Quasiparticle Calculations for a Three-Body Model with Local Potentials

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The quasiparticle concept introduced by Weinberg for two-particle scattering has recently been applied to the three-body problem. It leads to an exact representation of the scattering matrix in terms of a modified Born series, the lowest order of which yields just the widely used separable-potential approximation. To test this method, the binding energies of three identical spinless particles interacting via Yukawa potentials, and the elastic scattering of one particle off a bound state of the other two, are calculated in zeroth- and first-order quasi-Born approximation. Our calculation shows that the inclusion of the first quasi-Born correction greatly improves the zeroth-order, i.e., the separable-potential, results. Comparison is also made with other calculations recently performed in this model.

1. INTRODUCTION

DURING the last few years, the quantum-mechanical three-body problem has been studied with increasing interest. This interest originated in the numerical success of three-body calculations with separable potentials[1–5] on the one hand, and in Faddeev's mathematical studies[6] on the other hand. Both approaches have been combined in the work of Lovelace,[7] who emphasized that the results of the separable-potential model can be rather naturally derived in the framework of Faddeev's theory. Incorporating, in addition, a suitable version of Weinberg's quasiparticle idea, it was shown that these results represent the zeroth order of a general iteration scheme.[8] From this point of view, Faddeev's equations represent a rather suggestive, but neither a necessary nor even always a convenient starting point.[9]

Let us recall some aspects of this method. In the two-particle problem, Weinberg[10] has shown that, by extracting separable terms[11] from the potential, the "quasi-Born" series (i.e., the series containing only the small nonseparable rest potentials) can always be made convergent. The application of this concept to the three-body problem[12] provides us with effective two-particle Lippmann-Schwinger equations, where the occurring "potentials" and "propagators" are given in the form of convergent *quasi-Born series*. In particular, the zeroth-order quasi-Born approximation (QBA) reproduces the separable-potential model.

It is the purpose of the present paper to test this method by numerically calculating the *first* QBA in a model problem of three identical spinless particles, interacting via Yukawa potentials. The resulting three-body binding energies and the phase shifts for elastic scattering (below the three-particle threshold) off an s-wave two-particle bound state are compared with the separable-potential approach (zeroth-order QBA) as well as with "exact" values published recently for this model.

The latter have been calculated by two different methods of evaluating the Faddeev equations. One is based on expanding the two-particle transition operators occurring in them into sums of separable expressions built up from the eigenfunctions of the Lippmann-Schwinger kernel (i.e., by Sturmian functions).[13] The other one consists in solving directly the two-dimensional integral equations which are obtained after angular-momentum decomposition of the Faddeev equations.[14,15]

A further independent comparison is possible for the three-particle binding energy. Here, recent (Rayleigh-Ritz) variational results are available.[15,16]

In the following we consider three different splittings of the original two-particle interactions into separable and nonseparable parts. In all three cases we find that the first-order QBA yields an essential improvement over the separable approximation (zeroth-order QBA).

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[6] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory*, Publication No. 69 of the Steklov Mathematics Institute (Israel Program for Scientific Translation, Jerusalem, 1965).

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[10] S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963). Compare also K. Meetz, J. Math. Phys. **3**, 690 (1961).

[11] Such separable terms correspond to bound states, resonances, or, more generally, to "quasiparticles."

[12] S. Weinberg [Phys. Rev. **133**, B232 (1964)] extended his quasiparticle concept also to the multiparticle case but in a way completely different from ours.

[13] J. S. Ball and D. Wong, Phys. Rev. **169**, 1362 (1968).

[14] T. A. Osborn, Stanford Linear Accelerator Center Report No. 79, 1967 (unpublished).

[15] J. W. Humberstone, R. L. Hall, and T. A. Osborn, Phys. Letters **27B**, 195 (1968).

[16] L. Lovitch and S. Rosati, Nuovo Cimento **63B**, 355 (1969).

The agreement with the "exact" results of Refs. 13–16 is nearly perfect for the binding energies, and is reasonable in the scattering region.

Two further remarks should be added:

(i) It can be shown[17] that the first QBA provides us with rigorous upper bounds for the three-particle binding energy and with lower bounds for the elastic scattering phase shifts (below the three-particle breakup threshold).

(ii) None of the calculations of Refs. 13 and 14 is truly exact since in both cases only the s-wave part of the two-body interaction is taken into account,[18] in contrast to the present work. In fact, it is one of the main advantages of the quasi-Born approach that it automatically takes into account the effect of higher partial-wave interactions, without substantially increasing the computational effort. This point should be of considerable importance if more complicated problems are studied.

In Sec. 2, the general formulas are summarized. Our results are presented in Sec. 3, while details of the calculation are discussed in Appendices A and B. Finally, in Appendix C we show that the formulas of Ref. 9 which are used in the present calculation can also be derived from the approach of Ref. 8 based on Faddeev-type equations.

2. FORMALISM

A Yukawa potential of unit range,

$$V(r)=ge^{-r}/r, \tag{2.1}$$

is taken in the following as the original two-particle interaction. Starting from such a local potential, we apply the formalism of Ref. 9, which is best suited for this case (compare also Appendix C). Accordingly, we have to decompose the original potential into a separable part and a rest term:

$$V(\mathbf{p}'-\mathbf{p})=\chi(\mathbf{p}';z)\lambda(z)\chi^*(\mathbf{p};z^*)+V'(\mathbf{p}',\mathbf{p};z), \tag{2.2}$$

where the form factor χ is chosen as

$$\chi(\mathbf{p};z)=1/[\mathbf{p}^2+\beta(z)^2], \tag{2.3}$$

with $\mathbf{p}$ the relative momentum[19] and z the off-shell energy in the two-particle c.m. system. The functions $\lambda(z)$ and $\beta(z)$ will be specified later.

