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Algebraic Realization of Weinberg's Superconvergence Conditions*

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We investigate the algebraic realization of a pair of sum rules proposed by Weinberg for forward scattering of massless pions, for the case when only p-wave pions are taken into account. It is shown that the algebraic structure of the first superconvergence condition is given by the Lie algebra of the group $SU(2)\otimes SU(4)$. A few p-wave pion decay widths are calculated and found to agree satisfactorily with experiment. It is further shown that the mass spectra obtained from the second superconvergence conditon are unsatisfactory, as they predict that the hadron masses decrease with increasing isospin (or spin). Various possible reasons for this defect are discussed.

I. INTRODUCTION

TWO years ago it was shown by Capps[1] that superconvergence sum rules saturated with single-particle states give rise, under fairly general assumptions, to models in which the hadron states are connected with unitary representations of Lie groups. The fundamental requirements of the Capps bootstrap scheme are the following: (i) The dispersion integrals in the representation for the scattering amplitude at fixed momentum transfer t can be saturated with single-particle states in such a way that proper Regge behavior is not destroyed, and (ii) the set of internal and external hadrons are identical. These two conditions lead immediately to a Lie-algebraic formulation of the superconvergence conditions.

More recently, Weinberg[2] derived from a chiral Lagrangian a completely new class of sum rules for the forward scattering ($t=0$) of massless pions. Weinberg's sum rules have the elegant form

$$[X_\alpha,X_\beta]=i\epsilon_{\alpha\beta\gamma}I_\gamma \tag{1.1}$$

and

$$[X_\alpha,[m^2,X_\beta]]\sim\delta_{\alpha\beta}\,, \tag{1.2}$$

where the symbols I_γ denote the generators of the $SU(2)_I$ isospin group, the operator m^2 is a diagonal-mass-square operator, and the operators X_α are meson-source operators which will be defined presently. We shall refer to the sum rule equations (1.1) and (1.2) as Weinberg's first and second superconvergence relations even though some authors restrict the term "superconvergence relation" to results which follow from assumptions about high-energy behavior alone.

The matrix elements of the meson-source operators can be defined in forms of the Feynman amplitude M for the collinear process $B(i,i_z\lambda)\to B(I,I_z\lambda')+\pi_\alpha$ as follows[2]:

$$M[B(i,i_z,\lambda)\to B(I,I_z\lambda')+\pi_\alpha]
=2F_\pi^{-1}[m^2(i)-m^2(I)]\langle I,I_z,\lambda'|X_\alpha|i_\alpha,i_z,\lambda\rangle\,. \tag{1.3}$$

Here the symbol F_π is a constant ($F_\pi\approx190$ MeV), the symbols λ and λ' denote the helicities of the hadrons $B(i,i_z,\lambda)$, respectively, with their momenta along the z axis, and α is the isovector index of the pion. The meson-source operator is also diagonal in helicity,

$$\langle I,I_z,\lambda'|X_\alpha|i,i_z,\lambda\rangle=\delta_{\lambda\lambda'}\langle I,I_z|X_\alpha|i,i_z\rangle\,. \tag{1.4}$$

Equations (1.1) and (1.2) are necessary conditions for the requirement that the tree-graph contributions to the forward amplitude should not grow faster in energy than allowed by Regge-pole theory, which is the basic assumption in Weinberg's papers.[2] This requirement is equivalent to the saturation of dispersion-theoretic sum rules with single-particle states as was mentioned above, and also shown by Weinberg. In particular, Eq. (1.1) follows by considering the tree-graph contributions to the part of the amplitude (antisymmetric with respect to pion isovector indices) which has pure isospin $I=1$ exchanged in the t channel, under the further assumption that the single-pion couplings and the axial-vector current are related through the Goldberger-Treiman relation.

The commutator (1.2) follows by considering the tree-graph contributions to the part of the amplitude which is symmetric in the pion isovector indices, and which generally has isospin $I=0$ and $I=2$ exchanged in the t channel, under the additional assumption that there are no "exotic" meson states with $I=2$ exchanged in the t channel. If such a meson exchange is allowed, then one should add a term to the right-hand side of Eq. (1.2) which transforms as an isotensor with $I=2$. However, there is no convincing experimental indication for the existence of an $I=2$ meson resonance.

