Unitarity Upper and Lower Bounds on the Absorptive Parts of Elastic Scattering Amplitudes*

VIRENDRA SINGH AND S. M. ROY
Tata Institute of Fundamental Research, Bombay 5, India
(Received 3 December 1969)

We derive upper and lower bounds on the imaginary part of the elastic scattering amplitude of two spinless particles in the physical region, in terms of the elastic cross section σ_{el} and the total cross section σ_{tot}, using unitarity alone. The bounds derived are the best possible ones, given only the stated unitarity constraints. The upper bound for high energies and small values of the momentum transfer squared t has a particularly simple and "universal" form, $\mathrm{Im}F(s,t)/\mathrm{Im}F(s,0)\leqslant 1-\frac{1}{9}\rho+\frac{3}{8}(\rho/9)^2-(21/320)(\rho/9)^3+\cdots$ if $2.5\geqslant\rho\equiv(-t/4\pi)\sigma_{tot}^2(s)/\sigma_{el}(s)$, which depends on the particular scattering process and on the energy and momentum transfer only through the dimensionless parameter ρ. We give explicit formulas and numerical values for the upper bound up to $\rho=8.42$. We compare the experimental curve of $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ versus $4(-t)(d\sigma/dt)_{t=0}/\sigma_{el}$ with the theoretical upper bound on $[\mathrm{Im}F(s,t)/\mathrm{Im}F(s,t=0)]^2$ versus ρ. The quantities plotted in the experimental and theoretical curves are the same if the unpolarized cross sections are spin-independent and purely absorptive in the diffraction-peak region. We find that the experimental points for pp, $\bar{p}p$, π^+p, and π^-p scattering in the lab momentum range 6–13 GeV/c fall on a curve lying only slightly below the theoretical upper-bound curve, the difference being less than 10% for ρ in the range (0,3) and less than 25% for ρ in the range (3,5). We further notice that this experimental curve is universal. We also derive unitarity lower bounds on the nth derivatives of the absorptive part at $t=0$, and on the absorptive part for positive values of t within the Lehmann-Martin ellipse, in terms of σ_{el} and σ_{tot}. The corresponding bounds if σ_{tot} alone is known are also derived.

I. INTRODUCTION

ONE of the important features of high-energy elastic scattering is the presence of the diffraction peak in the low-momentum-transfer region. We would like to investigate how far one can understand this feature in terms of restrictions arising from direct-channel unitarity without recourse to any specific model. We begin by recalling some earlier results which suggest the importance of unitarity restrictions in the diffraction-peak region.

Let $F(s,t)$ be the elastic scattering amplitude for the process $A+B\to A+B$ with the particles A and B assumed spinless (e.g., $\pi\pi\to\pi\pi$), with s and t being, respectively, the squares of the c.m. energy and momentum-transfer variables. Martin[1] has established the following bounds, which follow from unitarity alone. In the physical region,

$$\mathrm{Im}F(s,t)\leqslant\mathrm{Im}F(s,0)\frac{4}{3}\frac{[1+N(N+1)\sin^2\theta]^{3/4}-1}{N(N+1)\sin^2\theta},\qquad \pi\geqslant\theta\geqslant 0 \quad (1.1)$$

and for small positive momentum transfers,

$$\mathrm{Im}F(s,t)\geqslant\mathrm{Im}F(s,0)\times\left[\frac{P_{N+1}'(1+t/2k^2)+P_N'(1+t/2k^2)}{(N+1)^2}\right],\qquad t_0\geqslant t>0 \quad (1.2)$$

where $t=-2k^2(1-\cos\theta)$, $k=$c.m. momentum, $(N+1)^2=k^2\sigma_{tot}(s)/(4\pi)$, $\sqrt{t_0}=$the mass of the lowest mass state that couples to the crossed t channel, i.e., $A\bar{A}\to B\bar{B}$, i.e., $t_0=4m_\pi^2$ for $\pi\pi$ and πN scattering, and $\sigma_{tot}(s)=$total cross section for $A+B$ at the c.m. energy square given by s. These bounds are purely in terms of the experimentally measurable quantity $\sigma_{tot}(s)$, since $\mathrm{Im}F(s,0)$ is completely specified by the normalization and $\sigma_{tot}(s)$, and as such are very important. The bound (1.1) is directly checkable in terms of experimental data. The bound (1.2), though not directly checkable since it refers to the unphysical region, is nevertheless important since it can be used[1] to establish the Froissart bound[2]

$$\sigma_{tot}(s)\underset{s\to\infty}{\leqslant}\mathrm{const}[\ln(s/s_0)]^2 \quad (1.3)$$

and also gives a bound[1,3] on the constant occurring in (1.3). From the point of view of the experimental comparison we must also mention here the unitarity upper bound on "diffraction-peak width" obtained by MacDowell and Martin,[4]

$$\left(\frac{d}{dt}\ln\mathrm{Im}F(s,t)\right)_{t=0}>\frac{1}{9}\left(\frac{\sigma_{tot}^2(s)}{4\pi\sigma_{el}(s)}-\frac{1}{k^2}\right), \quad (1.4)$$

where $\sigma_{el}(s)$ is the total elastic cross section. This bound is remarkably close to the observed experimental values when a comparison is made by neglecting the contribution of the real part of the amplitude to the diffraction-peak width.[4]

The MacDowell-Martin result leads us to hope that a unitarity upper bound on the differential cross section

* Some of the results of this paper were reported earlier: V. Singh and S. M. Roy, Phys. Rev. Letters 24, 28 (1970).
[1] A. Martin, Phys. Rev. 129, 1432 (1963).
[2] M. Froissart, Phys. Rev. 123, 1053 (1961).
[3] L. Łukaszuk and A. Martin, Nuovo Cimento 52A, 122 (1967).
[4] S. W. MacDowell and A. Martin, Phys. Rev. 135, B960 (1964).

in terms of σ_{tot} and σ_{el} might also be close to the experimental results in the diffraction-peak region. Indeed, an asymptotic upper bound on the differential cross section in terms of σ_{el} has been found recently by us which improves the asymptotic bounds due to Froissart,[2] Martin,[5] Łukaszuk and Martin,[3] Mahoux and Martin,[6] and Bell,[7] and is given by[8]

$$\left(\frac{d\sigma}{dt}\right)_{t=0} \underset{s\to\infty}{\leqslant} \frac{\sigma_{el}}{4t_0}\left(\ln\frac{s}{\sigma_{el}}\right)^2 \tag{1.5}$$

and

$$\left(\frac{d\sigma}{d\Omega}\right) \underset{\substack{s\to\infty\\ t\,\text{fixed}}}{\leqslant} \frac{1}{8\pi^2\sqrt{t_0}}\frac{s}{\sqrt{-t}}\sigma_{el}\ln\frac{s}{\sigma_{el}}, \quad t<0. \tag{1.6}$$

The assumptions needed to derive (1.5) and (1.6) are the same as in Łukaszuk and Martin's derivation[3] of the Froissart bound, namely, unitarity, analyticity in the domain derived from axiomatic field theory, polynomial boundedness within the Lehmann-Martin ellipse, and crossing symmetry. These asymptotic bounds contain smaller numbers of explicit ln$s$ factors than the corresponding Froissart bounds and can serve as important restrictions on theoretical models for hadronic scattering amplitudes. Unfortunately, they still contain some explicit ln$s$ factors, and we are unable to test these relations directly against experiment since present experimental accuracy is insufficient to prove or disprove the existence of such factors in the asymptotic cross sections.

We are therefore led to consider bounds on the absorptive part for which one is able to make a more efficient use of unitarity. One is then able to derive results valid at finite energies and not just asymptotic energies. These, incidentally, do not contain any explicit ln$s$ factors.

The physical-region results we present in this paper are consequences of unitarity alone like the results (1.1), (1.2), and (1.4). The unphysical-region results need, in addition, the use of analyticity within the Lehmann-Martin ellipse. In contrast with the derivation of the Froissart bound and the asymptotic bounds (1.5) and (1.6), assumptions regarding polynomial boundedness of the amplitude are unnecessary for the derivation of these results. Our most important result is an upper bound on the imaginary part of the amplitude in the physical region in terms of σ_{el} and σ_{tot}. It is very close to the experimental results for pion-nucleon, nucleon-nucleon, and nucleon-antinucleon scattering data in the diffraction-peak region when comparison is made by neglecting the real part of the amplitude and assuming spin independence of the unpolarized cross sections. The upper bound for small values of the momentum transfer squared t has a particularly simple form,

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \leqslant \left[1-\frac{\rho}{9}+\frac{3}{8}\left(\frac{\rho}{9}\right)^2-\frac{21}{320}\left(\frac{\rho}{9}\right)^3+\cdots\right] \tag{1.7}$$

if

$$0\leqslant \rho \equiv (-t)\sigma_{tot}^2(s)/4\pi\sigma_{el}(s) \leqslant 2.5. \tag{1.8}$$

It is enough to keep the first three terms in the series in (1.7) if an accuracy of 0.5% is desired. We give explicit formulas and numerical values for the upper bound up to $\rho=8.4$, and discuss detailed comparison with experiment in the text.