In operator form, Eq. (2.2) reads

$$V=V^s+V'=|\chi(z)\rangle\lambda(z)\langle\chi(z^*)|+V'. \tag{2.4}$$

The decomposition (2.2) or (2.4) leads to the well-known representation of the two-particle Green's function[12,9]

$$G(z)=G'(z)+G'(z)|\chi(z)\rangle\Delta(z)\langle\chi(z^*)|G'(z), \tag{2.5}$$

with

$$G'(z)=G_0(z)-G_0(z)V'(z)G'(z) \tag{2.6}$$

and[20]

$$\Delta(z)=-[\lambda^{-1}(z)+\langle\chi(z^*)|G'(z)|\chi(z)\rangle]^{-1}. \tag{2.7}$$

Let us define (m is the mass of the particles)

$$I(z)=\langle\chi(z^*)|G_0(z)|\chi(z)\rangle = \frac{\pi^2 m}{\beta(z)[\beta(z)+(-mz)^{1/2}]^2}. \tag{2.8}$$

Then the expression (2.7) becomes, to first order in $V'=V-V^s$ (see Fig. 1),

$$\begin{aligned}\Delta(z)&=-\{\lambda^{-1}(z)+\langle\chi(z^*)|G_0(z)\\ &\qquad -G_0(z)[V-V^s(z)]G_0(z)|\chi(z)\rangle\}^{-1}\\ &=-[\lambda^{-1}(z)+I(z)+\lambda(z)I^2(z)\\ &\qquad -\langle\chi(z^*)|G_0(z)VG_0(z)|\chi(z)\rangle]^{-1}.\end{aligned} \tag{2.9}$$

The last term on the right-hand side of Eq. (2.9) can be easily integrated for a Yukawa potential and a form factor as given by Eq. (2.3), if one goes over to configuration space.[21] Thereby we find

$$\langle\chi(z^*)|G_0(z)VG_0(z)|\chi(z)\rangle = \frac{2\pi^2m^2g}{[\beta^2(z)+mz]^2}\ln\frac{[1+\beta(z)+(-mz)^{1/2}]^2}{[1+2\beta(z)][1+2(-mz)^{1/2}]}. \tag{2.10}$$

It was shown in Ref. 9 (cf. also Ref. 8) that similar techniques, as have been used for the derivation of Eq. (2.5), can also be applied to the *three-body* problem. Starting again from a decomposition (2.2) or (2.4) of the three two-particle potentials occurring in the

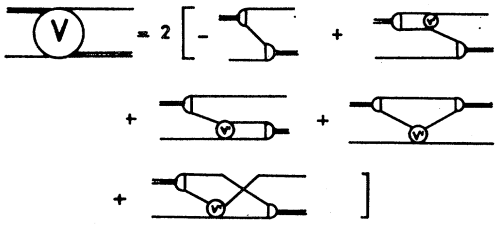

FIG. 2. Particle-quasiparticle "potential," up to first order in V', with symmetrization due to the identity of the three particles taken into account.

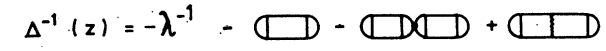

FIG. 1. Quasiparticle propagator, to first order in V' (semicircles indicate form factors, a wavy line the Yukawa potential).

[17] E. O. Alt, P. Grassberger, and W. Sandhas (unpublished).

[18] Without this approximation, one would arrive at a system of coupled equations, i.e., the numerical complexity of the problem would increase considerably. In order to justify the corresponding neglects, the d- and higher partial-wave contributions have been estimated to be small in the model under consideration (Ref. 13).

[19] The relative momenta are introduced as in Ref. 7 but with conventional normalization.

[20] A misprint in Eq. (2.16) of Ref. 9 should be noted. There, λ_α^{-1} has to be replaced by $-\lambda_\alpha^{-1}$.

[21] J. Wright and M. Scadron, Nuovo Cimento **34**, 1571 (1964).

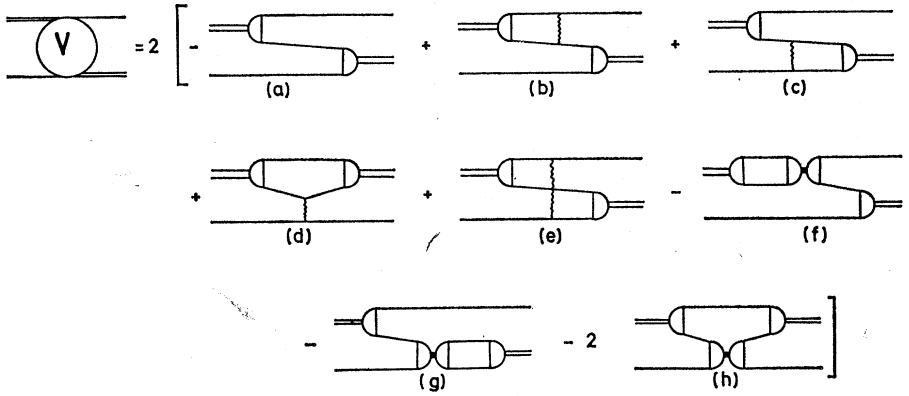

FIG. 3. Same as Fig. 2, but with V' reexpressed by the Yukawa and the separable potential: $V'=V-V^s$.

original equations, a system of integral equations has been derived which has the structure of coupled two-particle Lippmann-Schwinger equations. In the present case, the Pauli principle (incorporated by the method of Ref. 7, Sec. 3 c) and the fact that only *one* separable term is assumed in the decomposition (2.2) reduce this system to a single Lippmann-Schwinger-type equation

$$\mathbf{T}=\mathbf{V}-\mathbf{V}\mathbf{G}_0\mathbf{T} \tag{2.11}$$

for the transition operator $\mathbf{T}$. All the operators in Eq. (2.11) still act on the relative momenta $\mathbf{q}$ and $\mathbf{q}'$ (before and after the collision) between the elementary particle and the bound state. According to Eq. (3.10) of Ref. 9, we have

$$\mathbf{G}_0(\mathbf{q}',\mathbf{q};z)=\delta^{(3)}(\mathbf{q}'-\mathbf{q})\Delta(z-3\mathbf{q}^2/4m). \tag{2.12}$$

If there exists an s-wave two-particle bound state at the energy B_2, then $\Delta(z)$, defined by Eq. (2.7), has a pole at $z=B_2$ and the operator $\mathbf{T}(z)$ is an off-shell extension of the scattering amplitude describing the elastic collision of an elementary particle with this bound state. For calculational convenience we do not require the normalization condition (2.11) or (2.12) of Ref. 9 for the form factors $|\chi(z)\rangle$. Therefore, the residue R_Δ of $\Delta(z)$ will be different from unity. Consequently, our on-shell transition matrix $\mathbf{T}(q,\cos\Theta)$, with $\cos\Theta=\mathbf{q}\cdot\mathbf{q}'/qq'$, is related to the conventionally normalized amplitude of Ref. 9 by

$$\mathbf{T}(\mathbf{q}',\mathbf{q};z=3\mathbf{q}^2/4m+B_2=3\mathbf{q}'^2/4m+B_2)$$
$$=R_\Delta\mathbf{T}(q,\cos\Theta). \tag{2.13}$$