The first superconvergence relation (1.1), together with the commutation relations satisfied by the generators I_α of the isospin group $SU(2)_I$, defines the Lie

* Supported in part by the U. S. Atomic Energy Commission.
[1] R. H. Capps, Phys. Rev. 168, 1731 (1968); 171, 1591 (1969).
[2] S. Weinberg, Phys. Rev. 177, 2604 (1969); Phys. Rev. Letters 22, 1023 (1969); in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.

algebra of the chiral group $SU(2)\otimes SU(2)$, and tells us that the single-particle states involved in the tree graphs must, for each helicity λ, be assigned to a unitary representation of the chiral group. The commutator (1.1) then determines the coupling constants for the particles (with the same helicity but different isospins) which belong to the same unitary (irreducible or reducible) representation of the chiral group. The second superconvergence relation (1.2) then gives a condition on the masses of the particles in the representation. The algebraic structures of superconvergence relations proposed by Weinberg[2] are equivalent to those which were derived by Gilman and Harari.[3]

Unfortunately, the commutation relations (1.1) and (1.2) do not provide any information on how the representations with different helicities are related to each other. As was pointed out by Weinberg,[2] the helicity dependence of the operators X_α can be determined if one assumes that only a few partial waves are involved in the pion-decay processes $B(I,I_z,J,J_z)\rightarrow B(i,i_z,j,j_z)+\pi_\alpha$, where I, i and J, j denote the isospin and spin, respectively, of the hadrons in question. In particular, transitions between states of nearly the same mass and the same parity involve only p-wave pions, as, for example, in the decays $\Delta\rightarrow N\pi$, $Y_1^*\rightarrow\Lambda\pi$, $Y_1^*\rightarrow\Sigma\pi$, and $\Xi^*\rightarrow\Xi\pi$.

The purpose of the present paper is to investigate the algebraic realization of the superconvergence conditions (1.1) and (1.2) for the case when only p-wave pions are taken into account. Sections II and III deal with the algebraic realization of the first superconvergence relation. The results given in these sections have been previously described in a brief paper,[4] and are included here in a slightly expanded form for completeness. The main result obtained is that the algebraic structure of the first superconvergence condition (involving p-wave pions only) is given by the Lie algebra of the group $SU(2)\otimes SU(4)$. In the next two sections we derive the mass formulas for the hadrons which furnish irreducible unitary representations of the group $SU(2)\otimes SU(4)$. The representations we consider are obtained by using the most degenerate representations of the group $SU(4)$. The mass formulas obtained are unsatisfactory since they show that the hadron masses decrease with increasing isospin (or with increasing spin). Various possible reasons for this defect are discussed in the concluding section VII. We also calculate a few pion-decay widths in Sec. VI by using the representations for the group $SU(2)\otimes SU(4)$. These calculations compare favorably with experimental results.

[3] F. J. Gilman and H. Harari, Phys. Rev. Letters **19**, 723 (1967); Phys. Rev. **165**, 1803 (1968).

[4] C. Cronström and M. Noga, Nucl. Phys. **B15**, 61 (1970).

II. LIE-GROUP STRUCTURE OF FIRST SUPERCONVERGENCE RELATION IN THE CASE OF p-WAVE HADRON INTERACTION

The invariance symmetry group K for the p-wave pion-hadron interaction is given by $K=SU(2)_I\otimes SU(2)_J$, the direct product of the isospin $SU(2)_I$ and spin $SU(2)_J$ groups. The Lie algebra of the group K is

$$[I_\alpha,I_\beta]=i\epsilon_{\alpha\beta\gamma}I_\gamma\,, \tag{2.1}$$

$$[J_a,J_b]=i\epsilon_{abc}J_c\,, \tag{2.2}$$

and

$$[I_\alpha,J_a]=0\,. \tag{2.3}$$

The p-wave pion-source operator is denoted by $D_{\alpha a}$. This operator is an irreducible tensor of rank (1,1) with respect to the group K, i.e., the operator $D_{\alpha a}$ satisfies the commutation relations

$$[I_\alpha,D_{\beta b}]=i\epsilon_{\alpha\beta\gamma}D_{\gamma b} \tag{2.4}$$

and

$$[J_a,D_{\beta b}]=i\epsilon_{abc}D_{\beta c}\,. \tag{2.5}$$

Here the indices α, a denote the third components of the isospin and angular momentum of the p-wave pion, respectively.

The condition that Eqs. (1.1), (1.2), and (1.4) involve only p waves is enforced by requiring the operator X_α to transform as the component of the tensor $D_{\alpha a}$, which cannot change the helicity, so that

$$X_\alpha\equiv D_{\alpha 3}\,. \tag{2.6}$$

The fundamental commutation relation (1.1) then gives

$$[D_{\alpha 3},D_{\beta 3}]=i\epsilon_{\alpha\beta\gamma}I_\gamma\,. \tag{2.7}$$

In order to find the complete helicity dependence of the operator $D_{\alpha a}$, we must find the general expression for the commutator $[D_{\alpha a},D_{\beta b}]$. This quantity is odd with respect to the interchange of the pair of indices αa and βb; therefore, we have

$$[D_{\alpha a},D_{\beta b}]=i\epsilon_{\alpha\alpha\gamma}A_{\gamma(ab)}+i\epsilon_{abc}B_{c(\alpha\beta)}\,, \tag{2.8}$$

where $A_{\gamma(ab)}$ is a reducible isovector symmetric spin tensor and $B_{c(\alpha\beta)}$ is a reducible spin-vector symmetric isotensor. [We use the notation $A_{\gamma(ab)}\equiv\frac{1}{2}(A_{\gamma ab}+A_{\gamma ba})$.]

