadron states in the neutrino- and antineutrino-induced reactions are not the same. Since, however, we are restricting ourselves to forward elastic scattering, this difference is very small,[23] and as in the muon-capture case, one can readily show that an error of less than 1% is made by replacing $\langle\mathbf{q}\rangle_a^{(\bar{\nu})}$ by $\langle\mathbf{q}\rangle_a^{(\nu)}$ in Eq. (74). With this approximation, Eq. (74) becomes

$$\left.\frac{d\sigma^{(\bar{\nu})}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\cong\frac{G^2}{2\pi}\nu^2\mathcal{S}_{\alpha\beta}^{(\bar{\nu})}\frac{1}{2J_a+1}\sum_{M_a}\Big[\sum_{N_c(E_c\leqslant E)}\sum_{\mathbf{k}_c}\sum_{M_c}\langle N_a;\mathbf{k}_a=0,M_a|Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_c;\mathbf{k}_c,M_c\rangle \times\langle N_c;\mathbf{k}_c,M_c|Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle\Big].\quad(77)$$

If we use the closure realtion (27), Eqs. (72) and (77) can be rewritten as follows:

$$\left.\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\cong\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}^{(\nu)}}{2J_a+1}\sum_{M_a}\langle N_a,\mathbf{k}_a=0,M_a|Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle+\delta_1^{(\nu)}\quad(78)$$

and

$$\left.\frac{d\sigma^{(\bar{\nu})}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\cong\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}^{(\bar{\nu})}}{2J_a+1}\sum_{M_a}\langle N_a;\mathbf{k}_a=0,M_a|Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle+\delta_1^{(\bar{\nu})},\quad(79)$$

where

$$\delta_1^{(\nu)}=\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}^{(\nu)}}{2J_a+1}\sum_{M_a}\Big[\sum_{N_b(E_b\leqslant E)}\sum_{\mathbf{k}_b}\sum_{M_b}\langle N_a;\mathbf{k}_a=0,M_a|Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_b;\mathbf{k}_b,M_b\rangle \times\langle N_b;\mathbf{k}_b,M_b|Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle\Big]\quad(80)$$

and

$$\delta_1^{(\bar{\nu})}=\frac{G^2}{2\pi}\nu^2\frac{\mathcal{S}_{\alpha\beta}^{(\bar{\nu})}}{2J_a+1}\sum_{M_a}\Big[\sum_{N_c(E_c\leqslant E)}\sum_{\mathbf{k}_c}\sum_{M_c}\langle N_a;\mathbf{k}_a=0,M_a|Q_\alpha^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_c;\mathbf{k}_c,M_c\rangle \times\langle N_c;\mathbf{k}_c,M_c|Q_\beta^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})|N_a;\mathbf{k}_a=0,M_a\rangle\Big].\quad(81)$$

Let us choose the z axis along the incident neutrino (antineutrino) direction. Relations (78) and (79) can then be combined to give

$$\left.\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}-\left.\frac{d\sigma^{(\bar{\nu})}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}\cong\frac{G^2}{\pi}\nu^2\frac{1}{2J_a+1}\sum_{M_a}\{\langle N_a;\mathbf{k}_a=0,M_a|([Q_3^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)}),Q_3^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})] -[Q_4^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)}),Q_4^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})])|N_a;\mathbf{k}_a=0,M_a\rangle\}+\delta_1+\delta_2,\quad(82)$$

where

$$\delta_1=\delta_1^{(\nu)}-\delta_1^{(\bar{\nu})}\quad(83)$$

and

$$\delta_2=i\frac{G^2}{\pi}\nu^2\frac{1}{2J_a+1}\sum_{M_a}\langle N_a;\mathbf{k}_a=0,M_a|(\{Q_3^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)}),Q_4^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)})\}+\{Q_3^{(+)}(\langle\mathbf{q}\rangle_a^{(\nu)}),Q_4^{(-)}(\langle\mathbf{q}\rangle_a^{(\nu)})\}) \times|N_a;\mathbf{k}_a=0,M_a\rangle.\quad(84)$$

[23] We are, of course, assuming equal incident neutrino and antineutrino energies.