The potential $\mathbf{V}$ is defined in Eq. (3.11) of Ref. 9. Its first QBA [22] used in the following calculations is depicted in Fig. 2. By inserting the definition $V'=V-V^s$ we arrive at a form shown in Fig. 3. The contributions of the various graphs can be easily read off from this figure. Let us only recall that in $|\chi(z)\rangle$, $\lambda(z)$, and $I(z)$ the variable z has to be replaced[7] by the two-particle energy $z-3\mathbf{q}^2/4m$. With this in mind, the contribution of graph (a) of Fig. 3 is [see Fig. 4(a)]

$$\mathbf{V}_a(\mathbf{q}',\mathbf{q};z)=\chi\left(\mathbf{q}+\tfrac{1}{2}\mathbf{q}';z-\frac{3\mathbf{q}'^2}{4m}\right)\left(\frac{\mathbf{q}'^2}{2m}+\frac{(\mathbf{q}+\mathbf{q}')^2}{2m}\right.$$
$$\left.+\frac{\mathbf{q}^2}{2m}-z\right)^{-1}\chi\left(\mathbf{q}'+\tfrac{1}{2}\mathbf{q};z-\frac{3\mathbf{q}^2}{4m}\right) \tag{2.14}$$

and, for instance, the one of graph (e) of Fig. 3 [see Fig. 4(b)] is

$$\mathbf{V}_e(\mathbf{q}',\mathbf{q};z)=\int d^3k\,\chi\left(\mathbf{k}+\tfrac{1}{2}\mathbf{q}';z-\frac{3\mathbf{q}'^2}{4m}\right)$$
$$\times\left(\frac{\mathbf{q}'^2}{2m}+\frac{\mathbf{k}^2}{2m}+\frac{(\mathbf{k}+\mathbf{q}')^2}{2m}-z\right)^{-1}$$
$$\times V(\mathbf{k}+\mathbf{q}+\mathbf{q}')\left(\frac{(\mathbf{k}+\mathbf{q})^2}{2m}+\frac{\mathbf{k}^2}{2m}+\frac{\mathbf{q}^2}{2m}-z\right)^{-1}$$
$$\times\chi\left(\mathbf{k}+\tfrac{1}{2}\mathbf{q};z-\frac{3\mathbf{q}^2}{4m}\right). \tag{2.15}$$

Similar expressions are obtained for the other graphs of Fig. 3, which are, in fact, less complicated than graph (e). Fortunately, for a Yukawa potential and for the simple analytic form, Eq. (2.3), of the form factor,

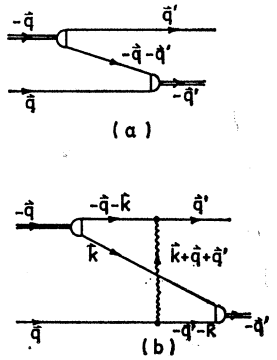

FIG. 4. Two of the terms on the right-hand side of Fig. 3.

[22] As already mentioned, the lowest-order term of $\mathbf{V}$ (zeroth-order QBA) equals the expression derived in the separable-potential model.

the integrals in Eq. (2.15) and in the expressions for $\mathbf{V}_b$, $\mathbf{V}_c$, and $\mathbf{V}_d$ can be done analytically. The details are given in Appendix A.

Up to now we have not yet specified the functions $\beta(z)$ and $\lambda(z)$. Actually, we take three different choices.

(i) The form most widely used in practical calculations is the Yamaguchi[23] ansatz

$$\chi(\mathbf{p})=1/(\mathbf{p}^2+\beta^2), \tag{2.16}$$

with β and λ [the coupling strength, see Eq. (2.4)] taken as constants independent of z. These two parameters are then determined from two characteristic quantities of the two-body system [with the potential (2.1)]. We choose the binding energy B_2 and the scattering length a_s. They are connected with λ and β by

$$\lambda=\beta(\alpha+\beta)^2/\pi^2, \tag{2.17}$$

$$a_s=2(\alpha+\beta)^2/\alpha\beta(\alpha+2\beta), \tag{2.18}$$

with

$$\alpha=(m|B_2|)^{1/2}.$$

(ii) We furthermore study Weinberg's ideal choice[10] which consists in taking for $|\chi(z)\rangle$ the eigenvector of the two-particle Lippmann-Schwinger kernel $VG_0(z)$ with the largest eigenvalue. The function $\lambda(z)$ would then be obtained by requiring

$$V'(z)G_0(z)|\chi(z)\rangle = \{V-|\chi(z)\rangle\lambda(z)\langle\chi(z^*)|\}G_0(z)|\chi(z)\rangle=0. \tag{2.19}$$

As a matter of fact, the exact Lippmann-Schwinger eigenfunctions have no simple analytic representation. But it can be shown[21] that the ansatz (2.3), with β depending on the energy, is a very good approximation for the s-wave function. The optimal choices for the functions $\beta(z)$ and $\lambda(z)$ may then be determined from variational principles. Details of this are given in Appendix B. The results are

$$\beta(z)=\tfrac{3}{4}+\tfrac{1}{2}(-mz)^{1/2} +\tfrac{1}{2}[9/4+5(-mz)^{1/2}-mz]^{1/2}, \tag{2.20}$$

$$\lambda(z)=(g/2\pi^2)[2\beta(z)-1]^2. \tag{2.21}$$

(iii) As a last choice, we consider again a Yamaguchi ansatz (2.3) but now with the two parameters fixed by enforcing the (nearly) correct binding energy B_2 and the residue at the bound-state pole as obtained by