The right-hand side of Eq. (2.8) is interpreted as a contribution from single particle exchanges in the t channel, i.e., $A_{\gamma(ab)}$ is connected with the exchange of particles having $I=1$ and $J=0$ and 2, while $B_{c(\alpha\beta)}$ is connected with exchange of particles having $J=1$ and I=0 and 2. In order to exclude the "exotic" state $(I,J)=(2,1)$, we require $B_{c(\alpha\beta)}$ to be an isospin scalar,

$$B_{c(\alpha\beta)}=\delta_{\alpha\beta}B_c\,. \tag{2.9}$$

The spin vector B_c then satisfies the commutation relations

$$[J_a,B_b]=i\epsilon_{abc}B_c \tag{2.10}$$

and

$$[I_\alpha,B_a]=0\,. \tag{2.11}$$

Equations (2.7) and (2.8) now imply that

$$A_{\gamma(33)}=I_\gamma. \tag{2.12}$$

The commutator of $A_{\gamma(33)}$ with J_a is zero; therefore, $A_{\gamma(ab)}$ is a spin scalar,

$$A_{\gamma(ab)}=\delta_{ab}I_\gamma. \tag{2.13}$$

Using Eqs. (2.9) and (2.13), we rewrite Eq. (2.8) as

$$[D_{\alpha a},D_{\beta b}]=i\delta_{\alpha\beta}\epsilon_{abc}B_c+i\delta_{ab}\epsilon_{\alpha\beta\gamma}I_\gamma. \tag{2.14}$$

From Eq. (2.14) it is now simple to calculate the commutators $[B_c,D_{\gamma d}]$ and $[B_a,B_b]$. The result is

$$[B_c,D_{\gamma d}]=i\epsilon_{cde}D_{\gamma e} \tag{2.15}$$

and

$$[B_a,B_b]=i\epsilon_{abc}B_c. \tag{2.16}$$

The commutators (2.1)–(2.5), (2.10), (2.11), and (2.14)–(2.16) show that the operators I_α, J_a, B_a, and $D_{\alpha a}$ form a closed Lie algebra which defines a certain Lie group G. In order to find the structure of this group it is convenient to introduce a generator V_a defined as follows:

$$V_a=J_a-B_a. \tag{2.17}$$

It is simple to verify that the generators V_a commute with all the generators I_α, B_a, and $D_{\alpha a}$, and that they satisfy the $SU(2)_V$ commutation relations

$$[V_a,V_b]=i\epsilon_{abc}V_c. \tag{2.18}$$

This means that the group G is the direct product of $SU(2)_V$ with a group G_0, which is defined by the commutators (2.1), (2.4), (2.11), and (2.14)–(2.16). The last-mentioned commutators define the Lie algebra of the group $SU(4)_{IB}$, which contains $SU(2)_I\otimes SU(2)_B$ defined by the commutators (2.1), (2.11), and (2.16), as a maximal compact subgroup.

We can conclude that the Lie-group structure of the first superconvergence condition, involving p waves only, is given by the group $G=SU(2)\otimes SU(4)$ which only for the special case $V=0$ reduces to the group $SU(4)$ considered by Weinberg.[2] This implies that hadron states involved in the tree graphs form the basis for the unitary representations of the group $SU(2)_V\otimes SU(4)_{IB}$.

III. ALGEBRAIC REALIZATION OF FIRST SUPERCONVERGENCE CONDITION

In order to calculate the p-wave pion decay rates we must find the representations of the algebra $SU(2)_V \otimes SU(4)_{IB}$ in the spherical basis $|I,I_z,J,J_z\rangle$. Since the operator $D_{\alpha a}$ is an irreducible tensor of rank (1,1) with respect to the group $K=SU(2)_I\otimes SU(2)_J$, we can write the matrix elements of the operator $D_{\alpha a}$ between the states $|I,I_z,J,J_z\rangle$ and $|i,i_z,j,j_z\rangle$ in the following form:

$$\langle I,I_z,J,J_z|D_{\alpha a}|i,i_z,j,j_z\rangle =G_{IJ}{}^{ij}\begin{pmatrix} i & 1 & I\\ i_z & \alpha & I_z\end{pmatrix}\begin{pmatrix} j & 1 & J\\ j_z & a & J_z\end{pmatrix}. \tag{3.1}$$

Here the symbols (:::) stand for the Clebsch-Gordan (CG) coefficients of the appropriate $SU(2)$ groups, and the quantity $G_{IJ}{}^{ij}$ is the reduced coupling constant for the $\pi_\alpha+B(i,i_z,j,j_z)\to B(I,I_z,J,J_z)$ vertex. The unitarity of the representation implies the so-called vertex symmetry

$$G_{IJ}{}^{ij}=\left[\frac{(2i+1)(2j+1)}{(2I+1)(2J+1)}\right]^{1/2}(-1)^{i+j-I-J}\bar{G}_{ij}{}^{IJ}, \tag{3.2}$$

where $\bar{G}$ denotes the complex conjugate of G. It now remains to determine the functional dependence of the reduced coupling constant $G_{IJ}{}^{ij}$ on the quantities i, j, I, and J. In order to solve this problem, we must know the representations of the groups $SU(2)_V$ and $SU(4)_{IB}$.