The main obstacle to obtaining good bounds on Im$F(s,t)$ in the physical region so far has been the complicated behavior of the required Legendre polynomials occurring in the partial-wave expansion. The Legendre polynomials $P_l(\cos\theta)$ for $\pi>\theta>0$ oscillate as a function of l (Fig. 1). Therefore one has as a rule tended to work with functions which majorize $|P_l(\cos\theta)|$ and have a nonoscillatory behavior in l. This entails loss of information. We have obtained the best possible bounds under the unitarity constraints stated by tackling this problem frontally. We have obtained both upper and lower bounds on Im$F(s,t)$ in the physical region. We are not aware of any previous significant results on the lower bounds in the physical region, and hence consider these to be of theoretical interest. The upper bounds in the physical region are found to be of immediate practical use.

The plan of the paper is as follows. In Sec. II we give all the exact results we have obtained in the form of a number of theorems. These are arranged as follows. First come the upper and lower bounds on Im$F(s,t)$ in the physical region (Theorems 1 and 2) which involve only the total cross sections. The upper bound is an improved version of Martin's theorem (1.1). Then comes Theorem 3 giving upper and lower bounds on Im$F(s,t)$ in the physical region involving both the total and elastic cross sections. This theorem is our best result from a practical point of view. This result can be improved under certain conditions to yield the bound

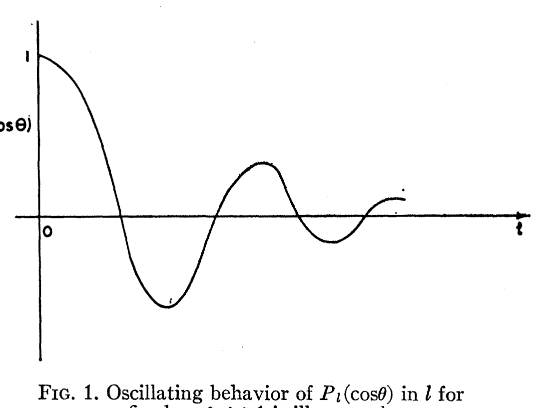

FIG. 1. Oscillating behavior of $P_l(\cos\theta)$ in l for a fixed $\cos\theta\neq\pm1$ is illustrated.

[5] A. Martin, Nuovo Cimento 42, 930 (1966).
[6] G. Mahoux and A. Martin, Phys. Rev. 174, 2140 (1968).
[7] J. S. Bell, Nuovo Cimento 61, 541 (1969).
[8] V. Singh and S. M. Roy, Ann. Phys. (N. Y.) (to be published).

given by Theorem 4. Theorem 5 gives lower bounds on the derivatives of $\mathrm{Im}F(s,t)$ at $t=0$ and yields the MacDowell-Martin result (1.4) as a special case. Lastly in Sec. II we give a number of theorems giving lower bounds on $\mathrm{Im}F(s,t)$ in the unphysical region $t_0>t>0$. These are of theoretical interest.

The exact results given in Sec. II are evaluated and discussed in the high-energy diffraction-peak region in Sec. III, and finally we compare these with the available experimental data in Sec. IV.

II. UNITARITY BOUNDS ON ABSORPTIVE PART OF ELASTIC AMPLITUDES

We have the following expansion for the elastic scattering amplitude $F(s,t)$ in terms of the partial-wave amplitudes $a_l(s)$ in the physical region:

$$F(s,t)=\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta). \tag{2.1}$$

The absorptive part is defined by

$$\mathrm{Im}F(s,t)\equiv A(s,t). \tag{2.2}$$

The normalization is fixed by stating the unitarity restriction on $a_l(s)$, i.e.,

$$1\geqslant \mathrm{Im}a_l(s)\geqslant |a_l(s)|^2\geqslant 0. \tag{2.3}$$

We then have

$$\sigma_{\mathrm{tot}}(s)=(4\pi/k\sqrt{s})\,\mathrm{Im}F(s,0)$$
$$=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}a_l(s), \tag{2.4}$$

$$\sigma_{\mathrm{el}}(s)=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|a_l(s)|^2. \tag{2.5}$$

When we consider the unphysical region of t, we will use the fact that the partial-wave expansion (2.1) is valid within the Lehmann-Martin ellipse whose size is fixed by the value of t_0.

We shall now state our results on the unitarity bounds in the form of a number of theorems. It is convenient for that purpose to divide the set of all positive integers (including zero) into three nonoverlapping sets as follows:

Definition 1. Let L be a positive integer and $\pi>\theta>0$. The positive integers l $(\geqslant 0)$ are said to belong to the set $U(L,\theta)$ if they satisfy $P_l(\cos\theta)>P_L(\cos\theta)$; to the set $V(L,\theta)$ if $P_l(\cos\theta)=P_L(\cos\theta)$; or to the set $W(L,\theta)$ if $P_L(\cos\theta)>P_l(\cos\theta)$.

The set $V(L,\theta)$ is finite. This follows from the inequality[9]

$$|P_l(\cos\theta)|<(2/\pi l\sin\theta)^{1/2},$$

valid for $\pi>\theta>0$, which implies that if $l\in V(L,\theta)$, then $l<2/\pi\sin\theta[P_L(\cos\theta)]^2$. Similarly, the set $U(L,\theta)$ is a finite set if $P_L(\cos\theta)>0$ and the set $W(L,\theta)$ is finite if $0>P_L(\cos\theta)$.

A. Upper and Lower Bounds on $A(s,t)$ in Physical Region Involving Only Total Cross Section

Theorem 1. An upper bound on $A(s,t)$ in the physical region is given by

$$A(s,t)\leqslant\left(\frac{k(\sqrt{s})\sigma_{\mathrm{tot}}(s)}{4\pi}\right)P_L(\cos\theta)$$
$$+\frac{\sqrt{s}}{k}\sum_{l\in U(L,\theta)}(2l+1)[P_l(\cos\theta)-P_L(\cos\theta)], \tag{2.6}$$

where the positive integer L is to be determined by

$$k^2\sigma_{\mathrm{tot}}(s)/4\pi=\sum_{l\in U(L,\theta)}(2l+1)+\xi\sum_{l\in V(L,\theta)}(2l+1) \tag{2.7}$$

and

$$1\geqslant P_L(\cos\theta)>0,\quad 1>\xi\geqslant 0.$$

We shall make a few comments on Theorem 1 before proceeding to give the proof.

(i) The introduction of the fractional number ξ, which does not appear in the bound (2.6) but occurs in (2.7), is necessary to take into account the fact that $k^2\sigma_{\mathrm{tot}}/(4\pi)$ is in general not integral, which it would be if ξ were taken to be zero. At high energies this becomes a pedantic point. The reader can also easily verify that (2.7) always leads to unique determination of L and ξ, since the right-hand side of (2.7) takes all possible values as $P_L(\cos\theta)$ and ξ vary in their allowed ranges. In particular, the value $P_L(\cos\theta)=1$ corresponds to $1>k^2\sigma_{\mathrm{tot}}/(4\pi)\geqslant 0$, while $P_L(\cos\theta)\to 0$ leads to values of $k^2\sigma_{\mathrm{tot}}/(4\pi)\to\infty$.

(ii) The upper bound (2.6) will be achieved for the following choice of the values of $\mathrm{Im}a_l$:

$$\begin{aligned}\mathrm{Im}a_l&=1 && \text{for } l\in U(L,\theta)\\ &=\xi_l && \text{for } l\in V(L,\theta)\\ &=0 && \text{for } l\in W(L,\theta),\end{aligned} \tag{2.8}$$

where

$$1>\xi_l\geqslant 0,$$

and

$$\xi=\sum_{l\in V(L,\theta)}(2l+1)\xi_l\Big/\sum_{l\in V(L,\theta)}(2l+1). \tag{2.9}$$

(iii) All the summations in Theorem 1 and in Eq. (2.7) are finite summations and are therefore well defined.

(iv) The upper bound given by (2.6) and (2.7) is the best possible given only σ_{tot} and the unitarity restriction $0\leqslant\mathrm{Im}a_l\leqslant 1$ since it is achieved by the choice (2.8). In particular, it is better than the upper bound (1.1) given

[9] G. Szego, *Orthogonal Polynomials* (American Mathematical Society Colloquim Publications, New York, 1959), p. 163, Eq. (7.3.8).

by Martin. A direct proof of this assertion will be given in Appendix A.

We now proceed to give the proof of Theorem 1. We shall use the direct subtraction method.

