$$\int \frac{z_4^4 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0^2}\left(\frac{c(5\Delta_0+3c^2)}{(a+c)^2} - 3(b+c)^2 G\right)+\cdots, \quad \text{(C3)}$$

$$\int \frac{z_4^3 z_5 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0^2}\left(-\frac{2\Delta_0+3c^2}{a+c} + 3c(b+c)G\right)+\cdots, \quad \text{(C4)}$$

$$\int \frac{z_4^2 z_5^2 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0^2}[3c-(\Delta_0+3c^2)G]+\cdots. \quad \text{(C5)}$$

The domain of integration is $0 \le z_4+z_5 \le K$ in all cases. Similar formulas for the integrals

$$\int \frac{z_5^2 dz_4 dz_5}{W^2}, \quad \int \frac{z_4 z_5^3 dz_4 dz_5}{W^3}, \quad \int \frac{z_5^4 dz_4 dz_5}{W^3}$$

are obtained from (C1), (C4), and (C3) by interchanging a and b.

PHYSICAL REVIEW D VOLUME 1, NUMBER 8 15 APRIL 1970

Bootstrap Conditions from the Veneziano Representation*

RICHARD H. CAPPS
Department of Physics, Purdue University, Lafayette, Indiana 47907
(Received 23 September 1969; revised manuscript received 6 January 1970)

A set of self-consistency conditions for ratios of hadron-hadron-hadron coupling constants is derived from the Veneziano model for scattering amplitudes and some simple assumptions. This set includes the crossing-matrix condition of the static meson-baryon model and the condition that the absence of resonances in exotic states implies a cancellation from Regge trajectories of opposite signatures. For simplicity, the spins of the external mesons and baryons are neglected. The conditions require that the particles correspond to representations of a Lie group, and that only certain sets of representations are allowed. The relation to previous bootstrap models is discussed. An illustrative solution corresponding to $SU(3)$ is obtained.

I. INTRODUCTION

In recent years, several dynamical models of hadrons have been based on various analyticity and crossing properties of hadron-hadron scattering amplitudes. Examples are the meson-nucleon static model, superconvergence relations, Regge-pole models, and the Veneziano model. The addition of the bootstrap hypothesis to the first three of these models has led to additional predictions, many of which are in approximate agreement with experiment. The main purpose of this paper is to obtain a set of bootstrap conditions from the Veneziano model, a set that includes the more successful conditions of earlier models.

Our definition of bootstrap includes two parts: (i) The subtraction constants or background terms are sufficiently small in number so that the dynamical equations determine the physical masses and interaction constants in terms of a very small set of external parameters; (ii) a complete set of two-hadron → two-hadron amplitudes should be considered, and the set of internal particles should be the same as the set of external particles. Two comments must be made concerning these points. First, statement (i) is an assumption that certain conceivable terms are negligible. Thus, at least in present models, the bootstrap is not implied by crossing and general analyticity requirements. It must be judged by its theoretical simplicity and the success of its predictions. Second, in any bootstrap model, it is reasonable to begin by considering amplitudes in which the lightest hadrons are external. In such a first approximation to a "complete" model, statement (ii) requires only that the set of virtual particles consists of the external particles plus heavier particles.

In Sec. II we review briefly the most successful bootstrap conditions of the static meson-nucleon model and of the combination of Regge theory with duality. Consistency conditions are derived from an idealized Veneziano model (with the spins of external particles treated like internal quantum numbers) in Sec. III. The conditions are summarized in Sec. III C. The fact that a solution must involve a Lie-group symmetry is discussed in Sec. IV, while Sec. V contains an illustrative solution involving $SU(3)$. The exact treatment of particle spin leads to a more complicated model, which will be discussed in a future paper.

II. BOOTSTRAP CONDITIONS OF PREVIOUS MODELS

The simplest type of conceivable hadron spectrum in a Reggeized model contains only a finite number of Regge trajectories. The experimental hadron data are consistent with such a spectrum. In such a model,

* Supported in part by the U. S. Atomic Energy Commission.

internal quantum numbers for which no resonances occur may be called "exotic." Some two-hadron states will be exotic. The bootstrap principle, that the sets of internal and external hadrons are the same, implies that the dynamical quantity that causes resonances must vanish in exotic hadron-hadron states. This "general exotic principle" is one of the most basic of bootstrap requirements.