As is known, the unitary irreducible representations of multiplicity 1 of the group $SU(4)$ are characterized by three numbers f_1, f_2, and f_3. For reasons of simplicity, we shall consider only the most degenerate representations for which $f_2=f_3=0$. In this case the states $|I,I_z,B,B_z\rangle$ which form the spherical basis for the representation of the group $SU(4)_{IB}$ are characterized by the condition $I=B$. The meson-source operator $D_{\alpha a}$ has the tensorial character $(I,B,V)=(1,1,0)$, whence its matrix elements in the $|I,I_z,B,B_z,V,V_z\rangle$ basis can be written as follows:

$$\langle I,I_z,B,B_z,V,V_z|D_{\alpha a}|i,i_z,B',B_z',v,v_z\rangle =A_I{}^i\begin{pmatrix} i & 1 & I\\ z & \alpha & I_z\end{pmatrix}\begin{pmatrix} B' & 1 & B\\ B_z' & a & B_z\end{pmatrix}\delta_{iB'}\delta_{IB}\delta_{Vv}\delta_{V_zv_z}, \tag{3.3}$$

where $A_I{}^i$ is defined as[5]

$$A_I{}^i=\left(\frac{2i+1}{2I+1}\right)^{1/2}\{r^2-\tfrac{1}{4}[i(i+1)-I(I+1)]^2\}^{1/2}. \tag{3.4}$$

From Eq. (3.4) one can infer that the number r, which characterizes the most degenerate representation of the group $SU(4)_{IB}$, is related to the maximum isospin $I_{\max}$ of the particles in the representation characterized by a given r as follows:

$$I_{\max}=r-1. \tag{3.5}$$

Taking into account Eq. (2.17), we can now pass from the basis $|I,I_z,B,B_z,V,V_z\rangle$ to the basis $|I,I_z,J,J_z\rangle$ as follows:

$$|I,I_z,J,J_z\rangle=\sum_{B_zV_z}\begin{pmatrix} V & B & J\\ V_z & B_z & J_z\end{pmatrix}|I,I_z,B,B_z,V,V_z\rangle. \tag{3.6}$$

[5] T. G. Kuriyan and E. C. G. Sudarshan, Phys. Rev. **162**, 1650 (1967); Phys. Letters **21**, 106 (1966).

Combining Eqs. (3.3) and (3.6), we find

$$\langle I,I_z,J,J_z|D_{\alpha a}|i,i_z j,j_z\rangle = \sum_{B_z,B_z',V_z} \begin{pmatrix} V & I & J \\ V_z & B_z & J_z \end{pmatrix} \begin{pmatrix} V & i & j \\ V_z & B_z' & j_z \end{pmatrix} \times \begin{pmatrix} i & 1 & I \\ i_z & \alpha & I_z \end{pmatrix} \begin{pmatrix} i & 1 & I \\ B_z' & a & B_z \end{pmatrix} A_I{}^i. \quad (3.7)$$

Using Eqs. (3.1) and (3.7), we obtain, after performing the sum in Eq. (3.7), the following formula for the reduced coupling constant $G_{IJ}{}^{ij}$:

$$G_{IJ}{}^{ij} = (-1)^{1+V+i+J}[(2i+1)(2j+1)]^{1/2} \times \begin{Bmatrix} 1 & i & I \\ V & J & j \end{Bmatrix} \{r^2 - \tfrac{1}{4}[i(i+1)-I(I+1)]^2\}^{1/2}, \quad (3.8)$$

where the symbol $\{:::\}$ stands for the $6j$ symbol[6] of the $SU(2)$ group. The triangular conditions implied by the $6j$ symbol in Eq. (3.8) show that the reduced coupling constant $G_{IJ}{}^{ij}$ is nonvanishing if

$$|i-j|,\ |I-J| = V,\ V-1,\ \ldots,\ \tfrac{1}{2} \text{ or } 0. \quad (3.9)$$

The solution (3.8) shows that the set of single-particle states which saturate the first superconvergence relation are characterized by two numbers r and V, the first giving the value of $I_{\max}$ in the set, and the second giving a correlation between the spin and the isospin in the set.