Proof of Theorem 1. Let

$$A_U(s,t)=\sum_{l\in U(L,\theta)}(2l+1)P_l(\cos\theta)+\xi\sum_{l\in V(L,\theta)}(2l+1)P_L(\cos\theta)\quad(2.10)$$

and consider the difference

$$\begin{aligned}A_U(s,t)-A(s,t)=&\sum_{l\in U(L,\theta)}(2l+1)(1-\mathrm{Im}a_l)P_l(\cos\theta)\\&+\sum_{l\in V(L,\theta)}(2l+1)(\xi-\mathrm{Im}a_l)P_L(\cos\theta)\\&-\sum_{l\in W(L,\theta)}(2l+1)\,\mathrm{Im}a_lP_l(\cos\theta)\\ \geqslant&\sum_{l\in U(L,\theta)}(2l+1)(1-\mathrm{Im}a_l)P_l(\cos\theta)\\&+\sum_{l\in V(L,\theta)}(2l+1)(\xi-\mathrm{Im}a_l)P_L(\cos\theta)\\&-P_L(\cos\theta)\sum_{l\in W(L,\theta)}(2l+1)\,\mathrm{Im}a_l.\end{aligned}\quad(2.11)$$

We now use (2.4) and (2.7) to obtain

$$\sum_{l\in W(L,\theta)}(2l+1)\,\mathrm{Im}a_l=\sum_{l\in U(L,\theta)}(2l+1)(1-\mathrm{Im}a_l)+\sum_{l\in V(L,\theta)}(2l+1)(\xi-\mathrm{Im}a_l).\quad(2.12)$$

Using then (2.12) in the inequality (2.11), we obtain

$$A_U(s,t)-A(s,t)\geqslant\sum_{l\in U(L,\theta)}(2l+1)(1-\mathrm{Im}a_l)\times[P_l(\cos\theta)-P_L(\cos\theta)].\quad(2.13)$$

We now use unitarity in the form

$$1\geqslant\mathrm{Im}a_l\geqslant0$$

to conclude that

$$A_U(s,t)-A(s,t)\geqslant0.\quad(2.14)$$

This concludes the proof of Theorem 1, since $A_U(s,t)$ is easily seen to be identical to the upper bound given there.

We now give a lower bound on $A(s,t)$ in the physical region involving the total cross section.

Theorem 2. A lower bound on $A(s,t)$ in the physical region ($\pi>\theta>0$) is given by

$$\begin{aligned}A(s,t)\geqslant[(\sqrt{s})/k]&\sum_{l\in W(M,\theta)}(2l+1)\times[P_l(\cos\theta)-P_M(\cos\theta)]\\&+k[(\sqrt{s})/4\pi]\sigma_{\mathrm{tot}}P_M(\cos\theta),\end{aligned}\quad(2.15)$$

where the positive integer M is to be determined by

$$k^2\sigma_{\mathrm{tot}}(s)/4\pi=\sum_{l\in W(M,\theta)}(2l+1)+\eta\sum_{l\in V(M,\theta)}(2l+1),\quad(2.16)$$

where

$$0>P_M(\cos\theta),\quad 1>\eta\geqslant0.$$

This theorem can be proved in exactly the same way as Theorem 1. We shall therefore omit the proof and forbear similar comments.

B. Upper and Lower Bounds on $A(s,t)$ Involving Both Total and Elastic Cross Sections

Let the set of positive integers ($\geqslant0$) be divided into two nonoverlapping sets as follows:

Definition 2. The positive integers l belong to the set $U(\alpha,A,\theta)$ if they satisfy

$$\alpha\left(\frac{P_l(\cos\theta)-A}{1-A}\right)\geqslant0$$

and to the set $V(\alpha,A,\theta)$ if they satisfy

$$0>\alpha\left(\frac{P_l(\cos\theta)-A}{1-A}\right).$$

Our next result can then be stated as follows:

Theorem 3. Let $(k^2/4\pi)\sigma_{\mathrm{tot}}^2/\sigma_{\mathrm{el}}\geqslant1$ and $\pi>\theta>0$ and let

$$A^{(\alpha,A)}(s,t)=\frac{\sqrt{s}}{k}\sum_{l\in U(\alpha,A,\theta)}(2l+1)\times\left(\frac{\alpha[P_l(\cos\theta)-A]}{1-A}\right)P_l(\cos\theta),\quad(2.17)$$

where α and A are determined by

$$\frac{k^2\sigma_{\mathrm{tot}}(s)}{4\pi}=\sum_{l\in U(\alpha,A,\theta)}(2l+1)\left(\frac{\alpha[P_l(\cos\theta)-A]}{1-A}\right),\quad(2.18)$$

$$\frac{k^2\sigma_{\mathrm{el}}(s)}{4\pi}=\sum_{l\in U(\alpha,A,\theta,)}(2l+1)\left(\frac{\alpha[P_l(\cos\theta)-A]}{1-A}\right)^2.\quad(2.19)$$

Then

$$A(s,t)\leqslant A^{(\alpha,A)}(s,t)\quad\text{if }\alpha/(1-A)>0\quad(2.20)$$

and

$$A(s,t)\geqslant A^{(\alpha,A)}(s,t)\quad\text{if }\alpha/(1-A)<0.\quad(2.21)$$

Further, both the upper and the lower bound on $A(s,t)$ do exist and are nontrivial. We shall need only the positivity of $\mathrm{Im}a_l$, i.e., $\mathrm{Im}a_l\geqslant0$ for proving this theorem.

Proof. We have

$$\Delta \equiv (k/\sqrt{s})[A(s,t)-A^{(\alpha,A)}(s,t)]$$
$$= \sum_{l\in U(\alpha,A,\theta)} (2l+1)(\mathrm{Im}a_l - \mathrm{Im}a_l^{(0)})P_l(\cos\theta) + \sum_{l\in V(\alpha,A,\theta)} (2l+1)\,\mathrm{Im}a_l P_l(\cos\theta), \quad (2.22)$$

where

$$\mathrm{Im}a_l^{(0)} \equiv [\alpha/(1-A)][P_l(\cos\theta)-A] \quad \text{for } l\in U(\alpha,A,\theta)$$

and

$$\mathrm{Im}a_l^{(0)} \equiv 0 \quad \text{for } l\in V(\alpha,A,\theta). \quad (2.23)$$

Further, using (2.4) and (2.18) we get

$$\sum_{l\in U(\alpha,A,\theta)} (2l+1)\,\mathrm{Im}a_l + \sum_{l\in V(\alpha,A,\theta)} (2l+1)\,\mathrm{Im}a_l = \sum_{l\in U(\alpha,A,\theta)} (2l+1)\,\mathrm{Im}a_l^{(0)}, \quad (2.24)$$

and from (2.5) and (2.19) we get

$$\sum_{l=0}^{\infty} (2l+1)(\mathrm{Re}a_l)^2 + \sum_{l=0}^{\infty} (2l+1)(\mathrm{Im}a_l)^2 = \sum_{l=0}^{\infty} (2l+1)(\mathrm{Im}a_l^{(0)})^2. \quad (2.25)$$

We now eliminate $P_l(\cos\theta)$ in the first sum on the right-hand side of (2.22) by using (2.23) in favor of $\mathrm{Im}a_l^{(0)}$ to obtain

$$\Delta = [(1-A)/2\alpha]\Big\{ \sum_{l\in U(\alpha,A,\theta)} (2l+1)[(\mathrm{Im}a_l)^2 - (\mathrm{Im}a_l^{(0)})^2 - (\mathrm{Im}a_l - \mathrm{Im}a_l^{(0)})^2]\Big\} + A \sum_{l\in U(\alpha,A,\theta)} (2l+1) \times(\mathrm{Im}a_l - \mathrm{Im}a_l^{(0)}) + \sum_{l\in V(\alpha,A,\theta)} (2l+1)\,\mathrm{Im}a_l P_l(\cos\theta).$$

We now use (2.24) and (2.25) here and obtain

$$\Delta = -\frac{1-A}{2\alpha}\Bigg[\sum_{l=0}^{\infty} (2l+1)(\mathrm{Re}a_l)^2 + \sum_{l\in U(\alpha,A,\theta)} (2l+1)(\mathrm{Im}a_l - \mathrm{Im}a_l^{(0)})^2 + \sum_{l\in V(\alpha,A,\theta)} (2l+1)(\mathrm{Im}a_l)^2 - 2\sum_{l\in V(\alpha,A,\theta)} (2l+1)(\mathrm{Im}a_l)\alpha\frac{P_l(\cos\theta)-A}{1-A}\Bigg].$$

The expression inside the square bracket is positive definite. It therefore follows that

$$A(s,t) \geqslant A^{(\alpha,A)}(s,t) \quad \text{if } \alpha/(1-A)<0$$

and

$$A(s,t) \leqslant A^{(\alpha,A)}(s,t) \quad \text{if } \alpha/(1-A)>0.$$

This concludes the proof of the theorem except for a discussion of the existence of solutions of (2.18) and (2.19) for α and A such that both the upper and lower bound do exist and the theorem is nonempty. The relevant result is given by the following lemma.

Lemma 1. Equations (2.18) and (2.19) have solutions for the pair (α,A) if $[(k^2/4\pi)\sigma_{\mathrm{tot}}^2/\sigma_{\mathrm{el}}]\geqslant 1$ with the general properties given by

(i) $\alpha>0,\quad 1\geqslant A>0;$
(ii) $\alpha<0,\quad 0>A\geqslant \min[P_l(\cos\theta)].$

This lemma will be proved in Appendix B.