As for the muon-capture case, we invoke the closure approximation[11] here also and neglect δ_1 [this is equivalent to assumption (b) of Sec. V B]. Furthermore, we shall now show that for elastic scattering δ_2 is also negligible. To demonstrate this, let us first rewrite δ_2 as follows:

$$\delta_2=i\frac{G^2}{\pi}\nu^2\frac{1}{2J_a+1}\sum_{M_a}\langle N_a;\mathbf{k}_a=0,M_a|\bigg(\bigg\{\int J_3^{(-)}(\mathbf{x},0)\exp(-i\langle\mathbf{q}\rangle_a^{(\nu)}\cdot\mathbf{x})d\mathbf{x},\int J_4^{(+)}(\mathbf{y},0)\exp(i\langle\mathbf{q}\rangle_a^{(\nu)}\cdot\mathbf{y})d\mathbf{y}\bigg\}$$
$$+\bigg\{\int J_3^{(+)}(\mathbf{x},0)\exp(i\langle\mathbf{q}\rangle_a^{(\nu)}\cdot\mathbf{x})d\mathbf{x},\int J_4^{(-)}(\mathbf{y},0)\exp(-i\langle\mathbf{q}\rangle_a^{(\nu)}\cdot\mathbf{y})d\mathbf{y}\bigg\}\bigg)|N_a;\mathbf{k}_a=0,M_a\rangle. \quad (85)$$

The main contributions to the integrals in relation (85) come from $|\mathbf{x}|$, $|\mathbf{y}|\leq R_A$, where $R_A\sim A^{1/3}/m_\pi$ is the nuclear radius of N_a (m_π is the pion mass). This is most easily seen by using the nonrelativistic impulse approximation. We have already pointed out that $|\langle\mathbf{q}\rangle_a^{(\nu)}|$ is at most 20 MeV/c. It is therefore clear that the main contributions to δ_2 come from values of $|\mathbf{x}|$ and $|\mathbf{y}|$ such that

$$\langle\mathbf{q}\rangle_a^{(\nu)}\cdot\mathbf{x}(\text{or }\mathbf{y})\leqslant|\langle\mathbf{q}\rangle_a^{(\nu)}|R_A\cong|\langle\mathbf{q}\rangle_a^{(\nu)}|A^{1/3}/m_\pi\ll 1. \quad (86)$$

A "dipole"-type approximation—in which the exponentials in Eq. (85) are expanded and only the leading term is kept—is thus quite feasible. With this approximation,

$$\delta_2\cong i\frac{G^2}{\pi}\nu^2\frac{1}{2J_a+1}\sum_{M_a}\langle N_a;\mathbf{k}_a=0,M_a|$$
$$\times\bigg(\bigg\{\int J_3^{(-)}(\mathbf{x},0)d\mathbf{x},\int J_4^{(+)}(\mathbf{y},0)d\mathbf{y}\bigg\}+\bigg\{\int J_3^{(+)}(\mathbf{x},0)d\mathbf{x},\int J_4^{(-)}(\mathbf{y},0)d\mathbf{y}\bigg\}\bigg)$$
$$\times|N_a;\mathbf{k}_a=0,M_a\rangle=0. \quad (87)$$

The vanishing of the right-hand side of Eq. (87) is easily demonstrated by using (i) the parity transformation properties of the vector and axial-vector currents and (ii) the Wigner-Eckart theorem.

Neglecting δ_1 and δ_2 and substituting the commutation relations (40a)–(40d) with $\eta=1$ (quark field algebra), relation (82) finally yields

$$\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\bigg|_{\theta=0}-\frac{d\sigma^{(\bar\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\bigg|_{\theta=0}\cong-8I^{(3)}(N_a)G^2\cos^2\theta_c\,\nu^2/\pi, \quad (88)$$

where use has been made of the analog of Eqs. (50)–(52) for the state $|N_a;\mathbf{k}_a=0,M_a\rangle$.

