Chiral $SU(3)\otimes SU(3)$ Mass Spectra and the Generalized "$\omega\rho\pi$" Interaction*

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We discuss the interaction between a pseudoscalar and two vector mesons within the context of chiral $SU(3)\otimes SU(3)$ symmetry. Assuming that the interaction transforms like a combination of zeroth and eighth components of a $(3, 3^*)\oplus(3^*, 3)$ representation, we are able, in a phenomenological Lagranigan treatment, to reproduce a previous $SU(3)$ parametrization of the related meson decay rates. In addition we obtain interaction terms for the decays of the X^0 meson. We study also the mass spectrum of the vector, axial-vector, and pseudoscalar mesons when these last belong to a nonlinear representation and when they are part of a linear representation which includes scalar mesons. In the latter case we find that the κ-K^* mixing is negligible.

I. INTRODUCTION

ABOUT one year ago we discussed[1] the interaction vertex involving two spin-1 mesons and one spin-zero meson in the framework of broken chiral $SU(2)\otimes SU(2)$ symmetry, using the effective Lagrangian approach. This vertex, which we shall here call "$\omega\rho\pi$" interaction, plays an important role in the vector-dominance picture[2–4] of production and decay processes of mesons. The generalization to broken chiral $SU(3)\otimes SU(3)$ is, however, not very simple to do, since it involves the fixing of parameters in a so-called[5] "super-Lagrangian," which contains nonets of spin-zero and spin-1 fields of both parities. While the choice of which terms to retain in the Lagrangian is (as always) arbitrary, our criterion of simplicity leads us to keep all chiral symmetric terms leading to bilinear and trilinear terms in the fields, and similarly for the symmetry-breaking part, which transforms as a $(3, 3^*)\oplus(3^*, 3)$ representation[6] of chiral $SU(3)\otimes SU(3)$. The problem of determining the parameters in the chiral Lagrangian and the breaking parameters, and consequent renormalization of the fields, in terms of the mass spectrum and partially conserved axial-vector current (PCAC) conditions turns out to be the main part of this work.

The formulation of the "$\omega\rho\pi$" interaction in the framework of chiral symmetry has been considered by several authors[7–9] who use different approaches and reach somewhat different conclusions. All authors are in agreement, however, that either the chiral field algebra[10] or strict PCAC must be relinquished in formulating $\omega\rho\pi$. Accordingly, in Ref. 9 a phenomenological model of PCAC breakdown is proposed and leads to the GMSW-type interaction. In Refs. 7 and 8, however, it is claimed that a stronger momentum dependence than is contained in GMSW follows from the requirements of current algebra, even when field algebra and strict PCAC are abandoned. This strong momentum dependence is not consistent with the experimental situation. Subsequently we examined the $\omega\rho\pi$ interaction in a convenient $O(4)$ formulation of chiral $SU(2)\otimes SU(2)$ symmetry, and were led to precisely the GMSW interaction, when the requirement of either explicit field algebra or strict PCAC was relaxed.

Earlier we considered radiative and strong decays of vector and pseudoscalar mesons,[11] as well as electromagnetic mass differences in the pseudoscalar octet,[12] using the vector-dominance approximation. In those works, as in the present one, the vector fields were treated as massive Yang-Mills gauge fields, while the symmetry assumed was octet-broken $SU(3)$. In this paper the generalized $\omega\rho\pi$ interaction transforms as a $(3, 3^*)\oplus(3^*, 3)$ representation of chiral $SU(3)\otimes SU(3)$ and involves the ninth pseudoscalar meson X^0, as well as axial vectors and scalars. For the interactions of the pseudoscalar octet and the vector mesons the results agree with those already given in Refs. 11 and 12.

II. EFFECTIVE LAGRANGIAN

From nine vector fields $V_\mu{}^a$ and nine axial-vector fields $A_\mu{}^a$ ($a=0, 1, 2, \ldots, 8$), we form matrices

$$\mathbf{V}_\mu=\sqrt{2}^{-1}\sum_0^8 V_\mu{}^a\lambda^a, \tag{1a}$$

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$$\mathbf{A}_\mu=\sqrt{2}^{-1}\sum_0^8 A_\mu{}^a\lambda^a, \tag{1b}$$

where λ^a are the three-dimensional $SU(3)$ matrices (for $a=1,\ldots,8$) and λ^0 is the unit matrix multiplied by $\sqrt{\frac{2}{3}}$. The fields

$$V_\mu{}^{\pm,a}=(1/\sqrt{2})(V_\mu{}^a\pm A_\mu{}^a) \tag{2}$$

form representations (8, 1) and (1, 8), respectively, of $SU(3)^+\otimes SU(3)^-$ for $a=1,\ldots,8$, and representations (1, 1) for $a=0$ (see Appendix A). Analogously, we define

$$\mathbf{V}_\mu{}^\pm=(1/\sqrt{2})(\mathbf{V}_\mu{}^a\pm\mathbf{A}_\mu{}^a) \tag{3}$$

and write for the spin-1 Lagrangian, with kinetic-energy breaking and current mixing,[13]

$$\mathfrak{L}_V=\mathfrak{L}_V{}^+ + \mathfrak{L}_V{}^-, \tag{4}$$

with

$$\mathfrak{L}_V{}^\pm=-\tfrac{1}{4}V_{\mu\nu}{}^{\pm,a}K^{ab}V_{\mu\nu}{}^{\pm,b}+\tfrac{1}{2}m^2V_\mu{}^{\pm,a}V_\mu{}^{\pm,a}, \tag{5}$$

where

$$\mathbf{V}_{\mu\nu}{}^\pm\equiv\partial_\mu\mathbf{V}_\nu{}^\pm-\partial_\nu\mathbf{V}_\mu{}^\pm+ig[\mathbf{V}_\mu{}^\pm,\mathbf{V}_\nu{}^\pm], \tag{6}$$

and K^{ab} is an $SU(3)$-breaking numerical matrix.

Similarly, we consider nine scalar fields σ^a and nine pseudoscalar fields π^a ($a=0,1,\ldots,8$) and introduce

$$\mathbf{P}^\pm=\boldsymbol{\sigma}\pm i\boldsymbol{\pi}, \tag{7}$$

where

$$\boldsymbol{\sigma}=\sqrt{2}^{-1}\sum_0^8\sigma^a\lambda^a, \tag{8a}$$

$$\boldsymbol{\pi}=\sqrt{2}^{-1}\sum_0^8\pi^a\lambda^a. \tag{8b}$$

The fields $\mathbf{P}^{\pm,a}$ belong, respectively, to representations $(3^*,3)$ and $(3,3^*)$ of $SU(3)^+\otimes SU(3)^-$. We introduce also the covariant derivatives of the $\mathbf{P}^\pm$,

$$\mathbf{P}_\mu{}^+\equiv D_\mu\mathbf{P}^+=\partial_\mu\mathbf{P}^++ig(\mathbf{V}_\mu{}^+\mathbf{P}^+-\mathbf{P}^+\mathbf{V}_\mu{}^-), \tag{9a}$$

$$\mathbf{P}_\mu{}^-\equiv D_\mu\mathbf{P}^-=\partial_\mu\mathbf{P}^-+ig(\mathbf{V}_\mu{}^-\mathbf{P}^--\mathbf{P}^-\mathbf{V}_\mu{}^+). \tag{9b}$$