For $V=0$ we have the set of baryons with $I=J$, and the set with $(I,J)=(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, and $(\frac{5}{2},\frac{5}{2})$ can be identified with the set comprised of the nucleon, Δ resonance, and the $(\frac{5}{2},\frac{5}{2})$ resonance.[7] The hyperons Λ, Σ, and $Y_1{}^*$ can be assigned to the representation with $V=\frac{1}{2}$. The cascade hyperons Ξ and Ξ^* can be accommodated in the representation with $V=1$, while the Ω^- can be accommodated in the representation with $V=\frac{3}{2}$.

We can now calculate the various baryon-decay rates by using the reduced coupling constants (3.8). Before proceeding with this calculation, we discuss the second superconvergence condition in the two next sections.

IV. SECOND SUPERCONVERGENCE RELATION

We consider the second superconvergence condition (1.2) following from the part of the scattering amplitude which is symmetric in the pion isovector indices α and β. The s- and u-channel tree-graph contributions to the corresponding superconvergence relation involving only p-wave pion-hadron interaction give rise to the double commutator

$$[D_{\alpha a},[m^2,D_{\beta b}]]. \quad (4.1)$$

The mass matrix m^2 must be helicity independent and must conserve isospin, so that

$$[J_a,m^2]=[I_\alpha,m^2]=0. \quad (4.2)$$

By applying the Jacobi identity to the double commutator (4.1) with the use of Eqs. (4.2) and (2.14), we find that

$$[D_{\alpha a},[m^2,D_{\beta b}]]-[D_{\beta b},[m^2,D_{\alpha a}]] = i\delta_{\alpha\beta}\epsilon_{abc}[m^2,B_c]. \quad (4.3)$$

The tensorial character of the left-hand side of this equation is given by $(I,J)=(0,1)$, $(1,0)$, $(1,2)$, and $(2,1)$ since the corresponding difference is antisymmetric in the pair of indices αa and βb. However, the tensorial character of the right-hand side of Eq. (4.3) is only $(I,J)=(0,1)$. This implies that the irreducible tensors constructed from the double commutator (4.1) with (I,J) characters equal to $(1,0)$, (1.2), and $(2,1)$ are identical to zero. The tree-graph contributions to the second superconvergence relation arising from the meson exchange in the t channel must have the same tensorial characters as the nonvanishing tensors constructed from the double commutator $[D_{\alpha a},[m^2 D_{\beta b}]]$. This means that generally only the mesons with the isospin and spin character

$$(I,J)=(0,0),\ (0,1),\ (0,2),\ (1,1),\ (2,0),\ (2,2) \quad (4.4)$$

can contribute to the t-channel tree graphs.

The second superconvergence relation arises from the requirement that the tree-graph contributions from all three channels to the symmetric part of the forward scattering amplitude should not destroy proper Regge behavior. In particular, we must require that the constant term in the asymptotic expansion of the tree-graph amplitude must be zero. This implies

$$[D_{\alpha a}[m^2 D_{\beta b}]] = \sum_{TL\gamma c} \begin{pmatrix} 1 & 1 & T \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 1 & L \\ a & b & c \end{pmatrix} Y_{\gamma c}{}^{TL}, \quad (4.5)$$

where we have passed on to a spherical basis and the summation runs over all slates (T,L) given by Eq. (4.4).

The right-hand side of Eq. (4.5) is physically interpreted as the contribution from the t-channel meson exchange, while the left-hand side represents the contributions from the single-particle exchange in the s and u channels as mentioned previously. The quantities $Y_{\gamma c}{}^{TL}$ in this equation are irreducible tensors of rank (T,L) of the group $SU(2)_I \otimes SU(2)_J$ and can be associated with the exchange of the following mesons[8]:

$$Y_{00}{}^{00} \sim \sigma(410), \quad Y_{0c}{}^{01} \sim D(1285), \quad Y_{0c}{}^{02} \sim f(1260)+f'(1515), \quad Y_{\gamma c}{}^{11} \sim \rho(765). \quad (4.6)$$

[6] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).

[7] A. Benvenuti, E. Marguit, and F. Oppenheimer, Phys. Rev. Letters 22, 970 (1969); E. Hegedüs, A. Abramovici, and L. Vékaś, Z. Physik (to be published).

[8] Particle Data Group, Rev. Mod. Phys. 41, 109 (1969).

The remaining irreducible tensors $Y_{\gamma 0}{}^{20}$ and $Y_{\gamma c}{}^{22}$ represent the contribution from the so-called exotic states which have isospin $I=2$. Weinberg's second superconvergence condition was derived under the assumption that such objects do not exist; therefore, we must require

$$Y_{\gamma 0}{}^{20}=0\,, \tag{4.7a}$$

$$Y_{\gamma c}{}^{22}=0\,. \tag{4.7b}$$

Without the constraints (4.7), the second superconvergence condition (4.5) would be a pure identity.