As mentioned earlier, we need only the positivity of $\mathrm{Im}a_l$ for a proof of Theorem 3. It may be of interest to take into account both the boundedness and positivity, i.e., $1\geqslant \mathrm{Im}a_l \geqslant 0$. We state the improved result as Theorem 4. For this purpose we split the set into two subsets as follows:

Definition 3. The integers $l\in U(\alpha,A,\theta)$ belong to the subset $U_1(\alpha,A,\theta)$ if they satisfy

$$\alpha\left(\frac{P_l(\cos\theta)-A}{1-A}\right)\geqslant 1 \quad \text{for } l\in U_1(\alpha,A,\theta)$$

and to the set $U_2(\alpha,A,\theta)$ if they satisfy

$$1>\alpha\left(\frac{P_l(\cos\theta)-A}{1-A}\right)\geqslant 0 \quad \text{for } l\in U_2(\alpha,A,\theta).$$

Theorem 4. Let

$$\bar{A}^{(\beta,B)}(s,t) = \frac{\sqrt{s}}{k}\Bigg[\sum_{l\in U_1(\beta,B,\theta)} (2l+1)P_l(\cos\theta) + \beta\sum_{l\in U_2(\beta,B,\theta)} (2l+1)P_l(\cos\theta)\left(\frac{P_l(\cos\theta)-B}{1-B}\right)\Bigg], \quad (2.26)$$

where β and B are determined by

$$\frac{k^2\sigma_{\mathrm{tot}}}{4\pi} = \sum_{l\in U_1(\beta,B,\theta)} (2l+1) + \beta\sum_{l\in U_2(\beta,B,\theta)} (2l+1)\left(\frac{P_l(\cos\theta)-B}{1-B}\right) \quad (2.27)$$

and

$$\frac{k^2\sigma_{\mathrm{el}}}{4\pi} = \sum_{l\in U_1(\beta,B,\theta)} (2l+1) + \beta^2\sum_{l\in U_2(\beta,B,\theta)} (2l+1)\left(\frac{P_l(\cos\theta)-B}{1-B}\right)^2. \quad (2.28)$$

Then

$$A(s,t) \leqslant \bar{A}^{(\beta,B)}(s,t) \quad \text{if } \beta/(1-B)>0 \quad (2.29)$$

and

$$A(s,t) \geqslant \bar{A}^{(\beta,B)}(s,t) \quad \text{if } \beta/(1-B)<0. \quad (2.30)$$

It may be noted that Theorem 4 is an improvement over Theorem 3 only if the set $U_1(\beta,B,\theta)$ is non-null. The theorem can be proved in the same way as Theorem 3, so we shall omit the proof.

C. Lower Bounds on Derivatives of $A(s,t)$ at $t=0$

Theorem 5. A lower bound on the nth derivative with respect to t of $A(s,t)$ at $t=0$ is given by

$$\left(\frac{d^n[A(s,t)]}{dt^n}\right)_{t=0} \geqslant \frac{k(\sqrt{s})\sigma_{\text{tot}}}{4\pi} \times\prod_{r=1}^{r=n}\left(R_n-\frac{r(r+1)}{4k^2}\right)\Big/ n!(2n+1) \quad (2.31)$$

for $n\geqslant 1$, where R_n is given by

$$R_n{}^2=\left(\frac{2n+2}{2n+1}\right)\left(\frac{\sigma_{\text{tot}}{}^2}{16\pi\sigma_{\text{el}}}\right)\left(R_n+\frac{n+1}{8k^2}\right). \quad (2.32)$$

For $n=1$ this theorem reduces to the MacDowell-Martin result (1.4) on the "diffraction-peak width." We were able to prove Theorem 5 by a direct subtraction method. This, incidentally, also provides a neater proof of the MacDowell-Martin result.

D. Lower Bounds on $A(s,t)$ in Unphysical Region $(t_0>t>0)$

Martin has already given one such bound, given by (1.2), which involves only σ_{tot}. We here give another involving both σ_{tot} and σ_{el}. The following bound follows from only the positivity of the $\operatorname{Im}a_l(s)$'s. The proof is by a direct subtraction method and will be omitted.

Theorem 6. A lower bound on $A(s,t)$ for $t_0>t>0$ and $(k^2/4\pi)\sigma_{\text{tot}}{}^2/\sigma_{\text{el}}\geqslant 1$ is given by

$$\frac{A(s,t)}{A(s,0)} \geqslant \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]P_l(z)\Big/ \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)], \quad (2.33)$$

where $z=1+t/2k^2$, $[K]$ is the largest positive integer less than or equal to K, and K is determined by the relation

$$\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}{}^2}{\sigma_{\text{el}}} =\Big\{\sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]\Big\}^2\Big/ \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]^2. \quad (2.34)$$

Incidentally, all the sums over l occurring in Theorem 6 can be evaluated in closed form by using

$$\sum_{l=0}^{R} (2l+1)=(R+1)^2,$$

$$\sum_{l=0}^{R} (2l+1)P_l(z)=P_{R+1}'(z)+P_R'(z), \quad (2.35)$$

$$\sum_{l=0}^{R} (2l+1)[P_l(z)]^2=(R+1)^2[P_R(z)]^2 +(1-z^2)[P_R'(z)]^2.$$

We can also give an explicit but somewhat weaker lower bound on $A(s,t)$ using the following lemma.

Lemma 2. Let $S(K,z)$ for $K>0$, $z>1$ be given by

$$S(K,z)=\sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]P_l(z)\Big/ \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)];$$

then $S(K,z)\geqslant S(K',z)$ if $K\geqslant K'$. This lemma is proved in Appendix B.

We also note the inequality, following from Schwartz's inequality applied to the right-hand side of (2.34),

$$\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}{}^2}{\sigma_{\text{el}}} \leqslant ([K]+1)^2. \quad (2.36)$$

Using Lemma 2 and the inequality (2.36), we obtain the following theorem.

Theorem 7. A lower bound on $A(s,t)$ for $t_0>t>0$ is given by

$$\frac{A(s,t)}{A(s,0)} \geqslant \sum_{l=0}^{[K_0]} (2l+1)[P_{K_0}(z)-P_l(z)]P_l(z)\Big/ \sum_{l=0}^{[K_0]} (2l+1)[P_{K_0}(z)-P_l(z)], \quad (2.37)$$

where

$$z=1+t/2k^2$$

and

$$(K_0+1)^2=\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}{}^2}{\sigma_{\text{el}}}. \quad (2.38)$$

Notice that Theorem 7 does not have the restriction $(k^2/4\pi)\sigma_{\text{tot}}{}^2/\sigma_{\text{el}}\geqslant 1$. The improvement comes about as follows. We first prove a lower bound on $[A(s,t)/A(s,0)]$ which is given by (2.37) and (2.38), except that $K_0\to K_0'$ and $\sigma_{\text{el}}\to\sigma_{\text{el,im}}$, where

$$\sigma_{\text{el,im}}\equiv\frac{4\pi}{k^2}\sum_{l=0}^{\infty} (2l+1)(\operatorname{Im}a_l)^2\leqslant\sigma_{\text{el}}. \quad (2.39)$$

We then show that use of the inequality $\sigma_{\mathrm{el}} \geqslant \sigma_{\mathrm{el,im}}$ allows us to replace $\sigma_{\mathrm{el,im}}$ by σ_{el} since it only worsens the bound.

Finally we state without proof the theorem on the lower bound on $A(s,t)$ in unphysical region when one takes into account $1 \geqslant \mathrm{Im}\,a_l \geqslant 0$.

Theorem 8. For $t_0 > t > 0$, we have

$$\frac{k}{\sqrt{s}} A(s,t) \geqslant \sum_{l=0}^{[K_1]} (2l+1) P_l(z) + \alpha \sum_{[K_1]+1}^{[K]} (2l+1) \left[\frac{P_K(z) - P_l(z)}{P_K(z) - 1}\right] P_l(z), \quad (2.40)$$

with

$$\frac{k^2 \sigma_{\mathrm{tot}}(s)}{4\pi} = \sum_{l=0}^{[K_1]} (2l+1) + \alpha \sum_{l=[K_1]+1}^{[K]} (2l+1) \left[\frac{P_K(z) - P_l(z)}{P_K(z) - 1}\right] \quad (2.41)$$

and

$$\frac{k^2 \sigma_{\mathrm{el}}}{4\pi} = \sum_{l=0}^{[K_1]} (2l+1) + \alpha^2 \sum_{l=[K_1]+1}^{[K]} (2l+1) \left[\frac{P_K(z) - P_l(z)}{P_K(z) - 1}\right]^2, \quad (2.42)$$

where

$$\alpha \frac{P_K(z) - P_{K_1}(z)}{P_K(z) - 1} = 1 \quad (2.43)$$

and

$$z = 1 + t/2k^2 .$$

III. HIGH-ENERGY AND LOW-MOMENTUM-TRANSFER BEHAVIOR OF UNITARITY BOUNDS

It is of practical interest to evaluate the upper bound given by Theorem 3 as a function of σ_{tot}, σ_{el}, and t in the diffraction-peak region. In order to illustrate the procedure, we shall first concentrate on the simpler case of the upper bound given by Theorem 1, and then quote the diffraction-peak-region results for Theorem 3. We also note the high-energy limit of Theorem 5, which again may be of practical use, and of Theorem 7, which has a theoretical interest.

A. Upper Bound on $A(s,t)$ given by Theorem 1 in Diffraction-Peak Region

As mentioned earlier, $P_l(\cos\theta)$, $\pi > \theta > 0$, is an oscillating function of l for a given value of θ (see Fig. 1). We have a number of extremum points given by

$$\partial P_l(\cos\theta)/\partial l = 0 .$$

Let us order the set of extremum values of $P_l(\cos\theta)$ as follows:

$$P_{L_0}(\cos\theta) \geqslant P_{L_1}(\cos\theta) \geqslant P_{L_2}(\cos\theta) \geqslant \cdots .$$

Let us now consider the various possible cases.