The gauge-invariant Lagrangian for the scalar and pseudoscalar fields, which consists of chiral invariants that are at most trilinear in the fields $\mathbf{P}^\pm$ and in their derivatives, together with the simplest terms which break chiral invariance, is

$$\begin{aligned}\mathfrak{L}_P=&\tfrac{1}{2}\alpha\,\mathrm{tr}\,[\mathbf{P}_\mu{}^+\mathbf{P}_\mu{}^-]-\tfrac{1}{2}\mu_0{}^2\,\mathrm{tr}\,[\mathbf{P}^+\mathbf{P}^-]\\&+\tfrac{1}{4}\gamma(\mathbf{P}_\mu{}^+\times\mathbf{P}_\mu{}^+\cdot\mathbf{P}^++\mathbf{P}_\mu{}^-\times\mathbf{P}_\mu{}^-\cdot\mathbf{P}^-)\\&+\tfrac{1}{2}\mu^2(\det\mathbf{P}^++\det\mathbf{P}^-)\\&+\mathrm{tr}\,[\mathbf{f}\boldsymbol{\sigma}]+\tfrac{1}{2}(\mathbf{P}^+\times\mathbf{P}^+\cdot\mathbf{f}'+\mathbf{P}^-\times\mathbf{P}^-\cdot\mathbf{f}').\end{aligned} \tag{10}$$

In expression (10),

$$\mathbf{f}=\sqrt{2}^{-1}\sum_0^8 f_a\lambda^a, \tag{11}$$

and similarly for $\mathbf{f}'$, while

$$\begin{aligned}\mathfrak{A}\times\mathfrak{A}\cdot\mathfrak{B}\equiv&(\mathrm{tr}\mathfrak{A})^2\,\mathrm{tr}\mathfrak{B}-2\,\mathrm{tr}\mathfrak{A}\,\mathrm{tr}(\mathfrak{A}\mathfrak{B})\\&-\mathrm{tr}(\mathfrak{A}\mathfrak{A})\,\mathrm{tr}\mathfrak{B}+2\,\mathrm{tr}(\mathfrak{A}\mathfrak{A}\mathfrak{B}).\end{aligned} \tag{12}$$

Equation (12) is a generalization of the determinant term and in the notation of Appendix A we would write it, for example,

$$\mathfrak{K}^+\times\mathfrak{L}^+\cdot\mathfrak{M}^+=\mathfrak{K}_{i_+}{}^{l_-}\mathfrak{L}_{j_+}{}^{m_-}\mathfrak{M}_{k_+}{}^{n_-}\epsilon^{i_+,j_+,k_+}\epsilon_{l_-,m_-,n_-}.$$

Our super-Lagrangian, Eqs. (4) and (10), does not contain invariant terms that, when expanded, have lowest-order pieces that are quartic in the fields and their derivatives, and differs in this respect from that of Ref. 5. It differs also in containing the last term in Eq. (10). When only the constants f_0' and f_8' are retained, the divergence of the axial-vector current receives contributions linear and bilinear in the spin-zero fields. Therefore we have a modification to strict PCAC which is added to the one produced by the $\omega\rho\pi$ interaction.

The $\omega\rho\pi$ interaction will be discussed in Sec. IV. For the present we remark that it transforms like a C- and P-invariant combination of zeroth and eighth components of $(3^*,3)\oplus(3,3^*)$ representations. Thus, our final Lagrangian will consist of

$$\mathfrak{L}=\mathfrak{L}_V+\mathfrak{L}_P+\mathfrak{L}_{\omega\rho\pi}, \tag{13}$$

with $\mathfrak{L}_V$ given by (4) and $\mathfrak{L}_P$ given by (10).

III. NONLINEAR THEORY

In this version of the theory, the σ^a are considered to be dependent on the pion field. We let[5]

$$\boldsymbol{\pi}^2+\boldsymbol{\sigma}^2=F^2 \tag{14}$$

and consider $\boldsymbol{\sigma}$ to be a function of $\boldsymbol{\pi}$ and the constant F^2. Accordingly, we drop the constant term

$$-\tfrac{1}{2}\mu_0{}^2\,\mathrm{tr}\,(\mathbf{P}^+\mathbf{P}^-)=-\tfrac{3}{2}\mu_0{}^2F^2 \tag{15}$$

from Eq. (10). Expanding (14) to bilinear terms in $\boldsymbol{\pi}$, we get

$$\sigma^a=\sqrt{3}F\delta^{0a}-(1/2\sqrt{2}F)d^{abc}\pi^b\pi^c. \tag{16}$$

We now use (16) to write the terms in Eq. (10) which are at most bilinear in the pseudoscalar fields. The result is

$$\begin{aligned}\mathfrak{L}_{P,\mathrm{free}}=&\tfrac{1}{2}\alpha\partial_\mu\pi^a\partial_\mu\pi^a+\tfrac{1}{2}\gamma F(\pi_\mu{}^a\pi_\mu{}^a-3\pi_\mu{}^0\pi_\mu{}^0)\\&-\tfrac{3}{2}\mu F\pi^0\pi^0-(f_0/2\sqrt{3}F)\pi^a\pi^a-(f_8/2\sqrt{2}F)d^{8bc}\pi^b\pi^c\\&-(f_0'/\sqrt{3})(\pi^a\pi^a+3\pi^0\pi^0)+2\sqrt{3}f_8'\pi^0\pi^8\\&-(f_8'/\sqrt{2})d^{ab8}\pi^a\pi^b.\end{aligned} \tag{17}$$

It is evident that there are more parameters in (17) than are needed to fit the pseudoscalar masses and satisfy PCAC. While the general case of diagonalizing a Lagrangian of the form of Eq. (17) is treated in Appendix B, we shall here, for simplicity, put $\gamma=\mu=0$

[13] S. Coleman and H. J. Schnitzer, Phys. Rev. **134**, B863 (1964).

and $\alpha=1$. Defining also

$$F_0=(\sqrt{\tfrac{1}{3}})[(f_0/2F)+f_0'], \quad (18)$$

$$F_8=(\sqrt{\tfrac{2}{3}})[(f_8/2F)+f_8'], \quad (19)$$

we have, in the notation of Appendix B,

$$\alpha_0=\alpha_1=1, \quad \beta_1=2F_0+F_8,$$
$$\beta_4=2F_0-\tfrac{1}{2}F_8, \quad \beta_8=2F_0-F_8, \quad (20)$$
$$\beta_0=2F_0+2\sqrt{3}f_0', \quad \beta^{08}=2\sqrt{2}F_8-4\sqrt{3}f_8'.$$

Using now the mass relations in Appendix B and the empirical values of the masses, we determine all the remaining parameters in Eq. (17) as follows (in GeV^2): $F_0=0.0845$, $F_8=-0.150$, $2\sqrt{3}f_0'=m_X^2+m_\eta^2-2m_K^2=0.732$, $4\sqrt{3}f_8'=-0.635$. The mixing angle is determined to be $\theta=9.6°$, from $\cos 2\theta=0.944$. Finally, from PCAC we get $F=-f_\pi/\sqrt{2}$, as well as $f_\pi=f_K=f_\eta=f_X$.