We wish to make the following remarks:

(a) The right-hand side of relation (88) depends on only one nuclear parameter, namely, $I^{(3)}(N_a)$ (or, equivalently, the neutron excess).

(b) In deriving Eq. (88), use was made of the equal-time commutators of the space components of the weak hadron current as well as the commutator of the time component.

(c) Relation (88) is in a form which is very convenient for comparison with experiment since nearly all neutrino and antineutrino scattering experiments are carried out on nuclei (and not on free nucleons). In order to distinguish between elastic and inelastic events, it is advantageous to test the relation for cases where the detector can also serve as a target. Excellent candidates are Al^{27} ($I=\frac{1}{2}$, $I^{(3)}=-\frac{1}{2}$) and Fe^{56} ($I=2$, $I^{(3)}=-2$), which are quite commonly used in spark chambers.

(d) It is interesting to note that relation (88) is quite similar to the one derived by Adler[24] for nucleons, but the methods of derivation are quite different. Although the Adler relation is not limited to elastic scattering or to production of muons in the forward direction alone, it does require taking the limit when neutrino and antineutrino energies tend to infinity. On the other hand, in our mode of derivation, relation (88) is expected to be applicable already for neutrino and antineutrino energies of 100 MeV or more when $l=e$ and of 0.5 GeV or more when $l=\mu$. The 100-MeV and 0.5-GeV lower limits that we are setting are partly due to the fact that in deriving this relation we have assumed the lepton mass to be negligible. That we do not have to consider the limit of infinite neutrino (antineutrino) energy is due to the fact that relation (88) applies to nuclei, and not to nucleons, for which a closure approximation may be meaningful and good saturation of the sum rules by low-lying nuclear states is perhaps possible.

In view of remarks (c) and (d), we believe that tests of relation (88) are at present quite feasible. Such tests are very important as they can provide further verification of the quark field algebra and the closure approximation [assumptions (a) and (b) of Sec. V B].

A useful lower bound for the total elastic differential cross section for forward scattering of neutrinos off *stable* nuclei N_a can be obtained by noting that, by definition, $d\sigma^{(\nu)}(N_a;\nu,\theta)/d(\cos\theta)\geq 0$, and that for nearly all[25] stable nuclei the number of excess neutrons is

$$A-2Z=-2I^{(3)}(N_a)\geqslant 0. \quad (89)$$

[24] S. L. Adler, Phys. Rev. **143**, 1144 (1966).
[25] The only exception is the He^3 nucleus.

For such stable nuclei, Eq. (88) clearly implies the following inequality:

$$\left.\frac{d\sigma^{(\nu)}(N_a;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \geqslant 4(A-2Z)\frac{G^2\cos^2\theta_c\,\nu^2}{\pi} \cong 1.6(A-2Z)\left.\frac{d\sigma^{(\nu)}(n\to p;\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0}, \quad (90)$$

where $d\sigma^{(\nu)}(n\to p;\nu,\theta)/d(\cos\theta)$ is the elastic differential cross section for scattering of neutrinos of energy $\nu\gg m_\mu$ off neutrons.[26] The lower bound defined by this inequality is 1.6 times larger than the differential cross section calculated by Goulard and Primakoff[27] using the nonrelativistic impulse-approximation limit. It should also be noted that if one uses in Eq. (82) the commutation relations based on the nonrelativistic impulse approximation [Eq. (44a)] instead of the quark field-algebra relations [Eqs. (40a)–(40d)], $8I^{(3)}$ in Eq. (88) is replaced by $5I^{(3)}$. We therefore see that in the nonrelativistic impulse approximation the value 1.6 in Eq. (90) should be replaced by unity. The corresponding lower bound on the cross section is then in agreement with that calculated by Goulard and Primakoff.