A more specific requirement occurs in the static model for the scattering of pseudoscalar or scalar mesons from baryons. This may be written as

$$X_u{}^i = X_s{}^i, \qquad (1)$$

where i denotes the quantum numbers of the amplitude, and X_s and X_u are the sums of resonance and bound-state residues in the s and u channels.[1,2] By definition, the X are also related by $X_u{}^i=(C^{su})_{ij}X_s{}^j$, where C^{su} is the $s \leftrightarrows u$ crossing matrix. The X are quadratic in the coupling constants. If i is exotic in one of the channels, then Eq. (1) is a form of the general exotic principle, and implies that the sum of the crossed channel residues vanish. However, this equation applies to nonexotic states as well, although only one partial wave and parity are included in the static model. This condition, that the $X_s{}^i$ are components of an eigenvector of the crossing matrix with eigenvalue one, is simple theoretically and also approximately correct experimentally in the pion-nucleon case.[1]

Recent application of the duality principle to Regge contributions had led to another significant extension of the exotic-state requirement on coupling constants. If i refers to an amplitude that is exotic in the u channel, the condition may be written as

$$X_s{}^{(+)i} - X_s{}^{(-)i} = 0, \qquad (2a)$$

where the X_s are sums of residues of Regge trajectories that correspond to particles in the s channel, and the sign refers to the parity of the trajectory.[3] (Parity means that of the physical states, or the signature of the trajectory.) If j refers to an amplitude that is exotic in the t channel, the corresponding condition is

$$X_s{}^{(+)j} + X_s{}^{(-)j} = 0. \qquad (2b)$$

We call these conditions the parity-cancellation conditions. If one assumes factorizability, the residues X are again quadratic combinations of coupling constants.

These conditions, Eqs. (2a) and (2b), have led to many predictions for meson-meson and meson-baryon systems that are approximately correct, experimentally.[3] They are more general than the crossing-matrix condition in that they are functions of momentum transfer and of the spin state, and they involve virtual states of both parities. However, they are limited to amplitudes exotic in some channel. We assume that any set of bootstrap conditions based on a more sophisticated and complete model should lead to both the crossing-matrix condition and the parity-cancellation conditions in the appropriate limits.

III. BOOTSTRAP CONDITIONS FROM VENEZIANO MODEL

A. Veneziano Form of Amplitudes

We consider meson-meson and meson-baryon scattering. For simplicity, all particles are taken to be spinless; the baryons are distinguished by the possession of a unit of the conserved baryon number. It is assumed that the masses μ of all external mesons are the same and the masses m of all external baryons are the same. It is assumed further that all baryonic Regge trajectories are degenerate, and that all mesonic trajectories are degenerate, and that their slopes are a universal constant b.

The s-, t-, and u-channel amplitudes are represented by

$$\begin{aligned} s&:\ a+b \to c+d\,,\\ t&:\ a+\bar{c} \to \bar{b}+d\,,\\ u&:\ \bar{c}+b \to \bar{a}+d\,, \end{aligned} \qquad (3)$$

where a and c represent mesons, and b and d represent either two mesons or two baryons. The Veneziano amplitude T^i corresponding to internal quantum numbers i is the sum of three terms,[4]

$$T^i = T_{su}{}^i + T_{st}{}^i + T_{ut}{}^i. \qquad (4)$$

Each $T_{\nu\omega}{}^i$ is taken as a finite linear combination of generalized β functions of the type

$$\beta_{\nu\omega,kln} = \Gamma(k-\alpha_\nu)\Gamma(l-\alpha_\omega)/\Gamma(n-\alpha_\nu-\alpha_\omega),$$

where k, l, and n are integers and α_ν is the trajectory function in the ν channel. The angular momentum analysis of the s-channel pole at $\alpha_s=j$ ($j \geqq k$) in the $\beta_{su,kln}$ term may be made from a Legendre expansion of the coefficient $\Gamma(l-\alpha_u)/\Gamma(n-j-\alpha_u)$, using the relations

$$\begin{aligned} t&=-2q_s{}^2(1-\cos\theta_s)\,,\\ u&=2(m^2+\mu^2)-t-s\,, \end{aligned} \qquad (5)$$

where q_s is the center-of-mass momentum in the s channel. (For meson-meson scattering, $m=\mu$.)

We are interested only in the leading trajectories. It is assumed that the spin of the lowest states on the leading baryon trajectories is the same as that of the lowest states on the leading meson trajectories; this spin is denoted by j_0. If all β functions that do not

[1] This requirement was first applied by G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[2] The crossing-matrix condition is discussed in some detail by R. H. Capps, Nuovo Cimento **34**, 932 (1964).