IV. "$\omega\rho\pi$" INTERACTION

Before considering the linear version of the spin-zero Lagrangian, we shall study the form of the $\omega\rho\pi$ interaction. For simplicity we exhibit first the space-time form of the interaction, suppressing internal indices:

$$\mathfrak{L}'\sim\epsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}V_{\mu\nu}P. \quad (21)$$

Next we consider the C- and P-invariant parts of $(3^*,3)\oplus(3,3^*)$ representations, formed of two V's and a P. These are three in number:

$$\mathbf{A}=-\tfrac{1}{2}i(\mathbf{P}^+-\mathbf{P}^-)\ \mathrm{tr}(\mathbf{V}^+\mathbf{V}^++\mathbf{V}^-\mathbf{V}^-), \quad (22)$$

$$\mathbf{B}=-i(\mathbf{V}^-\mathbf{P}^+\mathbf{V}^+-\mathbf{V}^+\mathbf{P}^-\mathbf{V}^-), \quad (23)$$

$$\mathbf{C}=-i(\mathbf{P}^+\mathbf{V}^+\mathbf{V}^++\mathbf{V}^-\mathbf{V}^-\mathbf{P}^+-\mathbf{P}^-\mathbf{V}^-\mathbf{V}^--\mathbf{V}^+\mathbf{V}^+\mathbf{P}^-). \quad (24)$$

Accordingly, we write

$$\mathfrak{L}'=\tfrac{1}{4}(a_0A_0+a_8A_8+b_0B_0+b_8B_8+c_0C_0+c_8C_8), \quad (25)$$

with

$$A_i=(1/\sqrt{2})\ \mathrm{tr}(\mathbf{A}\lambda_i), \quad (26)$$

and similarly for B_i and C_i. Thus, $\mathfrak{L}'$ transforms in the same way as the terms leading to the PCAC condition, and it modifies this condition by adding a multiparticle contribution to the standard divergence of the axial current. As discussed in the Introduction, this is essentially the only way of obtaining the $\omega\rho\pi$ interaction when one has a purely mesonic Lagrangian. If fermions are included, Adler[14] has shown that one also obtains a (model-dependent) modified PCAC condition. What the origin of $\mathfrak{L}'$ is does not have relevance to our ensuing discussion.

Selecting the interactions of two vector fields and a pseudoscalar, we obtain

$$\mathfrak{L}_{\omega\rho\pi}=[\delta^{ab}(a_0\delta^{c0}+a_8\delta^{c8})+(1/\sqrt{6})(b_0+2c_0)d^{abc}+b_8d^{8ak}d^{kbc}+\tfrac{1}{2}(2c_8-b_8)d^{abk}d^{kc8}]\times\tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}{}^aV_{\mu\nu}{}^b\pi^c. \quad (27)$$

We shall interpret Eq. (27) as being valid only for a, $b\neq0$, since otherwise we would be postulating a nonet form for V rather than the octet-plus-singlet form that we have assumed. For $a=b=0$, there is an interaction

$$\tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}{}^0V_{\mu\nu}{}^0(d_0\pi^0+d_8\pi^8) \quad (28)$$

that we shall not discuss further, since we assume there is no $SU(3)$ singlet electromagnetic current.

In Ref. 11 the $\omega\rho\pi$ interaction was written in the general octet-broken $SU(3)$ form:

$$\mathfrak{L}_{\omega\rho\pi}=\tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}(hD^{abc}V_{\alpha\beta}{}^aV_{\mu\nu}{}^b\pi^c+\lambda D^{ab}V_{\alpha\beta}{}^a\pi^bV_{\mu\nu}{}^0), \quad (29)$$

with

$$D^{abc}=d^{abc}+\sqrt{3}\epsilon_1d^{abk}d^{k8c}+\tfrac{1}{2}\sqrt{3}\epsilon_\beta(d^{ack}d^{k8b}+d^{bck}d^{k8a})+(\epsilon_3/\sqrt{3})\delta^{ab}\delta^{c8}, \quad (30)$$

$$D^{ab}=\delta^{ab}+\sqrt{3}\epsilon_4d^{a8b}, \quad (31)$$

and with a, b, etc. $=1,\ldots,8$. When one vector meson is the $SU(3)$ singlet we shall complete Eq. (27) with a term whose coupling matrix is given by (31), and it is then evident that Eq. (29) has the same form as Eq. (27), except that π^0 appears in Eq. (27) as part of the pseudoscalar nonet, permitting the discussion of η and X^0 decays.

To compare the parameters of Eqs. (27) and (29) we eliminate the explicit appearance of the $SU(3)$ index zero by using the identity

$$\tfrac{1}{3}(\delta^{a8}\delta^{bc}+\delta^{b8}\delta^{ac}+\delta^{c8}\delta^{ab})=d^{abm}d^{mc8}+d^{acm}d^{mb8}+d^{bcm}d^{ma8}, \quad (32)$$

the indices taking on values $1,\ldots,8$. There results

$$h=(\sqrt{\tfrac{1}{6}})(b_0+2c_0), \quad \sqrt{3}\epsilon_1h=c_8+\tfrac{1}{2}b_8,$$
$$\sqrt{3}\epsilon_2h=3b_8, \quad (\epsilon_3/\sqrt{3})h=a_8+\tfrac{2}{3}(c_8-b_8). \quad (33)$$

Using these identifications and the form (31) for the interaction when one of the vector mesons is an $SU(3)$ singlet, and neglecting the small η-X^0 mixing, the results of Ref. 11 are unchanged.

The interactions with the ninth pseudoscalar can now be written

$$\mathfrak{L}_{\pi^0VV}=\tfrac{1}{4}\epsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}{}^aV_{\mu\nu}{}^b\pi^0[(\sqrt{\tfrac{2}{3}})h(1+\epsilon_5)\delta^{ab}+\sqrt{2}h\epsilon_1d^{ab8}+\sqrt{2}\lambda\epsilon_4\delta^{a8}\delta^{b0}], \quad (34)$$

where, for convenience, we have let $a_0=(\sqrt{\tfrac{2}{3}})h\epsilon_5$. Again, we shall not be interested in the case $a=b=0$.

In a recent communication[15] Singer has considered the radiative decays involving the $X^0(958)$ meson, neglecting η-X^0 mixing, and using the formalism of Ref. 11. This interaction (for a, $b\neq0$) is that of Eq. (34), with the bracket replaced by

$$h_0(\delta^{ab}+\sqrt{3}\beta d^{ab8}). \quad (35)$$

The present formulation thus contains one less param-

[14] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[15] P. Singer (unpublished).

eter, for we identify

$$h_0=(\sqrt{\tfrac{2}{3}})h(1+\epsilon_5),\qquad \beta h_0=(\sqrt{\tfrac{2}{3}})h\epsilon_1. \tag{36}$$

Since it is shown in Ref. 11 that

$$(m_\pi^2h^2/4\pi)(1+\epsilon_1)^2=0.10,\qquad \epsilon_1=0.77 \text{ or } \epsilon_1=1.3, \tag{37}$$

the radiative decays involving the X^0 meson will depend on only one new parameter, namely, ϵ_5. In the following sections of this paper, we shall show that the field renormalizations and mixing parameters assumed in calculating the decay amplitudes of Ref. 11 and electromagnetic mass differences of Ref. 12 are essentially unchanged by the requirements of broken chiral symmetry.