1. $P_L(\cos\theta) > P_{L_0}(\cos\theta)$

The set $U(L,\theta)$ now becomes the set $(L-1) \geqslant l \geqslant 0$ and the set $V(L,\theta)$ contains only $l = L$. Therefore,

$$\frac{k}{\sqrt{s}} A(s,t) \leqslant \frac{k^2 \sigma_{\mathrm{tot}}}{4\pi} P_L(\cos\theta) + \sum_{l=0}^{L-1} (2l+1) \times [P_l(\cos\theta) - P_L(\cos\theta)] = P_{L-1}{}'(\cos\theta) + P_L{}'(\cos\theta) + P_L(\cos\theta)\left(\frac{k^2\sigma_{\mathrm{tot}}}{4\pi} - L^2\right), \quad (3.1)$$

where the positive integer L is determined by

$$\frac{k^2 \sigma_{\mathrm{tot}}}{4\pi} = \sum_{l=0}^{L-1} (2l+1) + \xi(2L+1) , \quad (3.2)$$

with

$$1 > \xi \geqslant 0 ,$$

i.e.,

$$L = [(k^2\sigma_{\mathrm{tot}}/4\pi)^{1/2}] \equiv \text{the largest integer equal to or less then } (k^2\sigma_{\mathrm{tot}}/4\pi)^{1/2} . \quad (3.3)$$

The bounds (3.1) and (3.3) are exact as long as the condition $P_L(\cos\theta) > P_{L_0}(\cos\theta)$ is satisfied. We now take the high-energy diffraction-peak limit, i.e.,

$$k \to \infty , \quad \theta \to 0 , \quad t = -2k^2(1 - \cos\theta) = \text{finite} .$$

In this limit, the Legendre functions can be replaced by Bessel functions,[10] and we obtain

$$\frac{A(s,t)}{A(s,0)} \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\leqslant} \frac{J_1([(-t)\sigma_{\mathrm{tot}}(s)/4\pi]^{1/2})}{\frac{1}{2}[(-t)\sigma_{\mathrm{tot}}(s)/4\pi]^{1/2}} , \quad (3.4)$$

if

$$3.46 \geqslant (-t)\sigma_{\mathrm{tot}}(s)/4\pi , \quad (3.5)$$

where (3.5) comes from the condition

$$P_L(\cos\theta) > P_{L_0}(\cos\theta) .$$

We could have obtained this final answer more directly by using the approximations

$$P_l(\cos\theta) \approx J_0((2l+1)\sin\tfrac{1}{2}\theta) ,$$

$$\sum \approx \int dl ,$$

[10] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York, 1954), p. 72.

and ignoring the pedantic introduction of the fraction ξ and simultaneously allowing L to assume nonintegral values. Then we have

$$\frac{k}{\sqrt{s}}A(s,t) \underset{\substack{s\to\infty\\ t\text{ fixed}}}{\leqslant} \int_0^L dl\,(2l+1)J_0((2l+1)\sin\tfrac{1}{2}\theta),$$

$$\frac{k^2\sigma_{\text{tot}}}{4\pi}=\int_0^L (2l+1)dl.$$

This leads to (3.4).

If $(-t)\sigma_{\text{tot}}$ does not lie in the range (3.5), then we have to consider the cases $P_{L_0}(\cos\theta)\geqslant P_L(\cos\theta)$.

2. $P_{L_0}(\cos\theta)\geqslant P_L(\cos\theta)>P_{L_1}(\cos\theta)$

The set $U(L,\theta)$ now consists of two pieces given by $0\leqslant l\leqslant L_{(1)}{}^{(2)}$ and $L_{(2)}{}^{(2)}\leqslant l\leqslant L_{(3)}{}^{(2)}$ satisfying $P_l(\cos\theta)>P_L(\cos\theta)$. Again the diffraction-peak limit is given by

$$\frac{k}{\sqrt{s}}A(s,t) \underset{\substack{s\to\infty\\ t\text{ fixed}}}{\leqslant} \left[\int_0^{L_{(1)}{}^{(2)}} dl(2l+1)J_0((2l+1)\sin\tfrac{1}{2}\theta) + \int_{L_{(2)}{}^{(2)}}^{L_{(3)}{}^{(2)}} dl(2l+1)J_0((2l+1)\sin\tfrac{1}{2}\theta)\right]. \quad (3.6)$$

Here

$$\frac{k^2\sigma_{\text{tot}}}{4\pi}=\int_0^{L_{(1)}{}^{(2)}} dl(2l+1)+\int_{L_{(2)}{}^{(2)}}^{L_{(3)}{}^{(2)}} dl(2l+1), \quad (3.7)$$

where we allow $L_{(i)}{}^{(2)}$ $(i=1, 2, 3)$ to assume nonintegral values and

$$J_0((2L_0+1)\sin\tfrac{1}{2}\theta)\geqslant J_0((2L_{(i)}{}^{(2)}+1)\sin\tfrac{1}{2}\theta) = J_0((2L+1)\sin\tfrac{1}{2}\theta)>J_0((2L_1+1)\sin\tfrac{1}{2}\theta).$$

This case covers the range, putting in numbers,

$$25.0\geqslant(-t)\sigma_{\text{tot}}(s)/4\pi. \quad (3.8)$$

We are now ready to give the answer for the general case in the diffraction-peak limit.

3. General case

Denoting by λ_i and λ_i' the values of $(2l+1)\sin\frac{1}{2}\theta$ at the beginning and end, respectively, of the various pieces in the l summation, we have

$$\frac{A(s,t)}{A(s,0)} \underset{\substack{s\to\infty\\ t\text{ fixed}}}{\leqslant} \sum_i \int_{\lambda_i}^{\lambda_i'} d\lambda\,\lambda J_0(\lambda) \Big/ \sum_i \int_{\lambda_i}^{\lambda_i'} d\lambda\,\lambda, \quad (3.9)$$

where

$$0=\lambda_0\leqslant\lambda_0'\leqslant\lambda_1\leqslant\lambda_1'\leqslant\lambda_2\leqslant\lambda_2'\cdots, \quad (3.10)$$

$$J_0(\lambda_i)=J_0(\lambda_i')=J_0(\lambda_0'), \quad \text{for all } i\neq 0, \quad (3.11)$$

and

$$\frac{(-t)\sigma_{\text{tot}}(s)}{4\pi}=2\sum_i\int_{\lambda_i}^{\lambda_i'}\lambda d\lambda. \quad (3.12)$$

The number of pieces i necessary in the summations occurring in (3.9) and (3.12) depends on the value of the left-hand side of (3.12).

B. Upper and Lower Bounds Given by Theorem 3 in Diffraction-Peak Region

Having illustrated the procedure, we now just quote the final results in this case. In the diffraction-peak region,

$$\frac{A(s,t)}{A(s,0)}\leqslant J_0(\mu_0')+\frac{1}{\rho}\sum_i\left[\mu_i'^2J_2(\mu_i')-\mu_i^2J_2(\mu_i)\right], \quad (3.13)$$

where

$$0=\mu_0\leqslant\mu_0'\leqslant\mu_1\leqslant\mu_1'\leqslant\mu_2\leqslant\mu_2', \quad (3.14)$$

$$J_0(\mu_i)=J_0(\mu_i')=J_0(\mu_0') \quad \text{for all } i\neq 0, \quad (3.15)$$

and

$$\rho=\frac{\{\sum_i[\mu_i'^2J_2(\mu_i')-\mu_i^2J_2(\mu_i)]\}^2}{\sum_i\{[\mu_i'J_1(\mu_i')]^2-[\mu_iJ_1(\mu_i)]^2\}-2J_0(\mu_0')[\sum_i\mu_i'^2J_2(\mu_i')-\mu_i^2J_2(\mu_i)]}, \quad (3.16)$$

where

$$\rho\equiv(-t)\sigma_{\text{tot}}{}^2(s)/4\pi\sigma_{\text{el}}. \quad (3.17)$$

For the range $0\leqslant\rho\leqslant 2.5$, the above relations reduce to

$$\frac{A(s,t)}{A(s,0)}\leqslant J_0(\mu_0')+\frac{\mu_0'^2J_2(\mu_0')}{\rho}, \quad (3.18)$$

with

$$\rho=\frac{[J_2(\mu_0')]^2(\mu_0')^2}{[J_1(\mu_0')]^2-2J_0(\mu_0')J_2(\mu_0')}. \quad (3.19)$$

More explicitly, we get

$$\frac{A(s,t)}{A(s,0)}\leqslant 1-\frac{\rho}{9}+\frac{3}{8}\left(\frac{\rho}{9}\right)^2-\frac{21}{320}\left(\frac{\rho}{9}\right)^3+\frac{43}{1280}\left(\frac{\rho}{9}\right)^4\cdots$$
$$\text{for } 0\leqslant\rho\leqslant 2.5, \quad (3.20)$$

a very simple formula. For $8.42\geqslant\rho>2.5$, one need only keep two terms corresponding to $i=0$ and $i=1$ in the summations in (3.13) and (3.16), and again we have to do some very elementary calculations.