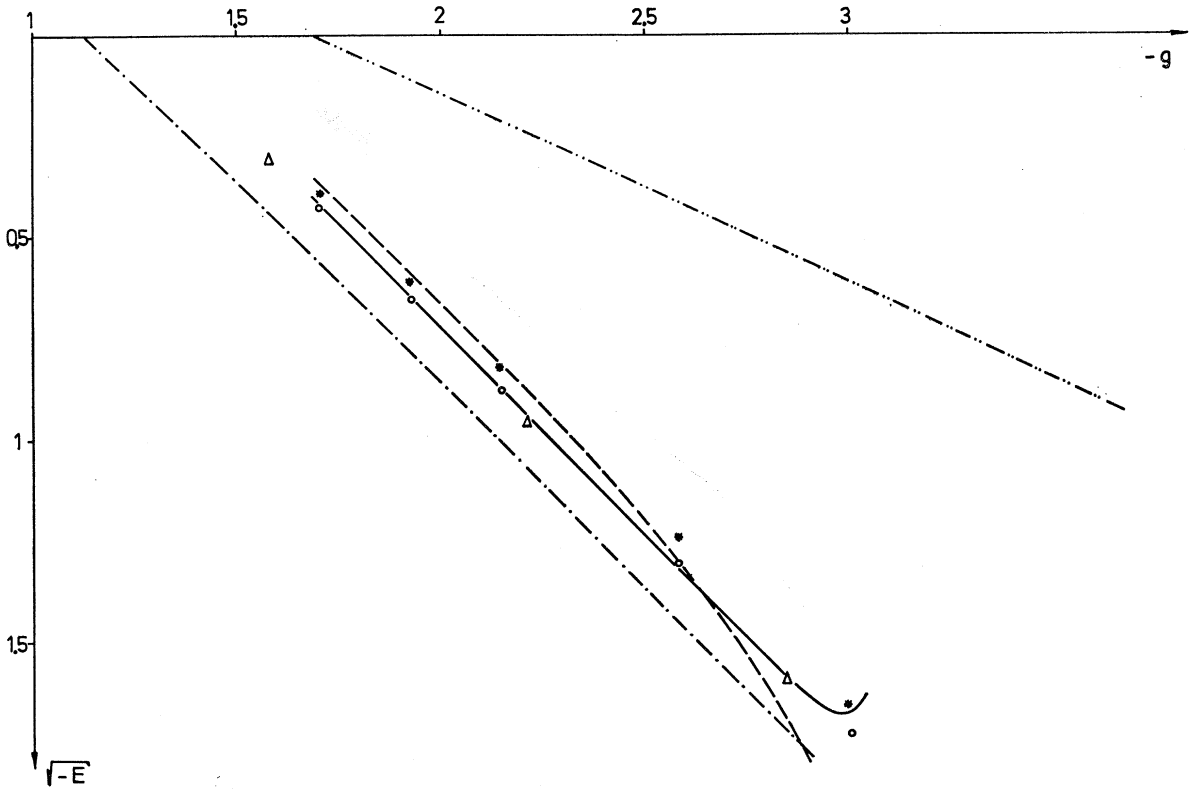

FIG. 5. Binding energies versus Yukawa coupling constant g. Separable parts according to choices (i) and (iii). Two-particle binding energy: double-dot–dashed line. Three-particle binding energy in zeroth-order QBA [choice (i): dashed line; choice (iii): $*$]. Three-particle binding energy in first-order QBA [choice (i): solid line; choice (iii): ○]. Lower bound on the three-particle binding energy: dot-dashed line. Three-particle binding energies obtained in Ref. 16 by a Rayleigh-Ritz variational calculation: △.

[23] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

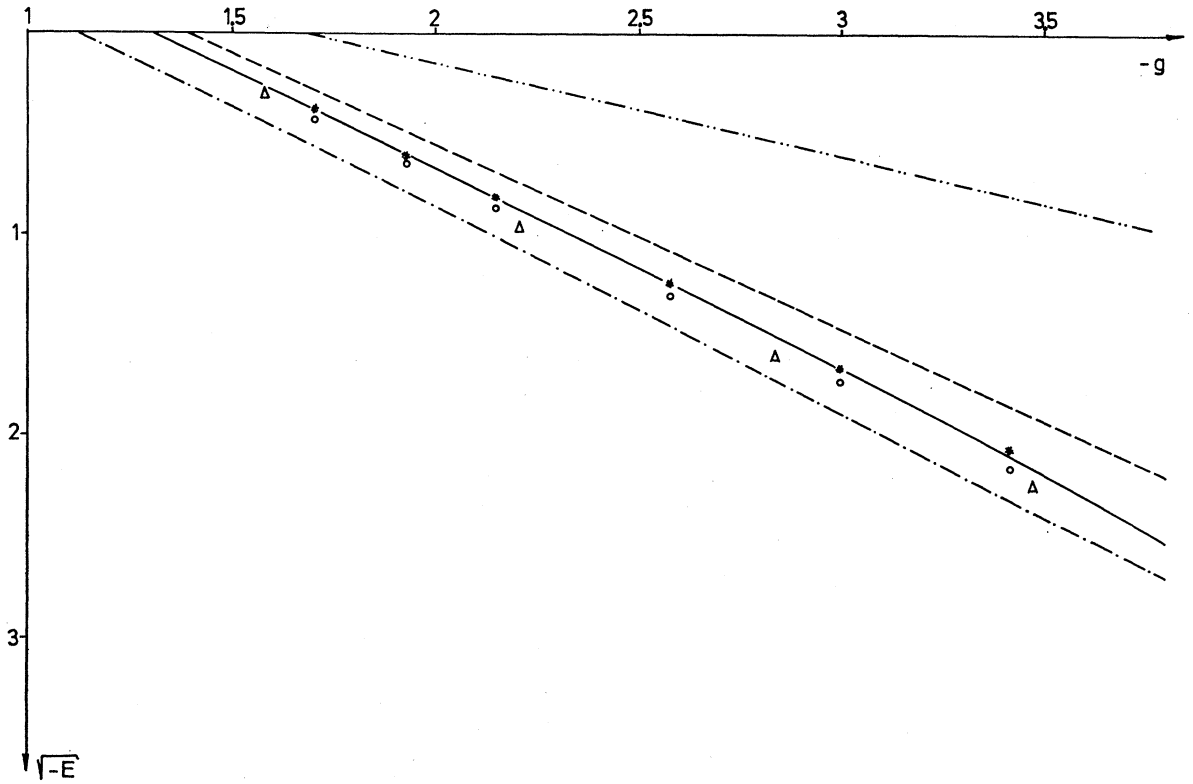

FIG. 6. Same as Fig. 5, but with separable choice (ii) instead of (i).

choice (ii):

$$\beta=\tfrac{3}{4}+\tfrac{1}{2}\alpha+\tfrac{1}{2}(9/4+5\alpha+\alpha^2)^{1/2}, \quad (2.22)$$

$$\lambda=(g/2\pi^2)(2\beta-1)^2. \quad (2.23)$$

3. SOLUTION AND RESULTS

To solve the integral equation, we had to project out the partial waves. For $\mathbf{V}_a$ (and, of course, for $\mathbf{V}_f$ and $\mathbf{V}_g$) this was worked out analytically, whereas for $\mathbf{V}_b$, $\mathbf{V}_c$, $\mathbf{V}_d$, and $\mathbf{V}_e$ it was done on the computer. The loop in $\mathbf{V}_h$ was integrated on the computer too, but only after (analytical) partial-wave projection, which reduced it to a one-dimensional integral.