V. SPIN-1 LAGRANGIAN IN NONLINEAR THEORY

In this section we consider the mass relations of the vector and axial-vector mesons. Spin-1 mass terms arise not only from Eq. (5), but also from Eq. (10) as a result of the use of the covariant derivative. Assuming the nonlinear relationship, Eq. (14) or Eq. (16), and assuming $\gamma=0$ as in Sec. III, the contributing term of the Lagrangian [Eq. (17)] is

$$\mathfrak{L}_P{}^0=\tfrac{1}{2}\alpha\,\mathrm{tr}(\mathbf{P}_\mu{}^+\mathbf{P}_\mu{}^-)$$
$$=\tfrac{1}{2}\alpha\,\mathrm{tr}\left[\left(\partial_\mu\boldsymbol{\pi}+\frac{g}{\sqrt{2}}\{\boldsymbol{\sigma},\mathbf{A}_\mu\}\right)\left(\partial_\mu\boldsymbol{\pi}+\frac{g}{\sqrt{2}}\{\boldsymbol{\sigma},\mathbf{A}_\mu\}\right)+\cdots\right], \tag{38}$$

where only the relevant terms are exhibited. Using Eq. (16), we have

$$\{\boldsymbol{\sigma},\mathbf{A}_\mu\}=2F\mathbf{A}_\mu, \tag{39}$$

and performing the indicated trace we get

$$\mathfrak{L}_P{}^0=\tfrac{1}{2}\alpha(\partial_\mu\pi^a\partial_\mu\pi^a+2\sqrt{2}gF\partial_\mu\pi^aA_\mu{}^a+2g^2F^2A_\mu{}^aA_\mu{}^a+\cdots). \tag{40}$$

The familiar mixing of the spin-1 and spin-zero fields can be eliminated by the substitution

$$\mathbf{A}_\mu=\mathfrak{A}_\mu-(\boldsymbol{\beta}/m)\partial_\mu\boldsymbol{\pi}; \tag{41}$$

in general $\boldsymbol{\beta}$ is a matrix.

We assume that there is an axial-vector $SU(3)$ singlet. Requiring the mixed terms to vanish, and the coefficient of $\partial_\mu\pi^a\partial_\mu\pi^a$ to be $\tfrac{1}{2}$, results in

$$\boldsymbol{\beta}=\sqrt{2}gF/m\equiv\beta \tag{42}$$

and

$$\alpha=(1-\beta^2)^{-1}. \tag{43}$$

The free Lagrangian of the spin-1 mesons takes the form

$$\mathfrak{L}^{(1)}=-\tfrac{1}{4}\bar{V}_{\mu\nu}{}^a\bar{V}_{\mu\nu}{}^bK^{ab}+\tfrac{1}{2}m^2V_\mu{}^aV_\mu{}^a-\tfrac{1}{4}\mathfrak{A}_{\mu\nu}{}^a\mathfrak{A}_{\mu\nu}{}^bK^{ab}$$
$$+\tfrac{1}{2}[m^2/(1-\beta^2)]\mathfrak{A}_\mu{}^a\mathfrak{A}_\mu{}^a, \tag{44}$$

with

$$\bar{V}_{\mu\nu}{}^a=\partial_\mu V_\nu{}^a-\partial_\nu V_\mu{}^a,$$
$$\mathfrak{A}_{\mu\nu}{}^a=\partial_\mu\mathfrak{A}_\nu{}^a-\partial_\nu\mathfrak{A}_\mu{}^a.$$

If $\beta=\tfrac{1}{2}$, we get $(m_A{}^a)^2=2(m_V{}^a)^2$, where $m_A{}^a$ and $m_V{}^a$ are the axial-vector and vector meson masses, respectively; $F=-(\sqrt{\tfrac{1}{2}})f_\pi$, and $gf_\pi=\sqrt{2}m$ [Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation].

In the case that no axial-vector singlet exists, the pseudoscalar kinetic energy takes the form

$$\tfrac{1}{2}\partial_\mu\pi^{a'}\partial_\mu\pi^{a'}+\tfrac{1}{2}\alpha\partial_\mu\pi^0\partial_\mu\pi^0, \tag{45}$$

where $a'=1,\ldots,8$. In this case the mass relations will be

$$(m_A{}^1)^2=2m_\rho{}^2,\qquad (m_{KA})^2=2(m_K{}^*)^2 \tag{46}$$

and

$$(m_A{}^8)^2=2\left(\frac{4}{3m_K{}^{*2}}-\frac{1}{3m_\rho{}^2}\right)^{-1}=1.816\text{ GeV}^2. \tag{47}$$

That is, $m_A{}^8=1350$ MeV, lying between the observed mesons D and E.

VI. LINEAR THEORY: K^*-κ MIXING

The linear theory, in which the scalar mesons are treated as real particles, is more demanding in that it requires us to fit (or predict) the scalar-meson mass spectrum. It also includes the possibility of mixing between the K^* and the κ, related to a possible nonzero vacuum expectation value of σ^8.

The terms in Eq. (10) which can contribute to the spin-zero kinetic energy are

$$\mathfrak{L}_P{}^0=\tfrac{1}{2}\alpha\,\mathrm{tr}(\mathbf{P}_\mu{}^+\mathbf{P}_\mu{}^-)$$
$$+\tfrac{1}{4}\gamma(\mathbf{P}_\mu{}^+\times\mathbf{P}_\mu{}^+\cdot\mathbf{P}^++\mathbf{P}_\mu{}^-\times\mathbf{P}_\mu{}^-\cdot\mathbf{P}^-). \tag{48}$$

Using Eqs. (9) and retaining terms at most linear in the spin-1 fields, we get

$$\mathbf{P}_\mu{}^\pm=(1/\sqrt{2})(R_\mu{}^a\pm iS_\mu{}^a)\lambda^a, \tag{49}$$

with

$$R_\mu{}^a=\partial_\mu\sigma^a-gF^{ca}V_\mu{}^c, \tag{50}$$
$$S_\mu{}^a=\partial_\mu\pi^a+g\Sigma^{ca}A_\mu{}^c, \tag{51}$$

where

$$F^{ca}=\sigma_8f^{8ca}, \tag{52}$$
$$\Sigma^{ca}=\sigma_bd^{abc}$$
$$=(\sqrt{\tfrac{2}{3}})\sigma_0\delta^{ac}+\sigma_8d^{ac8}. \tag{53}$$

Here $\sigma_0=\langle\sigma^0\rangle$ and $\sigma_8=\langle\sigma^8\rangle$, the other $\langle\sigma^a\rangle$ being zero.

After some calculation it becomes apparent that Eq. (48) has the form

$$\mathfrak{L}_P{}^0=X_-{}^{ab}R_\mu{}^aR_\mu{}^b+X_+{}^{ab}S_\mu{}^aS_\mu{}^b, \tag{54a}$$
$$X_\pm{}^{ab}=\tfrac{1}{2}\alpha\delta^{ab}\pm Q^{ab}, \tag{54b}$$

with

$$Q^{ab}=-\tfrac{1}{2}\sqrt{3}\gamma[\sigma_0(\delta^{a0}\delta^{b0}-\tfrac{1}{3}\delta^{ab})$$
$$-\sigma_8(\delta^{a8}\delta^{b0}+\delta^{a0}\delta^{b8}-(\sqrt{\tfrac{2}{3}})d^{ab8})]. \tag{55}$$

Since Eq. (54) contains terms which mix $\boldsymbol{\sigma}$ and $\mathbf{V}$ and also $\boldsymbol{\pi}$ and $\mathbf{A}$, our first task is to diagonalize the Lagrangian. Because of the space-time antisymmetry of the $\mathbf{V}_{\mu\nu}^{\pm}$ [Eq. (6)], the kinetic-energy part of the spin-1 Lagrangian will not be relevant to the diagonalization. We therefore consider only the spin-1 "mass terms" and define

$$\mathfrak{L}_{\sigma V}{}^0 = X_-{}^{ab}R_\mu{}^a R_\mu{}^b + \tfrac{1}{2}m^2 V_\mu{}^a V_\mu{}^a, \tag{56}$$

$$\mathfrak{L}_{\pi A}{}^0 = X_+{}^{ab}S_\mu{}^a S_\mu{}^b + \tfrac{1}{2}m^2 A_\mu{}^a A_\mu{}^a. \tag{57}$$

We diagonalize these expressions by the substitutions

$$V_\mu{}^a = \mathcal{V}_\mu{}^a + \Delta_-{}^{ac}\partial_\mu\sigma^c, \tag{58}$$

$$A_\mu{}^a = \mathfrak{A}_\mu{}^a + \Delta_+{}^{ac}\partial_\mu\pi^c. \tag{59}$$