To make use of the absence of exotic states, mathematically expressed by Eqs. (4.7), we convert Eqs. (4.5) using the orthogonality properties of the CG coefficients, to the form

$$Y_{\gamma c}{}^{TL}=\sum_{\alpha\beta a b}\begin{pmatrix}1 & 1 & T\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1 & 1 & L\\ a & b & c\end{pmatrix}\times[D_{\alpha a},[m^2,D_{\beta b}]]. \tag{4.8}$$

Then the absence of exotic states leads to the two following constraints:

$$\sum_{\alpha\beta a b}\begin{pmatrix}1 & 1 & 2\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ a & b & 0\end{pmatrix}[D_{\alpha a},[m^2,D_{\beta b}]]=0 \tag{4.9}$$

and

$$\sum_{\alpha\beta a b}\begin{pmatrix}1 & 1 & 2\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ a & b & c\end{pmatrix}[D_{\alpha a},[m^2,D_{\beta b}]]=0\,. \tag{4.10}$$

These two constraints determine the functional dependence of the mass matrix m^2 on I and J since the functional dependence of the meson-source operator $D_{\alpha a}$ on the isospin and spin variables is determined by the first superconvergence condition (2.14) and is given explicitly by Eqs. (3.1) and (3.8).

Taking the matrix element of Eqs. (4.9) and (4.10) between two hadron states $|ii_z,jj_z\rangle$ and $|i'i_z',j'j_z'\rangle$, we can rewrite these equations in terms of hadron masses $m^2(I,J)$ and the reduced coupling constants $G_{IJ}{}^{ij}$. After some tedious algebra using the orthogonality properties of CG coefficients and their relations to the $6j$ symbol, we finally get

$$\sum_{IJ}\bar{G}_{IJ}{}^{i'j'}G_{IJ}{}^{ij}[2m^2(I,J)-m^2(i',j')-m^2(i,j)]\times(2I+1)(2J+1)(-1)^I\begin{Bmatrix}1 & 1 & 2\\ i & i' & I\end{Bmatrix}\delta_{jj'}=0 \tag{4.11}$$

and

$$\sum_{IJ}\bar{G}_{IJ}{}^{i'j'}G_{IJ}{}^{ij}[2m^2(I,J)-m^2(i',j')-m^2(i,j)](2I+1)\times(2J+1)(-1)^{I+J}\begin{Bmatrix}1 & 1 & 2\\ i & i' & I\end{Bmatrix}\begin{Bmatrix}1 & 1 & 2\\ j & j' & J\end{Bmatrix}=0\,. \tag{4.12}$$

These two equations must be valid for any i, i' and j, j' belonging to the same representation characterized by the numbers r and V. Using the proper reduced coupling constants given explicitly by Eq. (3.8), we obtain a system of partial difference equations for the mass spectrum $m^2(I,J)$. These equations and their solutions will be discussed in the next section.

V. ALGEBRAIC REALIZATION OF MASS CONDITIONS

In this section we shall obtain the general solution to the mass conditions (4.11) and (4.12) by applying them for all possible values of the quantum numbers i, i', and j, j'. We shall first consider the implications of Eq. (4.12).

We shall first apply Eq. (4.12) for the case when $i'\to i+1$, $j'\to j+1$ and $i\to i-1$, $j\to j-1$. Because of the properties of the $6j$ symbol, the sum over I and J in Eq. (4.12) then reduces to a single term with $I=i$ and $J=j$. We then obtain the equation

$$m^2(i+1,\,j+1)-m^2(i,j)=m^2(i,j)-m^2(i-1,\,j-1)\,. \tag{5.1}$$

Using the standard method[9] for solving partial difference equations of this kind, we introduce two new variables s and t as follows:

$$s=i+j\,,\quad t=i-j\,. \tag{5.2}$$

The solution to Eq. (5.1) can then be written as

$$m^2(i,j)=sf(t)+g(t)\,, \tag{5.3}$$

where the functions $f(t)$ and $g(t)$ are arbitrary functions of t.

We then apply Eq. (4.12) for the case $i'\to i+1$, $j'\to j-1$ and $i\to i-1$, $j\to j+1$. In this case Eq. (4.12) becomes

$$m^2(i+1,\,j-1)-m^2(i,j)=m^2(i,j)-m^2(i-1,\,j+1)\,. \tag{5.4}$$

The general solution to Eq. (5.4) is

$$m^2(i,j)=tp(s)+r(s)\,, \tag{5.5}$$

where the functions $p(s)$ and $r(s)$ are arbitrary functions of s.

The solutions (5.3) and (5.5) must be identically equal, which implies that the functions $g(t)$, $f(t)$ and $r(s)$, $p(s)$, respectively, are linear functions of their arguments. It follows that

$$m^2(i,j)=m_0{}^2+bi+cj+d(i^2-j^2)\,, \tag{5.6}$$

where $m_0{}^2$, b, c, and d are arbitrary constants (independent of i and j). We then proceed further and apply Eq. (4.12) for the case $i'\to i+1$, $j'\to j+1$ and $i\to i$, $j\to j-1$. Inserting the solution (5.6) in Eq. (4.12) for this case, we obtain the equation

$$(i+2)[b+d(2i-1)]=i[b+d(2i+3)]\,. \tag{5.7}$$

[9] L. M. Milne-Thomson, *The Calculus of Finite Differences* (MacMillan, London, 1933).