For the bound state as well as for scattering energies our final Lippmann-Schwinger equations are one-dimensional integral equations. The bound-state equation for total angular momentum $J=0$,

$$\Phi(q)=-\mathbf{G}_0(q;z)\int_0^\infty dq'q'^2\mathbf{V}^{J=0}(q,q';z)\Phi(q'), \quad (3.1)$$

was solved directly by replacing the integral by a finite sum. For elastic scattering, the Lippmann-Schwinger equation was first transformed to a nonsingular integral equation by use of the method of Kowalski and Noyes[24] and then replaced by a matrix equation. Gaussian integration techniques were used throughout.

Three-body binding energies[25] obtained with the choices (i)–(iii), as discussed in Sec. 2, are shown in Figs. 5 and 6. There, we also exhibit the zeroth-order QBA, the variational values of Ref. 16, and a rigorous lower bound[26] on the true three-particle binding energy B_3.

Here we have to make the following remarks:

(a) For case (i), the rigorous lower bound is violated in zeroth-order QBA for $|g|>2.88$. Indeed, both the zeroth- and the first-order binding energies become singular for $|g|\sim 3.5$. This can easily be understood: From Eq. (2.18) it follows that for a separable potential of the Yamaguchi type $a_s\gtrsim 1/\alpha$ always. This condition is, however, violated by a Yukawa potential for $|g|\gtrsim 3.5$. Thus for these coupling strengths, a separable fit to the scattering lengths becomes impossible. This does not concern the choices (ii) and (iii).

(b) As will be shown elsewhere,[17] the first QBA represents a rigorous upper bound on the exact value of B_3. From this follows by inspection of Figs. 5 and 6 that for $1.683\leqslant|g|\lesssim 2.8$, the choices (i) and (iii) seem to be superior to (ii). For $|g|>2.8$, choice (iii) still seems to be better than (ii).

[24] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965); H. P. Noyes, *ibid.* **15**, 538 (1965).

[25] In the calculations, we set the mass $m=1$.

[26] R. L. Hall and H. R. Post, Proc. Phys. Soc. (London) **90**, 381 (1967).

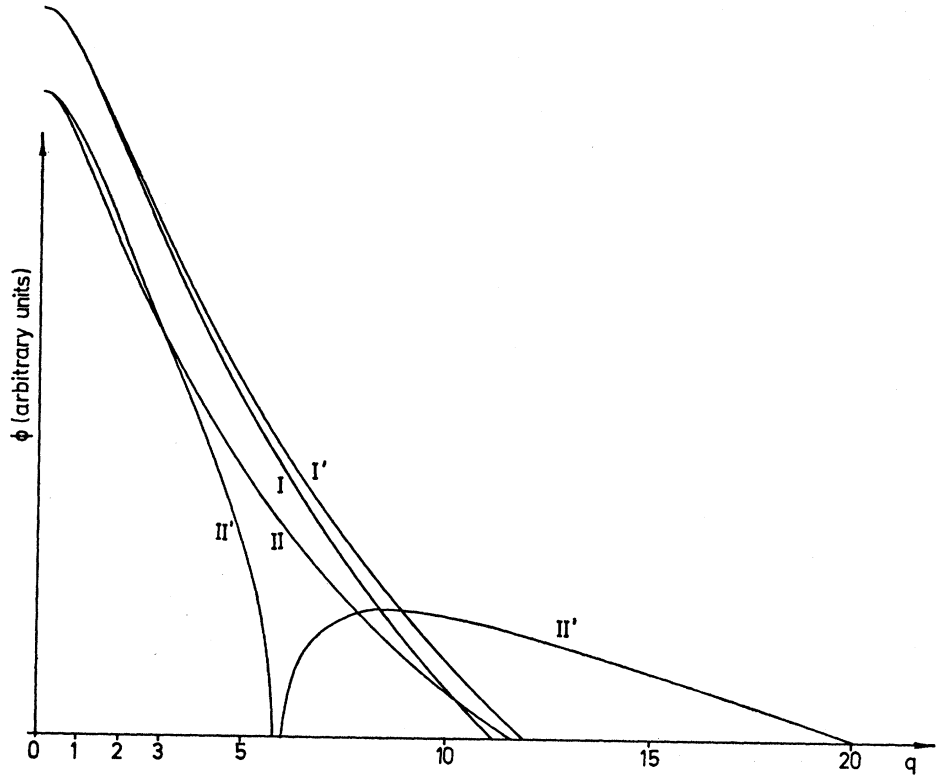

FIG. 7. "Wave function" $\phi(q)$, defined by Eq. (3.1), in the various approximations (for $g=-2.5$): I, choice (i) in zeroth-order QBA; I', choice (i) in first-order QBA; II, choice (ii) in zeroth-order QBA; II', choice (ii) in first-order QBA. Choice (iii) yields a wave function similar to the one of choice (i).