If we write Eq. (56) in the form

$$\mathfrak{L}_{\sigma V}{}^0 = X_-{}^{ab}\partial_\mu\sigma^a\partial_\mu\sigma^b + C_-{}^{ab}\partial_\mu\sigma^a V_\mu{}^b + D_-{}^{ab}V_\mu{}^a V_\mu{}^b, \tag{60}$$

we see that diagonalization requires

$$C_-{}^{ab} + 2D_-{}^{bc}\Delta_-{}^{ca} = 0, \tag{61}$$

and, using this condition, Eq. (60) becomes

$$\mathfrak{L}_{\sigma V}{}^0 = (X_-{}^{ab} + \tfrac{1}{2}C_-{}^{ac}\Delta^{cb})\partial_\mu\sigma^a\partial_\mu\sigma^b + D_-{}^{ab}\mathcal{V}_\mu{}^a\mathcal{V}_\mu{}^b. \tag{62}$$

From Eqs. (52) and (46), we see that

$$C_-{}^{ab} = -gX_-{}^{lm}(F^{bl}\delta^{ma} + F^{bm}\delta^{la}), \tag{63}$$

$$D_-{}^{ab} = g^2 X_-{}^{lm}F^{al}F^{bm} + \tfrac{1}{2}m^2\delta^{ab}. \tag{64}$$

Considering Eq. (59), we get the analogous results

$$C_+{}^{ab} + 2D_+{}^{bc}\Delta_+{}^{ca} = 0, \tag{65}$$

$$\mathfrak{L}_{\pi A}{}^0 = (X_+{}^{ab} + \tfrac{1}{2}C_+{}^{ac}\Delta_+{}^{cb})\partial_\mu\pi^a\partial_\mu\pi^b + D_+{}^{ab}\mathfrak{A}_\mu{}^a\mathfrak{A}_\mu{}^b, \tag{66}$$

$$C_+{}^{ab} = gX_+{}^{lm}(\Sigma^{bl}\delta^{ma} + \Sigma^{bm}\delta^{la}), \tag{67}$$

$$D_+{}^{ab} = g^2 X_+{}^{lm}\Sigma^{al}\Sigma^{bm} + \tfrac{1}{2}m^2\delta^{ab}. \tag{68}$$

At this point it will be advantageous to continue the development of Eqs. (56) and (57) separately because of the different properties of F^{ab} and Σ^{ab}. We note that F^{ab} is nonzero only if a, $b = 4, 5, 6, 7$, so that [e.g., in Eq. (63)]

$$\begin{aligned} X_-{}^{lm}F^{bl} &= X_-{}^{\bar{l}\bar{m}}F^{bl} \\ &= XF^{bm}, \end{aligned} \tag{69}$$

where we have used the notation $\bar{l}$ to mean an index which takes only the values 4, 5, 6, and 7. From Eq. (55) the quantity X is

$$X = \tfrac{1}{2}\alpha - (\gamma/2\sqrt{6})(\sqrt{2}\sigma_0 + \sigma_8). \tag{70}$$

Using also the relation

$$F^{am}F^{bm} = \tfrac{3}{4}\sigma_8{}^2\delta^{\bar{a}\bar{b}}, \tag{71}$$

Eq. (57) yields

$$\Delta_-{}^{ab} = 2gXF^{ab}(m^2 + \tfrac{3}{2}g^2\sigma_8{}^2 X)^{-1}, \tag{72}$$

which means that only the strange components of $\boldsymbol{\sigma}$ and $\mathbf{V}$ are mixed.

We note that Eqs. (69)–(71) showed that

$$C_-{}^{ab} = -2gXF^{ab}, \tag{73}$$

$$D_-{}^{ab} = \tfrac{3}{4}g^2\sigma_8{}^2\delta^{\bar{a}\bar{b}} + \tfrac{1}{2}m^2\delta^{ab}, \tag{74}$$

with consequent simplification of Eq. (62). Referring to Eq. (5) and letting

$$m'^2 = \tfrac{3}{2}g^2\sigma_8{}^2, \tag{75}$$

we write the (free) vector Lagrangian now as

$$\mathfrak{L}_V{}^0 = -\tfrac{1}{2}K^{ab}\mathcal{V}_{\mu\nu}{}^a\mathcal{V}_{\mu\nu}{}^b + \tfrac{1}{2}m^2\mathcal{V}_\mu{}^a\mathcal{V}_\mu{}^a + \tfrac{1}{2}m'^2\mathcal{V}_\mu{}^a\mathcal{V}_\mu{}^a, \tag{76}$$

with

$$\mathcal{V}_{\mu\nu}{}^a = \partial_\mu\mathcal{V}_\nu{}^a - \partial_\nu\mathcal{V}_\mu{}^a, \tag{77}$$

and the current-mixing matrix K^{ab} having the form

$$K^{ab} = \delta^{ab} + \sqrt{3}\epsilon d^{ab8} + \epsilon_0\delta^{a0}\delta^{b0} + \epsilon_8(\delta^{a0}\delta^{b8} + \delta^{a8}\delta^{b0}). \tag{78}$$

In terms of the empirical masses m_ρ and m^* of the ρ and the K^* mesons, we obtain

$$K^{11} = 1 + \epsilon = m^2/m_\rho{}^2, \tag{79}$$

$$K^{44} = 1 - \tfrac{1}{2}\epsilon = (m^2 + m'^2)/(m^*)^2, \tag{80}$$

which determine m^2 and ϵ as functions of m'^2.

Also, letting

$$\mathcal{V}_\mu{}^8 = -\sin\theta\,\Omega_\mu + \cos\theta\,\Phi_\mu, \tag{81a}$$

$$\mathcal{V}_\mu{}^0 = \cos\theta\,\Omega_\mu + \sin\theta\,\Phi_\mu, \tag{81b}$$

we obtain

$$\tan 2\theta = -2(\sqrt{2}\epsilon + \epsilon_8)/(\epsilon_0 + \epsilon), \tag{82}$$

$$\epsilon_0 - \epsilon = M_+ - 2, \tag{83}$$

$$\cos 2\theta = (\epsilon_0 + \epsilon)M_-, \tag{84}$$

where

$$M_\pm = m^2/m_\omega{}^2 \pm m^2/m_\phi{}^2. \tag{85}$$

In the case that $\sigma_8 = 0$, we solve these equations to get $\epsilon = 0.21$, $\epsilon_0 = 0.10$, $m = 847$ MeV, and $\theta = 27.5°$ as in Ref. 11. In general, requiring that $|\cos\theta| \leq 1$ leads to the restriction that $m'^2/m^2 \leq 0.38$. However, the equality corresponds to the case of no ω-φ mixing, and in the realistic case we expect $m'^2 \ll m^2$, small K^*-κ mixing and $g\sigma_8 \ll m$ [Eq. (75)].