TABLE I. Upper bounds on the imaginary part $A(s,t)$ and its square $A^2(s,t)$ in the diffraction-peak region as a function of $\rho=(-t)\sigma_{tot}^2(s)/4\pi\sigma_{el}(s)$.

ρ	Upper bound on $A(s,t)/A(s,0)$	Upper bound on $[A(s,t)/A(s,0)]^2$	ρ	Upper bound on $A(s,t)/A(s,0)]$	Upper bound on $[A(s,t)/A(s,0)]^2$
0	1.000	1.000	4.50	0.630	0.397
0.25	0.972	0.945	4.75	0.620	0.385
0.50	0.945	0.894	5.00	0.610	0.372
0.75	0.919	0.845	5.25	0.601	0.361
1.00	0.893	0.798	5.50	0.593	0.352
1.25	0.869	0.755	5.75	0.586	0.343
1.50	0.846	0.715	6.00	0.578	0.334
1.75	0.821	0.674	6.25	0.572	0.327
2.00	0.795	0.632	6.50	0.566	0.320
2.25	0.771	0.594	6.75	0.559	0.313
2.50	0.749	0.561	7.00	0.554	0.307
2.75	0.730	0.533	7.25	0.548	0.300
3.00	0.710	0.504	7.50	0.543	0.295
3.25	0.692	0.479	7.75	0.538	0.289
3.50	0.677	0.458	8.00	0.533	0.284
3.75	0.664	0.441	8.25	0.527	0.278
4.00	0.652	0.425	8.42	0.523	0.274
4.25	0.640	0.410			

The contribution of the absorptive part to the differential cross section is proportional to $A^2(s,t)$ and hence, for comparison with experiment, we need an upper bound on $A^2(s,t)$ and not just on $A(s,t)$. For this purpose we have to evaluate also the lower bound on $A(s,t)$ given by Theorem 3. The calculation proceeds exactly analogously. We find that in the region $0\leqslant\rho\leqslant 8.42$, the magnitude of the upper bound on $A(s,t)$ is larger than the magnitude of the lower bound. Hence the upper bound on $A^2(s,t)$ in this region is obtained simply by squaring the upper bound on $A(s,t)$. The values obtained are tabulated in Table I. For comparison we may mention that the square of the lower bound on $A(s,t)/A(s,0)$ goes from 0.162 to 0.138 monotonically as ρ goes from 0 to 11.5.

C. High-Energy Limits of Bounds Given by Theorems 5 and 7

From Theorem 5 we obtain

$$\left(\frac{d^nA(s,t)}{dt^n}\right)_{t=0} \underset{s\to\infty}{\geqslant} \frac{k(\sqrt{s})\sigma_{tot}}{4\pi}\frac{1}{n!(2n+1)!} \times\left(\left(\frac{2n+2}{2n+1}\right)\frac{\sigma_{tot}^2}{16\pi\sigma_{el}}\right)^n. \quad (3.21)$$

This result may be of practical interest in limiting the values of the coefficients $a, b,\ldots$ in the fits of the form $\exp(at+bt^2\cdots)$ to the differential cross section in the diffraction-peak region.

An amusing result follows by combining the high-energy limit of Theorem 7 with the polynomial boundedness of $A(s,t)$ within the Lehmann-Martin ellipse. Theorem 7 yields

$$\frac{A(s,t)}{A(s,0)} \underset{\substack{s\to\infty\\ t\text{ fixed}}}{\geqslant} \frac{I_1^2([\sigma_{tot}^2t/4\pi\sigma_{el}]^{1/2})-I_0([\sigma_{tot}^2t/4\pi\sigma_{el}]^{1/2})I_2([\sigma_{tot}^2t/4\pi\sigma_{el}]^{1/2})}{I_2([\sigma_{tot}^2t/4\pi\sigma_{el}]^{1/2})},\quad t_0\geqslant t>0. \quad (3.22)$$

On using

$$S^N \underset{s\to\infty}{\geqslant} A(s,t) \quad \text{for } t_0\geqslant t>0,$$

we obtain

$$(4\pi/t_0)(N-1)^2(\ln s)^2 \underset{s\to\infty}{\geqslant} \sigma_{tot}^2/\sigma_{el}. \quad (3.23)$$

This result is stronger than the earlier known result[1,3]

$$(4\pi/t_0)(N-1)^2(\ln s)^2 \underset{s\to\infty}{\geqslant} \sigma_{tot}(s)$$

but weaker than the result (1.5), which we have established recently.[8]

IV. COMPARISON WITH EXPERIMENTAL DATA

We would now like to compare the unitarity upper bounds on the absorptive part given by Theorem 3 and tabulated in Table I with the available experimental data in the diffraction-peak region. Since (i) it is the differential cross section and not the absorptive part which is directly measured and (ii) no precise data are available for the scattering of spin-zero particles for which our bounds apply, we are forced to make the following two approximations.

(a) We neglect the real part of the amplitude in comparison with the imaginary part in the diffraction-peak region. In this approximation,

$$\left[\frac{A(s,t)}{A(s,0)}\right]^2 \approx \frac{(d\sigma/dt)}{(d\sigma/dt)_{t=0}}, \quad (4.1)$$

and ρ given by (3.17) is approximated by

$$\rho\approx -4t\frac{(d\sigma/dt)_{t=0}}{\sigma_{el}}. \quad (4.2)$$

(b) We neglect the spin dependence of the unpolarized cross sections in the diffraction-peak region. This enables us to assume that our results derived for the scattering of spinless particles hold also for the unpolarized cross sections for the scattering of particles with spin.

For a discussion of the experimental validity of these two popular approximations, we refer the reader to the paper of MacDowell and Martin.[4]

Under these approximations the theoretical upper bound on the curve of $[A(s,t)/A(s,0)]^2$ versus ρ

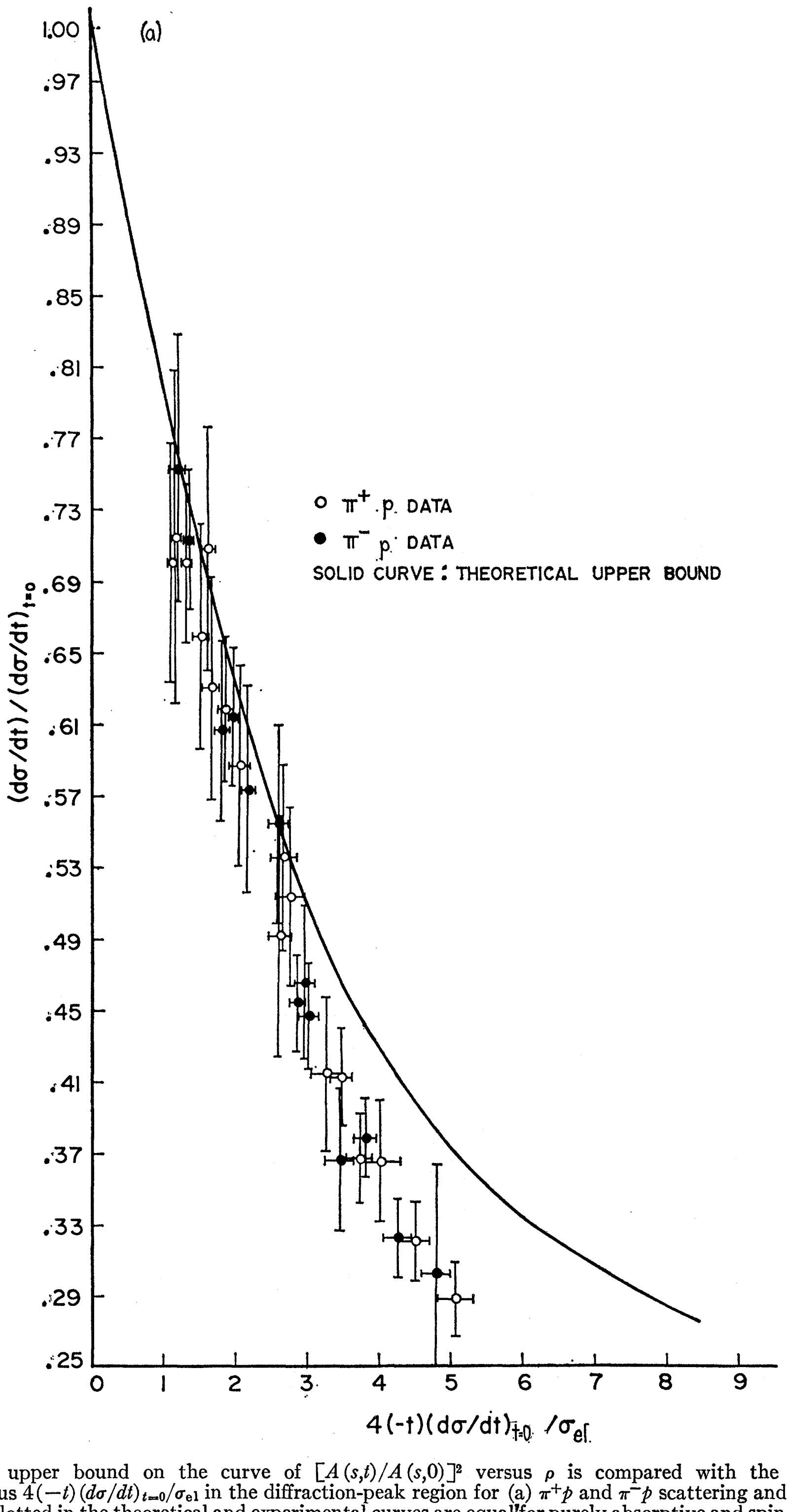

FIG. 2. Theoretical upper bound on the curve of $[A(s,t)/A(s,0)]^2$ versus ρ is compared with the experimental curve of $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ versus $4(-t)(d\sigma/dt)_{t=0}/\sigma_{el}$ in the diffraction-peak region for (a) π^+p and π^-p scattering and for (b) pp and $\bar{p}p$ scattering. The quantities plotted in the theoretical and experimental curves are equal for purely absorptive and spin-independent scattering. We use the data of Foley *et al.* (Ref. 11) for π^+p scattering at lab momenta 6.8, 8.8, 10.8, and 12.8 GeV/c, π^-p scattering at 7.0, 8.9, 10.8, and 13.0 GeV/c, pp scattering at 6.8, 8.8, 10.8, and 12.8 GeV/c, and $\bar{p}p$ scattering at 7.2, 8.9, and 12.0 GeV/c.