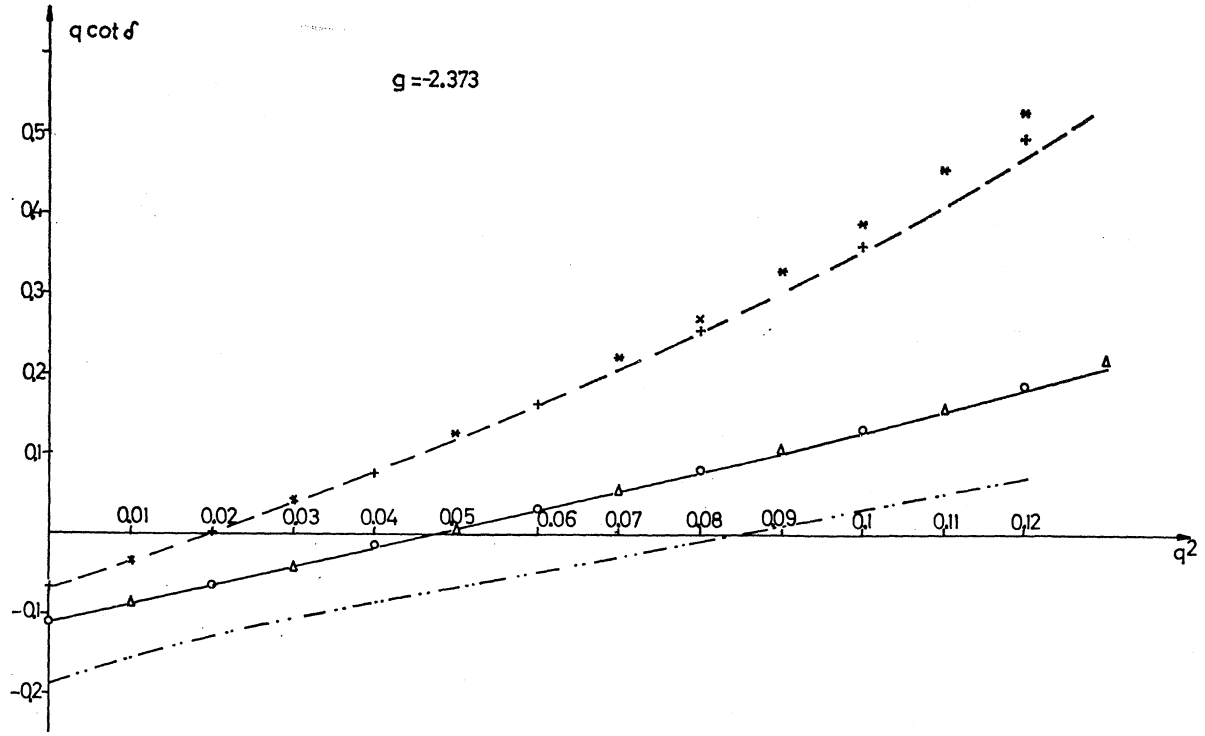

FIG. 8. Plot of $q\cot\delta$ for s waves versus q^2 for a Yukawa coupling constant $g=-2.373$. The curves are the zeroth-order QBA [+, choice (i); dashed line, choice (ii); *, choice (iii)] and the first-order QBA [○, choice (i); solid line, choice (ii); △, choice (iii)]. "Exact" results of Ref. 13 (obtained with four separable terms) are also shown: double-dot–dashed line.

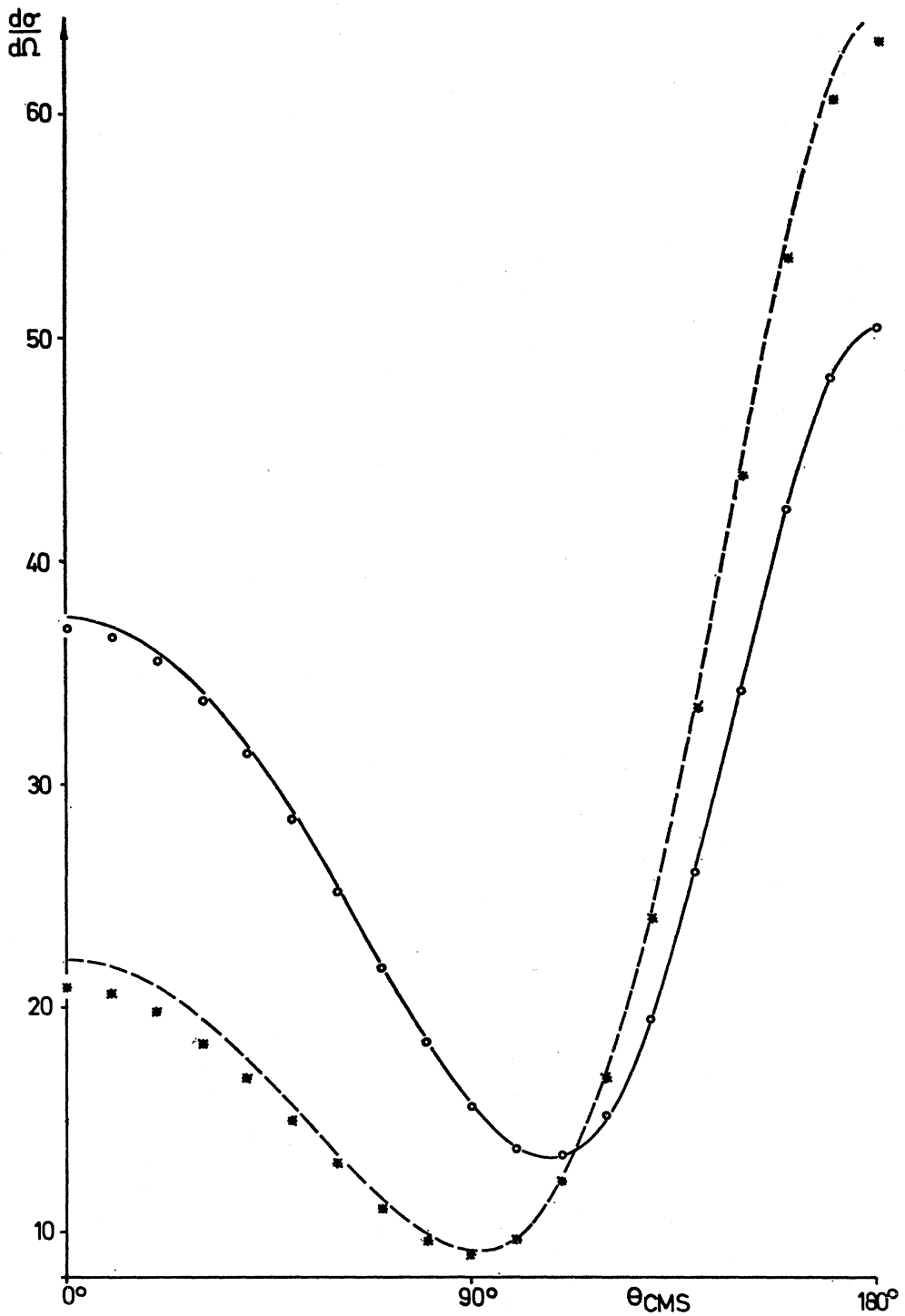

FIG. 9. Differential cross section in the c.m. system for elastic scattering ($q^2=0.06$) for Yukawa coupling constant $g=-2.373$ in zeroth-order QBA [$*$, choice (i); dashed line, choice (ii)] and in first-order QBA [○, choice (i); solid line, choice (ii)].