[3] See, for example, C. B. Chiu and J. Finkelstein, Phys. Letters **27B**, 510 (1968); V. Barger, Phys. Rev. **179**, 1371 (1969); R. H. Capps, Phys. Rev. Letters **22**, 215 (1969).

[4] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

contribute to any leading trajectory are omitted, each of the three terms of Eq. (4) may be written in the form

$$T_{\nu\omega}{}^i = \sum_{k\geqq j_0} \sum_{l\geqq j_0} \lambda_{\nu\omega,kl}{}^i \beta_{\nu\omega,lkn}, \quad n=\max(k,l) \tag{6}$$

where the λ's are real coefficients. The channel indices on the λ's and β's may be placed in either order, but the first of the angular momentum indices corresponds to the first of the channel indices, i.e., $\lambda_{\nu\omega,kl}=\lambda_{\omega\nu,lk}$.

In order to express simply the residues of the poles in T, we define the following sums of coefficients:

$$\Lambda_{\nu\omega,j}{}^i = \sum_{k\geqq j} \lambda_{\nu\omega,jk}{}^i .$$

The partial-wave amplitudes T_j in the s channel are defined in terms of T by the relation

$$T=(s^{1/2}/q_s)\sum_j(2j+1)T_jP_j(\cos\theta).$$

We denote by $R_j(s)^i$ the residue of the pole at $\alpha_s=j$ on the leading s-channel trajectory in the partial-wave amplitude T_j. If T^i is expressed in the form of Eqs. (4) and (6), this residue is

$$R_j(s)^i=-\rho_j[K_{st,j}{}^i+(-1)^jK_{su,j}{}^i], \tag{7}$$

where

$$\rho_j=\frac{b^{j-1}q_s(2q_s{}^2)^j}{s^{1/2}(2j+1)C_j} \tag{8}$$

and

$$K_{\nu\omega,j}{}^i=\sum_{l=j_0}^{j}(-1)^l\Lambda_{\nu\omega,l}{}^i/\Gamma(j-l+1). \tag{9}$$

The constant C_j is the coefficient of $(\cos\theta)^j$ in the polynomial P_j.

In Sec. III B, we make some assumptions concerning the proportionality of certain of the λ coefficients, assumptions that lead to bootstrap conditions of the desired kind. A simpler way to obtain these conditions is to make the strong assumption that only the one leading term with $k=l=j_0$ occurs in the Veneziano expansion of Eq. (6). This leads to the bootstrap conditions of Sec. III C if the ρ and X of Eqs. (18) and (19) are given by

$$\rho_j{}^B=\rho_j{}^M=\rho_j/\Gamma(j-j_0+1)$$

and

$$X_{\nu\omega}{}^i=(-1)^{j_0}\lambda_{\nu\omega,jj}{}^i(j=j_0).$$

We present the proportionality assumptions to show that the strong one-term assumption is not needed.[5] However, the reader interested only in the conditions and their solutions may skip to Sec. III C.

B. Proportionality Assumptions

We consider meson-baryon (MB) scattering, since a trivial modification makes the results applicable to meson-meson (MM) scattering as well. It is assumed that all baryon states and their trajectories are equivalent in the sense that their Regge residues are proportional (as a function of s or u) and that all meson residues are proportional in this sense. Specifically, the assumption is that the ratio defined by the following equation is independent of the internal state i:

$$r(\nu\omega,kl,k'l')=\lambda_{\nu\omega,kl}{}^i/\lambda_{\nu\omega,k'l'}{}^i. \tag{10}$$

This implies that the quantity $K_{\nu\omega,j}{}^i$ of Eq. (9) may be written as a product of a function independent of j and a function independent of i, i.e.,

$$K_{\nu\omega,j}{}^i=X_{\nu\omega}{}^i\mathcal{K}_{\nu\omega,j}. \tag{11}$$

Since a particular baryon state may be considered in either the s or u channel, crossing symmetry implies that the ratios r of Eq. (10) are invariant to the interchange $s\leftrightarrows u$, i.e.,

$$r(su,k\text{'s})=r(us,k\text{'s}), \tag{12a}$$

$$r(st,k\text{'s})=r(ut,k\text{'s}), \quad r(ts,k\text{'s})=r(tu,k\text{'s}), \tag{12b}$$

where k's is shorthand for kl, $k'l'$.