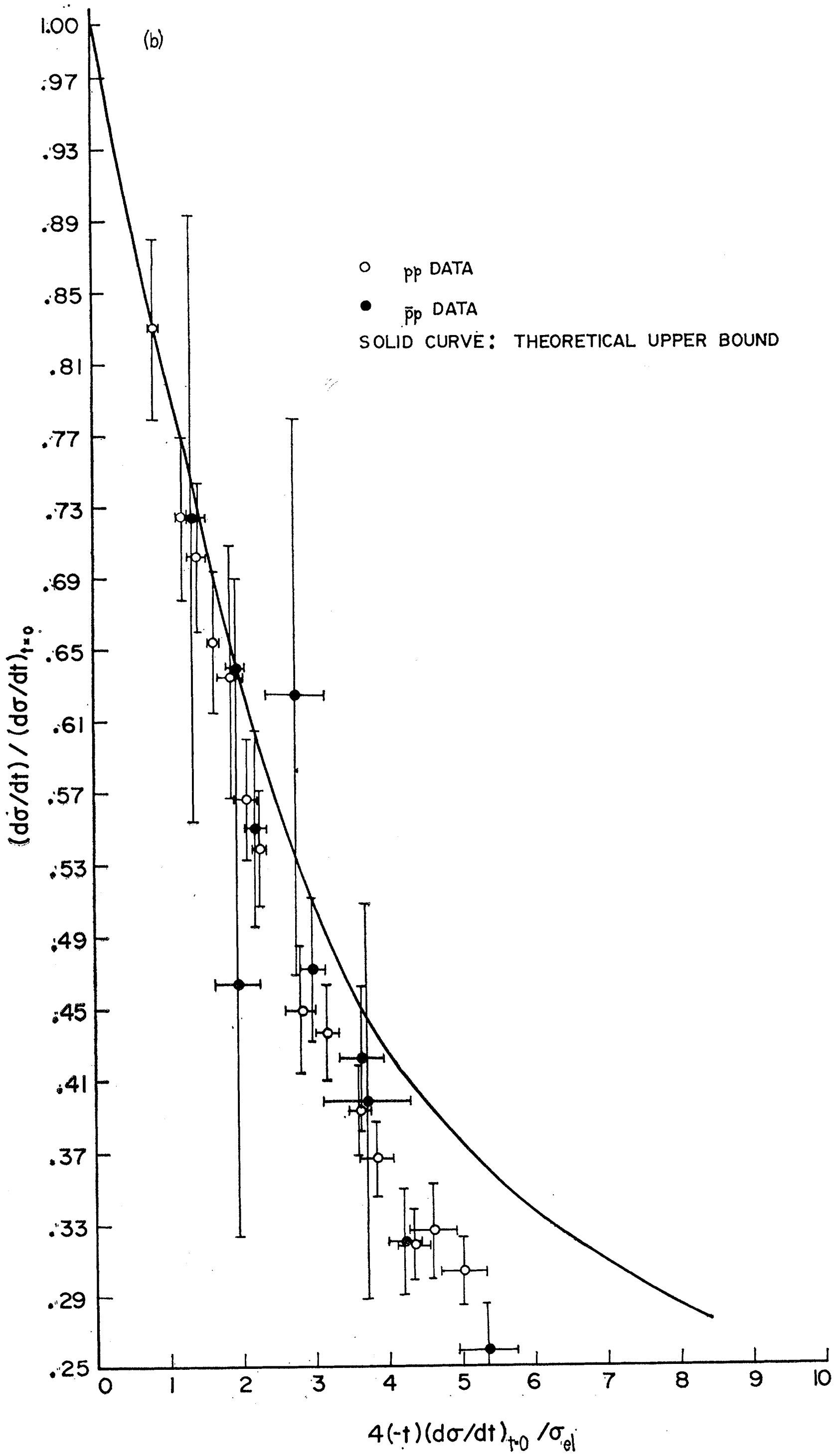

FIG. 2. (*Continued*)

can be compared with the experimental curve of $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ versus $4(-t)(d\sigma/dt)_{t=0}/\sigma_{\rm el}$, since the quantities plotted in the two curves are equal according to (4.1) and (4.2). Since the theoretical upper bound has the "universality" feature of depending only through the parameter ρ on the particular scattering process and the energy and momentum transfer, there is an enormous amount of data to be compared with a single theoretical point. To avoid overcrowding, the same theoretical curve has been plotted in Figs. 2(a) and 2(b) and the experimental points for π^+p and π^-p scattering shown in Fig. 2(a), for pp and $\bar{p}p$ scattering in Fig. 2(b). We have chosen the data of Foley *et al.*[11] in the lab momentum range 6–13 GeV/c.

We notice two striking facts. First, the theoretical upper-bound curve is very close to the experimental results for all these processes, the difference being less than 10% for ρ in the range (0,3) and less than 25% for ρ in the range (3,5). This means that in the part of the diffraction peak in which the differential cross section is greater than one-third of its value in the forward direction, it is substantially determined by the unitarity upper bound, given σ_{el} and $(d\sigma/dt)_{t=0}$. Secondly, we discover a somewhat unforeseen fact. The experimental results for these various processes, at the various energies and momentum transfers noted, fall on a universal curve lying slightly below the universal theoretical upper-bound curve. We are intrigued by the following question: Does this point to a special virtue of the variable ρ?

APPENDIX A: PROOF THAT THEOREM 1 IMPROVES THE MARTIN UPPER BOUND

We wish to establish by direct subtraction that the upper bound (1.1) on $A(s,t)$ given by Martin is weaker than the upper bound given by Theorem 1 for $A(s,t)$. Let

$$B_l(\cos\theta)\equiv[1+l(l+1)\sin^2\theta]^{-1/4}\geqslant|P_l(\cos\theta)|\,. \quad \text{(A1)}$$

The bound (1.1) is slightly weaker than the following Martin bound:

$$A(s,t)\leqslant\frac{\sqrt{s}}{k}\Big[\sum_{l=0}^{N_1}(2l+1)B_l(\cos\theta)+(2N_1+3)\epsilon_1B_{N_1+1}(\cos\theta)\Big]\equiv A_M(s,t)\,, \quad \text{(A2)}$$

where

$$\frac{k^2\sigma_{\text{tot}}}{4\pi}=\sum_{l=0}^{N_1}(2l+1)+\epsilon_1(2N_1+3)\,, \quad \text{(A3)}$$

with N_1 an integer and $1>\epsilon_1\geqslant0$. The bound given by Theorem 1 can be expressed as

$$A(s,t)\leqslant[(\sqrt{s})/k]\Big[\sum_{P_l(\cos\theta)>P_L(\cos\theta)}(2l+1)P_l(\cos\theta)+\xi\sum_{P_l(\cos\theta)=P_L(\cos\theta)}(2l+1)P_L(\cos\theta)\Big]\equiv A_P(s,t)\,, \quad \text{(A4)}$$

where

$$k^2\sigma_{\text{tot}}/4\pi=\sum_{P_l>P_L}(2l+1)+\xi\sum_{P_l=P_L}(2l+1)\,,\quad 1>\xi\geqslant0\,. \quad \text{(A5)}$$

We then have

$$\begin{aligned}\delta\equiv(k/\sqrt{s})[A_M(s,t)-A_P(s,t)]={}&\sum_{\substack{P_l(\cos\theta)>P_L(\cos\theta)\\ l\leqslant N_1}}(2l+1)[B_l(\cos\theta)-P_l(\cos\theta)]\\ &+\sum_{\substack{l\leqslant N_1\\ P_l(\cos\theta)\leqslant P_L(\cos\theta)}}(2l+1)B_l(\cos\theta)+(2N_1+3)\epsilon_1B_{N_1+1}(\cos\theta)-\sum_{\substack{P_l(\cos\theta)>P_L(\cos\theta)\\ l>N_1}}(2l+1)P_l(\cos\theta)\\ &-\xi\sum_{P_l(\cos\theta)=P_L(\cos\theta)}(2l+1)P_l(\cos\theta)\geqslant\sum_{\substack{l\leqslant N_1\\ P_l(\cos\theta)\leqslant P_L(\cos\theta)}}(2l+1)B_l(\cos\theta)+(2N_1+3)\epsilon_1B_{N_1+1}(\cos\theta)\\ &-B_{N_1+1}(\cos\theta)\sum_{\substack{P_l(\cos\theta)>P_L(\cos\theta)\\ l>N_1}}(2l+1)-\xi\sum_{P_l(\cos\theta)=P_L(\cos\theta)}(2l+1)P_l(\cos\theta)\,.\end{aligned} \quad \text{(A6)}$$