(c) All integrations were performed with a 10-point mesh. Their accuracy was tested both by varying the mapping of the momentum interval $(0,\infty)$ onto the range of integration and by varying the number of mesh points. In this way for $|g|\lesssim 3$, the error in the largest eigenvalue of $\mathbf{G_0V}$ was estimated to be smaller than 1%. Whereas this situation remained the same for the zeroth-order QBA, and for the first QBA in the case of choices (i) and (iii) even for $|g|\geqslant 4$, the results for the first QBA in the case of choice (ii) became rather

inexact. The reason for this behavior is that for choice (ii), $\mathbf{V}^{J=0}(q',q;z)$ is not a positive definite function for $z<0$, in contrast to its zeroth-order part $\mathbf{V}_a$ and to choices (i) and (iii). Indeed, the "wave function" $\phi(q)$ [Eq. (3.1)] has a zero for choice (ii) which is not present for the choices (i) and (iii), and the zeroth-order QBA.

These various "wave functions" are shown in Fig. 7, for $g=-2.5$. We finally mention that the main correction to the zeroth-order QBA, $\mathbf{V}_a$, comes from the fourth term of the right-hand side of Fig. 2. All other terms of that figure are much less important.

For scattering energies, calculations were performed only in the energy region below the three-particle threshold. In Fig. 8, $q\cot\delta$ of the s-wave phase shifts is depicted in zeroth- and first-order QBA for all three separable choices. The Yukawa coupling constant is fixed at a value $g=-2.373$. This allows comparison with the corresponding result of Ball and Wong[13] (we show their curve $n_0=4$). Differential cross sections for the same value of g and relative momentum $q^2=0.06$ are plotted in Fig. 9.

The numerical errors of the 10-point Gaussian integration are estimated from variation of the integration mesh to be at most about 0.003 [for choice (ii) the accuracy is again worse than for (i) and (iii)]. By comparison with Ball and Wong, we find that our results are not as good as in the bound-state region, although the improvement over the separable approximation (zeroth-order QBA) is still remarkable. The differences between all three choices nearly vanish in first-order QBA.

We conclude that the quasi-Born approximation scheme represents a practical tool for three-body calculations for local potentials.

ACKNOWLEDGMENTS

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APPENDIX A

The integral in Eq. (2.15) and the similar expressions for Figs. 3(b)–3(d) are special cases of an integral given in the literature,[27]

$$I_1(\xi;\mathbf{p}_1,\eta_1;\mathbf{p}_2,\eta_2)$$
$$=\int\frac{d^3k}{[(\mathbf{k}-\mathbf{p}_1)^2+\eta_1^2][(\mathbf{k}-\mathbf{p}_2)^2+\eta_2^2][\mathbf{k}^2+\xi^2]}$$
$$=\frac{\pi^2}{(\rho^2-\sigma)^{1/2}}\ln\left[\frac{\rho+(\rho^2-\sigma)^{1/2}}{\rho-(\rho^2-\sigma)^{1/2}}\right], \quad \text{(A1)}$$

where

$$\rho=\xi[(\mathbf{p}_1-\mathbf{p}_2)^2+(\eta_1+\eta_2)^2]+\eta_2[\xi^2+\mathbf{p}_1^2+\eta_1^2]+\eta_1[\xi^2+\mathbf{p}_2^2+\eta_2^2] \quad \text{(A2)}$$

and

$$\sigma=[(\mathbf{p}_1-\mathbf{p}_2)^2+(\eta_1+\eta_2)^2][\mathbf{p}_1^2+(\eta_1+\xi)^2]\times[\mathbf{p}_2^2+(\xi+\eta_2)^2]. \quad \text{(A3)}$$

In detail, the contributions of Figs. 3(b), 3(c), and 3(e) to the potential $\mathbf{V}$ are

$$\mathbf{V}_b(\mathbf{q}',\mathbf{q};z)=\frac{gm^2}{(p_1^2+x_1^2)(p_1^2+\beta_1^2)}I_1(\mu;\mathbf{p}_2,\beta_2;\mathbf{p}_2,x_2), \quad \text{(A4)}$$

$$\mathbf{V}_c(\mathbf{q}',\mathbf{q};z)=\mathbf{V}_b(\mathbf{q},\mathbf{q}';z), \quad \text{(A5)}$$

and

$$\mathbf{V}_e(\mathbf{q}',\mathbf{q};z)=\frac{gm^2}{(\beta_1^2-x_1^2)(\beta_2^2-x_2^2)}[I_1(\mu;\mathbf{p}_1,x_1;\mathbf{p}_2,x_2)-I_1(\mu;\mathbf{p}_1,x_1;\mathbf{p}_2,\beta_2)-I_1(\mu;\mathbf{p}_1,\beta_1;\mathbf{p}_2,x_2)+I_1(\mu;\mathbf{p}_1,\beta_1;\mathbf{p}_2,\beta_2)]. \quad \text{(A6)}$$

Here

$$\mathbf{p}_1=\mathbf{q}+\tfrac{1}{2}\mathbf{q}', \qquad \mathbf{p}_2=\mathbf{q}'+\tfrac{1}{2}\mathbf{q},$$
$$x_1=(\tfrac{3}{4}\mathbf{q}'^2-mz)^{1/2}, \quad x_2=(\tfrac{3}{4}\mathbf{q}^2-mz)^{1/2},$$

and

$$\beta_1=\beta(z-3\mathbf{q}'^2/4m), \quad \beta_2=\beta(z-3\mathbf{q}^2/4m).$$

APPENDIX B

The variational principle is the one described in Refs. 28 and 21. In Ref. 28 we have shown that

$$[\eta(z)]=\frac{\langle\chi(z^*)|G_0(z)|\chi(z)\rangle}{\langle\chi(z^*)|V^{-1}|\chi(z)\rangle} \quad \text{(B1)}$$

becomes stationary if $|\chi(z)\rangle$ is an eigenfunction of the Lippmann-Schwinger kernel $VG_0(z)$. The largest eigenvalue and the corresponding eigenfunctions are then found by extremizing $[\eta(z)]$.