VII. LINEAR THEORY: PSEUDOSCALAR–AXIAL-VECTOR MIXING

Although Eqs. (65) through (68) are rather complicated in general, inspection will show that the matrices X_+, C_+, D_+, and (therefore) Δ_+ are all diagonal in the isospin-1 and $\frac{1}{2}$ sectors. Accordingly we let $C_+{}^{11} \equiv C^{(1)}$, etc. (dropping the subscript plus), and similarly for the other diagonal matrices. Thus, Eq. (65) now reads

$$C^{(a)} + 2D^{(a)}\Delta^{(a)} = 0. \tag{86}$$

Referring first to the isospin-1 sector, we require the coefficient of $\partial_\mu\pi^1\partial_\mu\pi^1$ to be $\frac{1}{2}$ [Eq. (66)]. Thus,

$$2X^{(1)} + C^{(1)}\Delta^{(1)} = 1. \tag{87}$$

Next, we note that the (free) Lagrangian for the axial-vector mesons now reads

$$\mathfrak{L}_A{}^0 = -\tfrac{1}{2}K^{ab}\mathfrak{A}_{\mu\nu}{}^a\mathfrak{A}_{\mu\nu}{}^b + D_+{}^{ab}\mathfrak{A}_\mu{}^a\mathfrak{A}_\mu{}^b, \tag{88}$$

so that

$$m_{A_1}{}^2 = 2D_+{}^{11}/K^{11} = 2D^{(1)}/K^{11} = 2(m_\rho{}^2/m^2)D^{(1)}, \tag{89}$$

where we have used Eq. (79). Setting $m_{A_1}{}^2 = 2m_\rho{}^2$, we obtain

$$D^{(1)} = m^2. \tag{90}$$

Since $C^{(1)} = 2gX^{(1)}\Sigma^{(1)}$ and $D^{(1)} = g^2X^{(1)}\Sigma^{(1)2} + \frac{1}{2}m^2$, we conclude from Eqs. (86), (87), and (90) that

$$X^{(1)} = 1, \qquad C^{(1)} = (2m^2)^{1/2}. \tag{91}$$

(For $\gamma = 0$, this corresponds to $\alpha = 2$.) Also, letting $\sigma = \sqrt{2}\sigma_0$, $X = (3m^2/2g^2)^{1/2}$, we have

$$3(\Sigma^{(1)})^2 = (\sigma + \sigma_8)^2 = x^2. \tag{92}$$

By a similar argument, using Eq. (80), and letting $r^2 = (m_{K_A}/m^*)^2$,

$$2D^{(4)} = r^2(m^2 + m'^2). \tag{93}$$

For $m_{K_A} = 1240$ MeV (1330 MeV), $r^2 = 1.94$ (2.24), so that for our present purpose it will be sufficient to assume $r^2 = 2$. We then get [from Eq. (75)]

$$D^{(4)} = g^2X^{(4)}(\Sigma^{(4)})^2 + \tfrac{1}{2}m^2 = \tfrac{1}{2}r^2m^2[1 + (9/4)(\sigma_8{}^2/x^2)]. \tag{94}$$

For $\gamma = 0$, using (92), we get

$$(1 \mp \tfrac{3}{2}\sigma_8/x)^2 = +\tfrac{9}{2}(\sigma_8/x)^2, \tag{95}$$

so that $(\sigma_8/x) = 0, \mp\frac{4}{3}$. The nonzero values are excluded by the requirement of a real mixing angle for the vector mesons. For $m_{K_A} = 1240$ MeV (1330 MeV), we obtain

$$\sigma_8/x = 0.02\ (-0.08), \tag{96}$$

and, accordingly, we shall assume $(\sigma_8/x) \approx 0$. With this assumption, Eq. (94) gives

$$X^{(4)} = r^2 - 1, \tag{97}$$

and since

$$X^{(4)} = \tfrac{1}{2}\alpha + \gamma\sigma/2\sqrt{6} = X^{(1)} = 1, \tag{98}$$

we are required to have $r^2 = 2$. We see also that $(m')^2$ is negligible, and reach the important conclusion that K^*-κ mixing is very small.

In view of the experimental uncertainties with regard to the isospin-zero axial-vector mesons, we shall not consider further the problem of mixing them, but pass instead to the spin-zero mass spectrum.

VII. LINEAR THEORY: SPIN ZERO

In the previous sections we found $\sigma_8 \approx 0$, in which case Eqs. (54b) and (55) simplify to diagonal matrices:

$$X_\pm{}^{AB} = (\tfrac{1}{2}\alpha \pm \gamma\sigma/2\sqrt{6})\delta^{AB}, \tag{99}$$

$$X_\pm{}^{00} = \tfrac{1}{2}\alpha \pm \gamma\sigma/\sqrt{6}, \tag{100}$$

where $A, B = 1, \ldots, 8$, and, in addition [Eq. (91)],

$$\tfrac{1}{2}\alpha + \gamma\sigma/2\sqrt{6} = 1. \tag{101}$$

In consequence, Eqs. (65) through (69) give for the pseudoscalar-meson kinetic energy

$$\tfrac{1}{2}\partial_\mu\pi^A\partial_\mu\pi^A + \tfrac{1}{2}z\partial_\mu\pi^0\partial_\mu\pi^0, \tag{102}$$

$$z = X_+{}^{00}(1 + X_+{}^{00})^{-1} = (\alpha - \tfrac{4}{3})(\alpha - \tfrac{2}{3})^{-1}. \tag{103}$$

Since, in Eq. (66),

$$D_+{}^{00} = \tfrac{1}{2}m^2(1 + X_+{}^{00}), \tag{104}$$

$X_+{}^{00}$ is determined by the masses of the isoscalar axial-vector mesons, but in view of the experimental uncertainties we leave z undetermined, at least for the present.

Similarly, for the scalar-meson kinetic energy we find

$$(\alpha - 1)\partial_\mu\sigma^A\partial_\mu\sigma^A + (2 - \tfrac{1}{2}\alpha)\partial_\mu\sigma^0\partial_\mu\sigma^0. \tag{105}$$

Requiring positivity for each kinetic-energy term in Eqs. (102) and (105), we find

$$4 > \alpha > \tfrac{4}{3}. \tag{106}$$

Turning now to the remaining terms in the spin-zero Lagrangian [Eq. (10)], we find first that linear terms in the scalar field ("tadpoles") are eliminated by choosing

$$\sqrt{2}f_0 = (\mu_0{}^2 - 2g_0)\sigma, \tag{107}$$

$$2f_8 = g_8\sigma, \tag{108}$$

where we have set

$$g_0 = 4f_0'/\sqrt{3}, \tag{109}$$

$$g_8 = 4\sqrt{2}f_8'/\sqrt{3}. \tag{110}$$

With $\sigma_8 = 0$, the term $\frac{1}{2}\mu^2(\det P^+ + \det P^-)$ gives the same type of mass contribution as the last term in Eq. (10), so we set it to zero.

We thus obtain, as the free pseudoscalar Lagrangian,

$$\mathfrak{L}_\pi = \tfrac{1}{2}\partial_\mu\pi^A\partial_\mu\pi^A + \tfrac{1}{2}z\partial_\mu\pi^0\partial_\mu\pi^0 - \tfrac{1}{2}\mu_0{}^2\pi^a\pi^a - \tfrac{1}{2}g_0(3\pi^0\pi^0 - \pi^a\pi^a) - \tfrac{1}{2}g_8(\sqrt{3}d^{ab8}\pi^a\pi^b - 3\sqrt{2}\pi^0\pi^8), \tag{111}$$

while the free scalar Lagrangian is

$$\mathfrak{L}_\sigma = (\alpha - 1)\partial_\mu\sigma^A\partial_\mu\sigma^A + (2 - \tfrac{1}{2}\alpha)_\mu\partial\sigma^0\partial_\mu\sigma^0 - \tfrac{1}{2}\mu_0{}^2\sigma^a\sigma^a + \tfrac{1}{2}g_0(3\sigma^0\sigma^0 - \sigma^a\sigma^a) + \tfrac{1}{2}g_8(\sqrt{3}d^{ab8}\sigma^a\sigma^b - 3\sqrt{2}\sigma^0\sigma^8), \tag{112}$$