In (A6) we substitute for the third term on the right-hand side using (A3) and (A5) and obtain

$$\begin{aligned}\delta\geqslant{}&\sum_{\substack{l\leqslant N_1\\ P_l(\cos\theta)\leqslant P_L(\cos\theta)}}(2l+1)[B_l(\cos\theta)-B_{N_1+1}(\cos\theta)]+\xi\sum_{P_l(\cos\theta)=P_L(\cos\theta)}(2l+1)[B_{N_1+1}(\cos\theta)-P_l(\cos\theta)]\\ \geqslant{}&\sum_{\substack{l\leqslant N_1\\ P_l(\cos\theta)=P_L(\cos\theta)}}(2l+1)[B_l(\cos\theta)-B_{N_1+1}(\cos\theta)]+\xi\sum_{\substack{P_l(\cos\theta)=P_L(\cos\theta)\\ l\leqslant N_1}}(2l+1)[B_{N_1+1}(\cos\theta)-B_l(\cos\theta)]\\ ={}&\sum_{\substack{l\leqslant N_1\\ P_l(\cos\theta)=P_L(\cos\theta)}}(2l+1)[B_l(\cos\theta)-B_{N_1+1}(\cos\theta)](1-\xi)\geqslant0\,.\end{aligned} \quad \text{(A7)}$$

This completes the proof that the present upper bound is better than the Martin upper bound.

[11] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **11**, 503 (1963).

APPENDIX B: PROOF OF NEEDED LEMMAS

Proof of Lemma 1

Let

$$C_l \equiv \frac{\alpha[P_l(\cos\theta)-A]}{1-A} \geqslant 0 \quad \text{for } l\in U(\alpha,A,\theta). \tag{B1}$$

We then have

$$k^2\sigma_{\text{tot}}/4\pi = \sum_{C_l\geqslant 0} (2l+1)C_l, \tag{B2}$$

$$k^2\sigma_{\text{el}}/4\pi = \sum_{C_l\geqslant 0} (2l+1)C_l^2. \tag{B3}$$

It follows that

$$\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}^2}{\sigma_{\text{el}}} = [\sum_{C_l\geqslant 0} (2l+1)C_l]^2 / \sum_{C_l\geqslant 0} (2l+1)C_l^2. \tag{B4}$$

The right-hand side of (B4) is larger than or equal to 1. Therefore we can have a solution of Eqs. (B2) and (B3), i.e., of Eqs. (2.18) and (2.19), only if

$$\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}^2}{\sigma_{\text{el}}} \geqslant 1. \tag{B5}$$

The origin of the condition (B5) is thus clear.

Let us now discuss the cases of upper and lower bounds separately.

(*i*) *Upper bound.* We need

$$\alpha/(1-A)>0,$$

and therefore

$$P_l(\cos\theta)\geqslant A \quad \text{for } l\in U(\alpha,A,\theta).$$

Therefore we must have $1\geqslant A$. Further, A must satisfy $A>0$; otherwise σ_{tot}, given by

$$\frac{k^2\sigma_{\text{tot}}}{4\pi} = \left(\frac{\alpha}{1-A}\right) \sum_{P_l(\cos\theta)>A} (2l+1)[P_l(\cos\theta)-A],$$

cannot be finite. Therefore, if a solution for the pair (α,A) exists which assures the existence of an upper bound, it must have

$$\alpha>0, \quad 1\geqslant A>0. \tag{B6}$$

Let us now look at Eqs. (2.18) and (2.19) for this range of parameters. We then have

$$\frac{k^2}{4\pi}\frac{\sigma_{\text{tot}}^2}{\sigma_{\text{el}}} = \{\sum_{P_l(\cos\theta)\geqslant A} (2l+1)[P_l(\cos\theta)-A]\}^2 / \sum_{P_l(\cos\theta)\geqslant A} (2l+1)[P_l(\cos\theta)-A]^2 \tag{B7}$$

and

$$\frac{\alpha}{1-A} = \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \sum_{P_l(\cos\theta)\geqslant A} (2l+1)[P_l(\cos\theta)-A] / \sum_{P_l(\cos\theta)\geqslant A} (2l+1)[P_l(\cos\theta)-A]^2. \tag{B8}$$

Equation (B8) obviously corresponds to $\alpha>0$ if Eq. (B7) does have a solution with $1\geqslant A>0$. The right-hand side of (B7) is easily seen to be a continuous and monotonic function of A. It takes the value 1 as $A\to 1$ and tends to $+\infty$ as $A\to 0+$. We therefore always have a required solution if the condition (B5) is satisfied.

(*ii*) *Lower bound.* A similar discussion can be given for this case also.

Proof of Lemma 2

The $S(K,z)$, $(K>0, z>1)$, is given by

$$S(K,z) = \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]P_l(z) / \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)].$$

Now let $K\geqslant K'$. Then

$$S(K,z)\geqslant S(K',z)$$

if

$$\sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)]P_l(z) / \sum_{l=0}^{[K]} (2l+1)[P_K(z)-P_l(z)] \geqslant \sum_{l=0}^{[K']} (2m+1)[P_{K'}(z)-P_m(z)]P_m(z) / \sum_{l=0}^{[K']} (2m+1)[P_{K'}(z)-P_m(z)],$$

i.e., if

$$\sum_{l=0}^{[K]} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_K(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]\geqslant 0$$

or

$$\sum_{l=0}^{[K']} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_K(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)] + \sum_{l=[K']+1}^{[K]} \sum_{m=0}^{[K']} (2l+1)(2m+1)[P_K(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]\geqslant 0. \tag{B9}$$

Now using

$$P_l(z)\geqslant P_{l'}(z) \quad \text{if } l\geqslant l' \quad \text{and} \quad z\geqslant 1,$$

we see that the second term on the left-hand side of (B9) is positive. Let us therefore concentrate on the first term on the left-hand side of (B9). Now

$$\sum_{l=0}^{[K']}\sum_{m=0}^{[K']}(2l+1)(2m+1)[P_K(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$=\sum_{l=0}^{[K']}\sum_{m=0}^{[K']}(2l+1)(2m+1)[P_K(z)-P_{K'}(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$+\sum_{l=0}^{[K']}\sum_{m=0}^{[K']}(2l+1)(2m+1)[P_{K'}(z)-P_l(z)][P_{K'}(z)-P_m(z)][P_l(z)-P_m(z)]$$

$$=[P_K(z)-P_{K'}(z)]\sum_{l=0}^{[K']}\sum_{m=0}^{[K']}(2l+1)(2m+1)[P_m^2(z)-P_l(z)P_m(z)],$$

and this is easily seen to be positive by using $P_K(z)\geqslant P_{K'}(z)$ and Schwartz's inequality. This proves Lemma 2

PHYSICAL REVIEW D VOLUME 1, NUMBER 9 1 MAY 1970

Applications of Current Commutation Relations to Muon Capture and Neutrino (Antineutrino) Reactions in Nuclei

C. W. Kim*
Department of Physics, The Johns Hopkins University, Baltimore, Maryland 21218

AND

Michael Ram†
Department of Physics, State University of New York, Buffalo, New York 14214

(Received 17 November 1969)

Relations among total muon-capture rates in nuclei and the equal-time commutators of the space and time components of the strangeness-conserving weak hadron current are derived. Using the quark field algebra and the closure approximation, this relation yields a total muon-capture rate in He^3 of $\Gamma(He^3)\approx 2.36\times 10^3\ \text{sec}^{-1}$, in very good agreement with experiment. We demonstrate that application of the gauge field algebra to our relations does not yield a result that can be compared with experiment, since we cannot justify the use of the closure approximation in the context of this algebra. Using the quark field algebra and the closure approximation, similar relations are also derived for the total "elastic" differential cross section for forward scattering of neutrinos off nuclei.

I. INTRODUCTION

In a recent series of Letters,[1–3] we have discussed the application of the Gell-Mann algebra of currents[4] to the calculation of the total muon-capture rate in complex nuclei as well as the derivation of relations between cross sections for elastic neutrino and antineutrino scattering by nuclei.

The purpose of the present paper is twofold:

(i) To reproduce in all detail the derivation of the results quoted in Refs. 1–3. This was not done in our previous brief communications and is essential for the proper understanding of our results.

(ii) To investigate how results are modified when the quark field algebra[4] is replaced by the gauge field algebra.[5]

* Research supported in part by a grant from the National Science Foundation.

† Research supported in part by grants from the National Science Foundation and from the Research Foundation of the State of New York.

[1] C. W. Kim and M. Ram, Phys. Rev. Letters **18**, 508 (1967).

[2] C. W. Kim and M. Ram, Phys. Rev. Letters **20**, 35 (1968).

[3] C. W. Kim and M. Ram, Phys. Letters **27B**, 306 (1968).

[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967); T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).