If we insert our ansatz (2.3) for $\chi(\mathbf{p};z)$, the numerator of Eq. (B1) becomes

$$\langle\chi(z^*)|G_0(z)|\chi(z)\rangle=\frac{\pi^2 m}{\beta(z)\{\beta(z)+(-mz)^{1/2}\}^2}, \quad \text{(B2)}$$

while the denominator is most easily calculated in configuration space, yielding

$$\langle\chi(z^*)|V^{-1}|\chi(z)\rangle=\frac{2\pi^2}{g\{2\beta(z)-1\}^2}$$

and, therefore,

$$[\eta(z)]=\frac{mg\{2\beta(z)-1\}^2}{2\beta(z)\{\beta(z)+(-mz)^{1/2}\}^2}. \quad \text{(B3)}$$

[27] R. R. Lewis, Phys. Rev. **102**, 537 (1956).

[28] E. O. Alt, P. Grassberger, and W. Sandhas, Phys. Rev. C **1**, 85 (1969).

The latter becomes an extremum if $\beta(z)$ is given by Eq. (2.20). The "coupling strength" $\lambda(z)$ is obtained from Eq. (2.19):

$$\{V-|\chi(z)\rangle\lambda(z)\langle\chi(z^*)|\}G_0(z)|\chi(z)\rangle=0. \quad \text{(B4)}$$

Using $VG_0(z)|\chi(z)\rangle\approx[\eta(z)]|\chi(z)\rangle$, which is fulfilled with sufficient accuracy, we thus obtain

$$\lambda^{-1}(z)=\langle\chi(z^*)|V^{-1}|\chi(z)\rangle, \quad \text{(B5)}$$

leading directly to Eq. (2.21).

APPENDIX C

In Ref. 9 it is shown that by splitting the two-particle interactions into separable terms and a nonseparable rest, the three-body resolvent equation can be reduced to exact effective two-particle equations. Previously, an analogous technique, making use of a splitting of the two-particle transition operators T_γ into separable parts and a nonseparable remainder, has been applied to Faddeev-type equations.[8] With respect to this approach, the one of Ref. 9 has the advantage that it provides us directly with a particularly simple form of the occurring potentials. To clarify the relation between both formalisms, we show that by a special splitting of T_γ the simple expression for the "potentials" obtained in Ref. 9 can also be derived from the results of the Faddeev approach.[8]

In Eq. (2.11) of Ref. 8, we defined transition operators $U_{\beta\alpha}$ connected with the full Green's function by

$$G=\delta_{\beta\alpha}G_\alpha-G_\beta U_{\beta\alpha}G_\alpha. \quad \text{(C1)}$$

The $U_{\beta\alpha}$ fulfill the Faddeev-type equations

$$U_{\beta\alpha}=-(1-\delta_{\beta\alpha})G_0^{-1}-\sum_{\gamma\neq\beta}T_\gamma G_0 U_{\gamma\alpha}. \quad \text{(C2)}$$

Splitting T_γ according to [Eq. (4.5) of Ref. 8]

$$T_\gamma=-\sum_{r,s}|\gamma,r\rangle\Delta_{\gamma,rs}\langle\gamma,s|+T_\gamma', \quad \text{(C3)}$$

we found the multichannel Lippmann-Schwinger equations of Ref. 8,

$$\mathbf{T}_{\beta n,\alpha m}=\mathbf{V}_{\beta n,\alpha m}-\sum_{\gamma,r,s}\mathbf{V}_{\beta n,\gamma r}\Delta_{\gamma,rs}\mathbf{T}_{\gamma s,\alpha m} \quad \text{(C4)}$$

for the transition amplitudes

$$\mathbf{T}_{\beta n,\alpha m}=\langle\beta n|G_0U_{\beta\alpha}G_0|\alpha m\rangle. \quad \text{(C5)}$$

The "potentials" are given by

$$\mathbf{V}_{\beta n,\alpha m}=\langle\beta n|G_0U_{\beta\alpha}'G_0|\alpha m\rangle, \quad \text{(C6)}$$

where $U_{\beta\alpha}'$ fulfills Eq. (C2) with T_γ replaced by T_γ'.

If we now start from a decomposition of the two-particle interaction as in Eq. (2.4) of Ref. 9,

$$V_\gamma=\sum_r|\chi_{\gamma r}\rangle\lambda_{\gamma r}\langle\chi_{\gamma r}|+V_\gamma', \quad \text{(C7)}$$

we find, via the quasiparticle method,[10] the explicit expression

$$T_\gamma=-\sum_{r,s}(1-T_\gamma'G_0)|\chi_{\gamma r}\rangle \times\Delta_{\gamma,rs}\langle\chi_{\gamma s}|(1-G_0T_\gamma')+T_\gamma', \quad \text{(C8)}$$

which is of the form (C3). Here we have

$$(\Delta_\gamma^{-1})_{rs}=-\delta_{rs}\lambda_\gamma^{-1}-\langle\chi_{\gamma r}|G_0-G_0T_\gamma'G_0|\chi_{\gamma s}\rangle, \quad \text{(C9)}$$

that is, the quasiparticle propagator, Eq. (2.9) of Ref. 9· T_γ' is defined as T_γ, but with V_γ replaced by V_γ'. Consequently, we have, in analogy to Eq. (C1),

$$G'=\delta_{\beta\alpha}G_\alpha'-G_\beta'U_{\beta\alpha}'G_\alpha', \quad \text{(C10)}$$

with

$$G'(z)=(H_0+\sum_\gamma V_\gamma'-z)^{-1}$$

and

$$G_\alpha'(z)=(H_0+V_\alpha'-z).$$

Inserting in Eq. (C6) the $|\chi_{\gamma r}\rangle$ following from comparison of Eq. (C3) with Eq. (C8), we have (with $G_\gamma'=G_0-G_0T_\gamma'G_0$)

$$\mathbf{V}_{\beta n,\alpha m}=\langle\chi_{\beta n}|G_\beta'U_{\beta\alpha}'G_\alpha'|\chi_{\alpha m}\rangle =-\langle\chi_{\beta n}|G'-\delta_{\beta\alpha}G_\alpha'|\chi_{\alpha m}\rangle. \quad \text{(C11)}$$

This is, indeed, the form of the potentials derived in Ref. 9.