it now being understood that in Eq. (112), $\langle\sigma^a\rangle = 0$, $a = 0, \ldots, 8$. From Eq. (111) we see immediately that

$$m_\pi{}^2 = \mu_0{}^2 - g_0 + g_8, \tag{113}$$

$$m_K{}^2 = \mu_0{}^2 - g_0 - \tfrac{1}{2}g_8, \tag{114}$$

and, from Eq. (108),

$$m_{\pi'}^2=\tfrac{1}{2}(\mu_0^2+g_0-g_8)/(\alpha-1), \qquad (115)$$

$$m_{K'}^2=\tfrac{1}{2}(\mu_0^2+g_0+\tfrac{1}{2}g_8)/(\alpha-1). \qquad (116)$$

Thus,

$$m_{\pi'}^2-m_{K'}^2=\tfrac{1}{2}(m_K^2-m_\pi^2)/(\alpha-1) \qquad (117)$$

and we necessarily have $m_{\pi'}>m_{K'}$. Considering only the restriction on α in Eq. (106), they can be nearly degenerate or differ in squared mass by as much as 0.34 BeV^2. In the latter extreme case, $m_{K'}\approx 800$ MeV if $m_{\pi'}\approx 980$ MeV; however, we shall see that we cannot take α as small as $\frac{4}{3}$.

Using the mixing theory of Appendix B, we define the particle fields η and X through

$$(\sqrt{z})\pi^0=-\eta\sin\varphi+X\cos\varphi, \qquad (118)$$

$$\pi^8=\eta\cos\varphi+X\sin\varphi, \qquad (119)$$

and obtain

$$\tan 2\varphi=(\beta^{08}/\sqrt{z})(\beta_0/z-\beta_8)^{-1}, \qquad (120)$$

$$m_X^2+m_\eta^2=\beta_0/z+\beta_8, \qquad (121)$$

$$m_X^2-m_\eta^2=(\beta^{08}/\sqrt{z})(\sin 2\varphi)^{-1}, \qquad (122)$$

with

$$\beta_0=\mu_0^2+2g_0,$$
$$\beta_8=\mu_0^2-g_0-g_8, \qquad (123)$$
$$\beta^{08}=-g_8/\sqrt{2}.$$

Using Eqs. (113) and (114) gives (in GeV^2) $\mu_0^2-g_0=0.169$, $g_8=-0.150$, determining β_8, and $\varphi=10.4°$. All remaining parameters are also determined: $g_0=0.0143$, $\mu_0^2=0.183$, and $z=0.236$. (This last is equivalent to $\alpha=1.54$.)

These parameters, and in particular the large value of α, result in disastrously low values for the scalar meson masses $m_{\pi'}$ and $m_{K'}$, Eqs. (115) and (116), namely, $m_{\pi'}^2=0.321$, $m_{K'}^2=0.113$. The masses of the isoscalar-scalar mesons are still lower, which is an unacceptable situation.

Remaining within the limitations set in this work, there appears to be no way out of this difficulty. However, if we allow the chiral-invariant mass term $\frac{1}{2}\mu_0^2\,\mathrm{tr}(\mathbf{P}^+\mathbf{P}^+ +\mathbf{P}^-\mathbf{P}^-)$ to be further broken by allowing, for example, an additional $SU(3)$-invariant term $\frac{1}{2}\mu'^2\sigma^a\sigma^a$ the situation can be much improved. Since the scalar-meson masses are not really known, we give the following two examples of what results.

If $\mu'^2=0.5$ GeV^2,

$m_{\pi'}=885$ MeV, $m_{K'}=755$ MeV, $m_{\eta'}=710$ MeV,

$m_{X'}=515$ MeV, $\varphi_S=15°40'$.

If $\mu'^2=0.6$ GeV^2,

$m_{\pi'}=935$ MeV, $m_{K'}=815$ MeV, $m_{\eta'}=775$ MeV,

$m_{X'}=550$ MeV, $\varphi_S=6°20'$.

VIII. SUMMARY

One of our motivations in this work was to relate the generalized $\omega\rho\pi$ interaction to an effective Lagrangian with broken chiral symmetry. Although we considered more general effective Lagrangians, including both linear and nonlinear treatments of the scalar mesons and the possibility of mixing between κ and K^* mesons, nevertheless, the broken-$SU(3)$ meson decay amplitudes given in Ref. 11 are unchanged. For in each case considered, the pseudoscalar *octet* mesons do not require field renormalization and the nine vector mesons are renormalized in the same way as in Ref. 11.

We have shown also that in the context of our Lagrangian, the simplest one that retains all trilinear chiral-symmetric and -breaking terms, the mixing between κ and K^* mesons is negligible. We have also obtained the mass spectrum of scalar mesons in the linear version of the theory and find, as already noted[5] by Gasiorowicz and Geffen, that it is difficult to avoid the scalar mesons having masses which appear to be smaller than the existing experimental candidates.

APPENDIX A: CHIRAL TRANSFORMATIONS

We consider commuting groups $SU(3)^+$ and $SU(3)^-$ which are assumed to transform into each other by a parity transformation. Representations of $SU(3)^+\otimes SU(3)^-$ are denoted by tensors $A_{l_+l_-}{}^{u_+u_-}$, where u_+ stands for a set of upper $SU(3)$ indices, each index taking on values 1, 2, 3 in the $SU(3)^+$ space, u_- is an analogous set of upper indices in the $SU(3)^-$ space, and $l_\pm$ designate similar sets of lower indices. We are particularly interested in the eight-component representations (8,1) and (1,8) and the nine-component representations (3*, 3) and (3, 3*).

Denoting (8, 1) and (1, 8), respectively, by $\mathbf{R}^\pm\equiv R_{i_\pm}{}^{j_\pm}$, with $\mathrm{tr}\mathbf{R}^\pm=0$, we form combinations like $\mathrm{tr}(\mathbf{R}^+\mathbf{S}^+ +\mathbf{R}^-\mathbf{S}^-)$, which are invariant under parity and $SU(3)\otimes SU(3)$ transformations. The representations (3*, 3) and (3, 3*), denoted, respectively, by by $\mathfrak{L}^\pm\equiv\mathfrak{L}_{i_\pm}{}^{j_\mp}$, give typical invariants of the form $\mathrm{tr}(\mathfrak{L}^+\mathfrak{M}^- +\mathfrak{L}^-\mathfrak{M}^+)$. Under rotations characterized by the infinitesimal parameters $\boldsymbol{\alpha}^+\equiv\alpha_{i_+}{}^{j_+}$ in $SU(3)^+$ and $\boldsymbol{\alpha}^-\equiv\alpha_{i_-}{}^{j_-}$ in $SU(3)^-$, we have, e.g., for $\mathfrak{L}^-$,

$$\delta\mathfrak{L}_{i_-}{}^{j_+}=i(\alpha_{i_-}{}^{k_-}\mathfrak{L}_{k_-}{}^{j_+}-\mathfrak{L}_{i_-}{}^{k_+}\alpha_{k_+}{}^{j_+}) \qquad (A1)$$

and similarly for $\mathfrak{L}^+$; alternatively, we write

$$\delta\mathfrak{L}^\pm=i(\boldsymbol{\alpha}^\pm\mathfrak{L}^\pm-\mathfrak{L}^\pm\boldsymbol{\alpha}^\pm). \qquad (A2)$$