Using essentially the same techniques as those described in this section and in Sec. III, and in the context of the quark field algebra, we have also derived the following relations[2]:

$$\left.\frac{d\sigma^{(\nu)}(N(A,I,-I^{(3)});\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} - \left.\frac{d\sigma^{(\nu)}(N(A,I,I^{(3)});\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \cong \left.\frac{d\sigma^{(\bar\nu)}(N(A,I,I^{(3)});\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} - \left.\frac{d\sigma^{(\bar\nu)}(N(A,I,-I^{(3)});\nu,\theta)}{d(\cos\theta)}\right|_{\theta=0} \cong 8I^{(3)}G^2\cos^2\theta_c\,\nu^2/\pi \quad (I^{(3)}>0), \quad (91)$$

where $d\sigma^{(\nu)}(N(A,I,I^{(3)});\nu,\theta)/d(\cos\theta)$ is the total elastic differential cross section for scattering of neutrinos of energy $\nu\gg m_\mu$ off a nucleus $N(A,I,I^{(3)})$ of mass number A, isospin I, and third component of isospin $I^{(3)}$. These relations are not as useful as relation (88) since one of the nuclei appearing in Eq. (88) is always quite unstable (the one with the proton excess), and, as a result, experimental tests of the relations are not very practical. It is to be noted, however, that Eqs. (91) also yield the useful inequality (90).

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APPENDIX

From Eqs. (30), (35), and (37), and the definition of the disconnected part,[28] we have

$$\Delta_{\rm dis}=\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\sum_N\sum_{\mathbf{k}}\sum_M [\langle 0|Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{\rm He^3})|N;\mathbf{k},M\rangle \times\langle N;\mathbf{k},M|Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{\rm He^3})|0\rangle -\langle 0|Q_\sigma^{(+)}(-\langle\mathbf{v}\rangle_{\rm He^3})|N;\mathbf{k},M\rangle \times\langle N;\mathbf{k},M|Q_\lambda^{(-)}(-\langle\mathbf{v}\rangle_{\rm He^3})|0\rangle], \quad (A1)$$

where $|0\rangle$ represents the physical vacuum state. Using the closure relation (27), Eq. (31c), and the invariance of the vacuum state under isotopic-spin rotations, Eq. (A1) becomes

$$\Delta_{\rm dis}=\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\{\langle 0|[Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{\rm He^3})Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{\rm He^3}) -Q_\sigma^{(-)}(\langle\mathbf{v}\rangle_{\rm He^3})Q_\lambda^{(+)}(\langle\mathbf{v}\rangle_{\rm He^3})]|0\rangle\}. \quad (A2)$$

Since only the symmetric part of $\mathcal{L}_{\lambda\sigma}$ contributes to Eq. (A2), we can write

$$\Delta_{\rm dis}=\int\frac{d\Omega_\nu}{4\pi}\mathcal{L}_{\lambda\sigma}\langle 0|[Q_\sigma^{(+)}(\langle\mathbf{v}\rangle_{\rm He^3}), \times Q_\lambda^{(-)}(\langle\mathbf{v}\rangle_{\rm He^3})]|0\rangle. \quad (A3)$$

Using Eq. (12) and the commutators (40a)–(40f) (with $\eta=1$ or 0) and using the parity-transformation properties of the vector and axial-vector currents, relation (57) immediately follows.[28]

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[26] T. D. Lee and C. N. Yang, Phys. Rev. Letters **4**, 307 (1960); Phys. Rev. **119**, 1410 (1960); **126**, 2239 (1962); Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto) **23**, 1117 (1960); N. Cabibbo and R. Gatto, Nuovo Cimento **15**, 304 (1960); B. Goulard and H. Primakoff, Phys. Rev. **135**, B1139 (1964).

[27] B. Goulard and H. Primakoff (Ref. 26).

[28] See, for example, C. W. Kim, W. Repko, and A. Sato, Phys. Rev. D 1, 434 (1970).