Equation (5.7) is identically satisfied if an only if

$$b=d. \tag{5.8}$$

Applying Eq. (4.12) for the case $i'\rightarrow i+1$, $j'\rightarrow j$ and $i\rightarrow i-1$, $j\rightarrow j+1$, we get the relation

$$(j+2)[-c+d(2j-1)]=j[-c+d(2j+3)], \tag{5.9}$$

which is satisfied if and only if

$$c=-d. \tag{5.10}$$

Inserting the results (5.8) and (5.10) in Eq. (5.6), we get

$$m^2(i,j)=m_0^2+c[j(j+1)-i(i+1)]. \tag{5.11}$$

Applying Eq. (4.12) for the remaining combinations i, i' and j, j' does not give any restrictions on the free parameters m_0^2 and c in Eq. (5.11). We can thus conclude that Eq. (5.11) represents the general solution of Eq. (4.12).

However, we must also take into account the constraints implied by Eq. (4.11). This equation can be solved in the same manner as Eq. (4.12) with the result

$$\begin{aligned}m^2(I,J)=m_0'^2-(a+b'r^2)\\ \times[\Psi(I+1+r)+\Psi(I+1-r)]\\ +b'[J(J+1)-I(I+1)],\end{aligned} \tag{5.12}$$

where $m_0'^2$, a, and b' are arbitrary constants, and the function $\Psi(x)$ is the logarithmic derivative of the gamma function. Equations (5.11) and (5.12) must be identically equal, which implies that

$$m_0'^2=m_0^2,\quad b'=c,\quad a+cr^2=0. \tag{5.13}$$

We can thus conclude that the mass spectrum which follows from Eqs. (4.11) and (4.12) is given by Eq. (5.11), with m_0^2 and c as arbitrary constants.

This means that Eq. (5.11) gives the mass spectrum of the hadrons which are eigenstates of those irreducible representations of the group $SU(2)\otimes SU(4)$ which have been considered above.

Unfortunately, the mass spectrum (5.11) cannot be considered satisfactory, as Eq. (5.11) shows that the hadron masses either decrease with increasing isospin i (if $c>0$) or decrease with increasing spin j (if $c<0$).

One could obtain a mass spectrum which increases with increasing spin and isospin simply by assuming the existence of an exotic state with $I=J=2$. In that case we would only have the mass condition given by Eq. (4.11), the solution of which is given by Eq. (5.12). It is clear that if the parameter a in Eq. (5.12) is negative but large enough in absolute value, then the mass spectrum given by Eq. (5.12) is increasing with increasing i and j. The existence of the exotic $I=J=2$ state has been assumed by Arnold and Uretsky[10] in order to explain the A_2 splitting. However, this assumption is not very attractive simply because the experimental indication for the existence of an $A_2(I=2)$ meson is very small, and also because this assumption would make the derivation of the second Weinberg superconvergence condition a little obscure.

There is also the possbility that the representations of the group $SU(2)\otimes SU(4)$ investigated here are inadequate, and that a more satisfactory mass spectrum would follow from a consideration of more complicated representations. Unfortunately, the matrix elements of the generators of the group $SU(4)$ in the physical reduction chain $SU(4)\supset SU(2)\otimes SU(2)$ are not known, since only the most degenerate representations are given in the literature.[5,11]

Finally, it is possible that the p-wave approximation is inadequate, and that one has to consider at least both s and p waves in order to get a reasonable mass spectrum.

We must thus conclude that it is necessary to consider representations of the group $SU(2)\otimes SU(4)$ more general than those investigated here, and also consider other partial waves besides the p wave, in order to verify the validity of the second superconvergence condition.

VI. PREDICTIONS AND COMPARISON WITH EXPERIMENT

Given the explicit form for the reduced coupling constants $G_{IJ}{}^{ij}$, we can calculate the decay widths Γ of various p-wave decay processes $B(I,J)\rightarrow B(i,j)+\pi$. The decay width Γ is given as[2]

$$\Gamma(B(I,J)\rightarrow B(i,j)+\pi)=\frac{2|G_{IJ}{}^{ij}|^2p_\pi{}^3}{\pi F_\pi{}^2(2J+1)}, \tag{6.1}$$

where p_π is the three-momentum of the pion, which is determined through the masses $m(I,J)$, $m(i,j)$ and the pion mass m, which now is given its (nonzero) physical value. The reduced coupling constants are given by Eq. (3.8). As mentioned previously, the nucleon, Δ resonance, and $(\frac{5}{2},\frac{5}{2})$ resonance[7] are assigned to the representation characterized by $V=0$ and $r=\frac{7}{2}$. The reduced coupling constant $G_{\frac{1}{2}\frac{1}{2}}{}^{\frac{1}{2}\frac{1}{2}}$ for the πNN vertex can be fitted to experiment as follows:

$$g_A/g_V=-1.231=\langle p|\pi^+|n\rangle=-\tfrac{2}{3}G_{\frac{1}{2}\frac{1}{2}}{}^{\frac{1}{2}\frac{1}{2}}, \tag{6.2}$$

where the ratio $g_A/g_V\approx-1.231$ is taken from Ref. 8. We must now introduce a renormalization of the reduced coupling constants given by Eq. (3.8) in order to take Eq. (6.2) into account. We multiply the reduced coupling constants $G_{IJ}{}^{ij}$ in Eq. (3.8) with a parameter z, so that

$$G_{\frac{1}{2}\frac{1}{2}}{}^{\frac{1}{2}\frac{1}{2}}=rz. \tag{6.3}$$

The parameter z is then fixed by Eq. (6.2). The decay width of the Δ resonance determined by Eqs. (6.1)–(6.3)

[10] R. C. Arnold and J. L. Uretsky, Phys. Rev. Letters **23**, 444 (1969).

[11] K. T. Hecht and S. C. Pang, J. Math. Phys. **10**, 1571 (1969).

TABLE I. Comparison of decay widths of p-wave baryons calculated from superconvergence conditions with experiment, $SU(6)$, and bootstrap values.

			Γ (MeV)			
Decay	V	r	Experiment	Superconv.	$SU(6)$	Bootstrap
$\Delta \to N\pi$	0	$\frac{7}{2}$	120 ± 2	101	76	125
$Y_1^* \to \Sigma\pi$	$\frac{1}{2}$	3	3.4 ± 1.2	4.18	3.3	4.8
$Y_1^* \to \Lambda\pi$	$\frac{1}{2}$	3	33.6 ± 12	39.25	24	48
$^* \to \Xi\pi$	1	$\frac{3}{2}$	7.3 ± 1.7	5.21	8.9	19

turns out to be 101 MeV, which is reasonably close to the experimental value[8] 120 ± 2 MeV.

The hyperons Λ, Σ, and Y_1^* and the possible exotic Y_2^* resonance[12] belong to the representation characterized by $V=\frac{1}{2}$ and $r=3$. We can then calculate the ratio $\Gamma(Y_1^* \to \Sigma\pi)/\Gamma(Y_1^* \to \Lambda\pi)$ which turns out to be 11%, which is close to the experimental value[8] $(11\pm2)\%$. If we make the assumption that the scale of all the reduced coupling constants (3.8) is determined by the πNN coupling constant given by Eqs. (6.2) and (6.3), then we can predict the decay widths of the Δ, Y^*, and Ξ^* resonances in terms of the πNN coupling constant. The resulting decay widths are collected in Table I, together with the experimental values[8] and bootstrap[13] and $SU(6)$ predictions.[14] The agreement with experiment is satisfactory.

VII. SUMMARY AND CONCLUSIONS

We have derived the algebraic structure of a pair of superconvergence relations originally proposed by Weinberg[2] for the case of p-wave pion-hadron interaction. The first superconvergence condition together with the algebra of the spin and isospin generators defines the algebra of the group $SU(2)\otimes SU(4)$. This result follows from exactly those assumptions that were made in Ref. 2. This means that the hadron states are eigenstates of the unitary irreducible representations of the group $SU(2)\otimes SU(4)$, and that their decay widths can be calculated by using the explicit expressions for the matrix elements of the generators of the group in question. We have calculated the decay widths for the well-established p-wave baryons, using the simplest representations of the group $SU(2)\otimes SU(4)$, i.e., those representations that can be obtained from a knowledge of the most degenerate representations of the group $SU(4)$. The agreement with the experimental decay widths is satisfactory.

The second superconvergence relation gives the mass spectrum of the hadrons in a given representation of the group $SU(2)\otimes SU(4)$. However, the mass spectra obtained are disappointing in the sense that they imply that the hadron masses must decrease with increasing isospin (or decrease with increasing spin). This indicates either that the assumptions leading to the second superconvergence relation are basically unsound or that the simplifications made in considering only p-wave pion-hadron interaction are unjustified. There is also the possibility that a reasonable mass spectrum can be obtained in the p-wave approximation by using more complicated representations of the group $SU(2)\otimes SU(4)$.

Let us finally mention the similarity between the first superconvergence relation in the p-wave approximation, and the intermediate coupling model[15,16] which has its origin in the static model of the meson-baryon interaction.[17]

The group $SU(2)\otimes SU(4)$ was suggested by Rangwala[16] as the intermediate coupling group on the phenomenological grounds. However, the second superconvergence condition has not counterpart in previous models. It is interesting to note that the superconvergence conditions considered in this paper give the results which represent one class of the possible solutions to the static bootstrap models[13,18,19] and strong-coupling theory if we take the limit $r \to \infty$.

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