Also,

$$\delta\mathbf{R}^\pm=i[\boldsymbol{\alpha}^\pm,\mathbf{R}^\pm]. \qquad (A3)$$

The vector and axial-vector gauge fields treated in the text belong to the representations $(8,1)\oplus(1,8)$ and (1, 1), while the scalar and pseudoscalar fields belong to $(3^*,3)\oplus(3,3^*)$. The physical $SU(3)$ transformations are those with $\boldsymbol{\alpha}^+=\boldsymbol{\alpha}^-=\boldsymbol{\alpha}$. Defining $\mathbf{P}^\pm\equiv\boldsymbol{\sigma}\pm i\boldsymbol{\pi}$ and using (A2), we have

$$\delta\boldsymbol{\sigma}=i[\boldsymbol{\alpha},\boldsymbol{\sigma}], \qquad \delta\boldsymbol{\pi}=i[\boldsymbol{\alpha},\boldsymbol{\pi}]. \qquad (A4)$$

Under the chiral transformations, for which $\boldsymbol{\alpha}^+ = -\boldsymbol{\alpha}^- = \boldsymbol{\beta}$, we have

$$\bar{\delta}\boldsymbol{\sigma} = -\{\boldsymbol{\beta}, \boldsymbol{\pi}\}, \qquad \bar{\delta}\boldsymbol{\pi} = \{\boldsymbol{\beta}, \boldsymbol{\sigma}\}. \tag{A5}$$

Results given in the text for the vector fields are obtained in a similar way.

APPENDIX B: MIXING OF π^0 AND π^8 AND PCAC

We wish to diagonalize the Lagrangian [compare Eq. (17) of the text]

$$\mathfrak{L} = \tfrac{1}{2}\alpha^{ab}\partial_\mu\pi^a\partial_\mu\pi^b - \tfrac{1}{2}\beta^{ab}\pi^a\pi^b, \tag{B1}$$

where α^{ab} is a diagonal matrix with

$$\alpha^{jj} = \alpha_0, \qquad j = 0$$

$$= \alpha_1, \qquad j = 1, \ldots, 8 \tag{B2}$$

and

$$\beta^{jj} = \beta_j, \qquad j = 0, \ldots, 8 \tag{B3}$$

with β^{08} the only nonzero nondiagonal term.

Working only in the nondiagonal 0–8 sector, we let

$$\pi^0 = \lambda'(a\eta + bX), \qquad \pi^8 = b\eta - aX, \tag{B4}$$

where $a = -\sin\theta$, $b = \cos\theta$. To keep the kinetic energy diagonal, we require $\lambda' = (\alpha_1/\alpha_0)^{1/2}$, and get

$$\mathfrak{L}_{\mathrm{KE}} = \tfrac{1}{2}\alpha_1(\partial_\mu\eta\partial_\mu\eta + \partial_\mu X\partial_\mu X), \tag{B5}$$

while diagonalizing the mass terms gives

$$\tan 2\theta = (\alpha_0\alpha_1)^{1/2}\beta^{08}(\alpha_1\beta_0 - \alpha_0\beta_8)^{-1}, \tag{B6}$$

$$m_X^2 + m_\eta^2 = (\alpha_1\beta_0 + \alpha_0\beta_8)/\alpha_0\alpha_1, \tag{B7}$$

$$m_X^2 - m_\eta^2 = \beta^{08}[(\alpha_0\alpha_1)^{1/2}(\sin 2\theta)]^{-1}. \tag{B8}$$

We note also that

$$m_\pi^2 = \beta_1/\alpha_1, \qquad m_K^2 = \beta_4/\alpha_1, \tag{B9}$$

and that the renormalized (physical) fields are

$$\pi_p^1 = (\sqrt{\alpha})\pi^1, \qquad K_p^4 = (\sqrt{\alpha_1})\pi^4, \text{ etc.}, \tag{B10}$$

and

$$\eta_p = a(\sqrt{\alpha_0})\pi^0 + b(\sqrt{\alpha_1})\pi^8, \tag{B11}$$

$$X_p = b(\sqrt{\alpha_0})\pi^0 - a(\sqrt{\alpha_1})\pi^8. \tag{B12}$$

For the nonlinear Lagrangian, Eq. (17), with definitions (18) and (19), we have

$$\alpha_0 = \alpha - 2\gamma F, \qquad \alpha_1 = \alpha + \gamma F,$$

$$\beta_0 = 3\mu F + (2/\sqrt{3})(f_0/2F + 4f_0'), \qquad \beta_8 = 2F_0 - F_8,$$

$$\beta^{08} = -4\sqrt{3}f_8' + 2\sqrt{2}F_8, \tag{B13}$$

$$\beta_1 = 2F_0 + F_8, \qquad \beta_4 = 2F_0 - \tfrac{1}{2}F_8.$$

Calculating the divergence of the axial-vector current and retaining only linear terms,

$$\partial_\mu J_{5\mu}{}^1 = -\sqrt{2}F\beta_1\pi'$$

$$= -(2\alpha_1)^{1/2}Fm_\pi^2\pi_p^1, \tag{B14}$$

so that

$$F = -f_\pi(2\alpha_1)^{-1/2}. \tag{B15}$$

Similarly,

$$\partial_\mu J_{5\mu}{}^4 = -(2\alpha_1)^{1/2}Fm_K^2K_p^4. \tag{B16}$$

Also,

$$\partial_\mu J_{5\mu}{}^8 = -\sqrt{2}F(\beta_8\pi^8 + \tfrac{1}{2}\beta^{08}\pi^0), \tag{B17}$$

$$\partial_\mu J_{5\mu}{}^0 = -\sqrt{2}F(\beta_0\pi^0 + \tfrac{1}{2}\beta^{08}\pi^8). \tag{B18}$$

If we form

$$\partial_\mu J_{5\mu}{}^\eta = a\lambda'\partial_\mu J_{5\mu}{}^0 + b\partial_\mu J_{5\mu}{}^8, \tag{B19}$$

$$\partial_\mu J_{5\mu}{}^X = b\lambda\partial'_\mu J_{5\mu}{}^0 - a\partial_\mu J_{5\mu}{}^8, \tag{B20}$$

we get

$$\partial_\mu J_{5\mu}{}^\eta = -(2\alpha_1)^{1/2}F(a^2\lambda'^2\beta_0 + b^2\beta_8 + ab\lambda'\beta^{08})\eta_p \tag{B21}$$

and

$$\partial_\mu J_{5\mu}{}^X = -(2\alpha_1)^{1/2}F(b^2\lambda'\beta_0 + a^2\beta_8 - ab\lambda'\beta^{08})X_p. \tag{B22}$$

As the quantities in parentheses in the last two equations are, respectively, m_η^2 and m_X^2, we have

$$f_X = f_\eta = f_K = f_\pi. \tag{B23}$$

Similarly, in the linear case, applying Eq. (A5) to Eq. (10), we calculate $\bar{\delta}\mathfrak{L}$ and then

$$\partial_\mu J_{5\mu}{}^a = \partial\bar{\delta}\mathfrak{L}/\partial\beta^a. \tag{B24}$$

In this way we get, e.g. (with $\sigma_8 = 0$),

$$\partial_\mu J_{5\mu}{}^{(1)} = -(\sigma/\sqrt{3})m_\pi^2\pi^1, \tag{B25}$$

$$\partial_\mu J_{5\mu}{}^{(4)} = -(\sigma/\sqrt{3})m_K^2\pi^4, \tag{B26}$$

so that

$$f_\pi = f_K = -\sigma/\sqrt{3}. \tag{B27}$$

From Eq. (92) of the text we get (KSRF relation)

$$\sigma = \pm x = \pm(3m^2/2g^2)^{1/2}, \tag{B28}$$

which gives (for a ρ-meson width of 130 MeV) $f_\pi = 95$ MeV.