The simple requirement that at least one λ of the type $\lambda_{su,kk}{}^i$ is nonzero, together with Eq. (12a), implies that the $K_{su,j}{}^i$ of Eq. (11) are symmetric in the interchange $s\leftrightarrows u$. We may then choose $\mathcal{K}$ and X to have this symmetry also:

$$\mathcal{K}_{su,j}=\mathcal{K}_{us,j}, \tag{13}$$

$$X_{su}{}^i=X_{us}{}^i. \tag{14}$$

Further, we may choose $\mathcal{K}$'s that satisfy the conditions

$$\mathcal{K}_{st,j}=\mathcal{K}_{ut,j}, \quad \mathcal{K}_{ts,j}=\mathcal{K}_{tu,j}. \tag{15}$$

The proportionality condition of Eq. (10) implies further that $K_{t\nu,j}{}^i/K_{\nu t,j'}{}^i$ is independent of i, and Eq. (12b) implies that this ratio is the same when ν is s or u. We choose $\mathcal{K}_{t\nu,j}/\mathcal{K}_{\nu t,j}$ so that

$$X_{t\nu}{}^i=X_{\nu t}{}^i. \tag{16}$$

We assume that there are some states i for which s-channel resonances exist only of even parity, and other states i' for which s-channel resonances exist only of odd parity. It is seen from Eq. (11) that this implies

$$X_{st}{}^i\mathcal{K}_{st,j}-X_{su}{}^i\mathcal{K}_{su,j}=0, \quad j \text{ odd}$$

$$X_{st}{}^{i'}\mathcal{K}_{st,j}+X_{su}{}^{i'}\mathcal{K}_{su,j}=0, \quad j \text{ even}$$

for such states i and i'. These equations imply that the ratio $\mathcal{K}_{st,j}/\mathcal{K}_{su,j}$ is the same for all odd j, and also is the same for all even j. We assume the even-j and odd-j ratios are the same. We may then choose the X's so that $\mathcal{K}_{st,j}=\mathcal{K}_{su,j}$. Then there are only two independent $\mathcal{K}$ functions,

$$\mathcal{K}_{su,j}=\mathcal{K}_{us,j}=\mathcal{K}_{st,j}=\mathcal{K}_{ut,j} \text{ and } \mathcal{K}_{ts,j}=\mathcal{K}_{tu,j}.$$

We define new phase-space factors

$$\rho_j{}^B=\rho_j\mathcal{K}_{su,j}, \quad \rho_j{}^M=\rho_j\mathcal{K}_{ts,j}, \tag{17}$$

[5] A set of assumptions that is similar to our set of proportionality assumptions, and which leads to the same results, is used in the meson model of S. Mandelstam, Phys. Rev. **184**, 1625 (1969). Mandelstam points out that his assumptions are equivalent to the requirement that the leading trajectories bootstrap themselves.

so that the s-channel residue of Eq. (7) and the corresponding t-channel residue may be written in the form

$$R_j^i(s)=-\rho_j^B[X_{st}^i+(-1)^jX_{su}^i], \quad (18)$$

$$R_j^i(t)=-\rho_j^M[X_{st}^i+(-1)^jX_{ut}^i], \quad (19)$$

where all the $X_{\nu\omega}$ are symmetric in ν and ω. These equations also apply to MM scattering; in this case, ρ_j^B and ρ_j^M are the same. The effective residues $X^{(\pm)}$ for even- and odd-j values are defined as follows:

$$X_s^{(\pm)i}=X_{st}^i\pm X_{su}^i, \quad (20)$$

$$X_t^{(\pm)i}=X_{st}^i\pm X_{ut}^i. \quad (21)$$

C. Consistency Conditions

We take the residues to be in the form of Eqs. (18) and (19), and list below the bootstrap conditions.

1. Factorization condition. The residue functions $X_\nu^{(\pm)}$ of Eqs. (18) and (19) must be proportional to the sums over ν-channel trajectories of parity $\pm$ of products of constants of coupling to the initial and final states in question. For the amplitude of Eq. (18),

$$X_s^{(+)}=\sum_r D_{cdr}D_{abr}^*, \quad (22a)$$

$$X_s^{(-)}=\sum_r F_{cdr}F_{abr}^*, \quad (22b)$$

where the D and F are coupling constants.[6] Similar equations hold for the t- and u-channel residues. We have defined the magnitude of the constants by using an equality rather than a proportionality sign in these equations. This is not important here, since the consistency conditions concern only ratios of coupling constants. If we take a and c to be self-conjugate states, then

$$D_{abr}=D_{arb}^*, \quad F_{abr}=F_{arb}^*, \quad (23)$$

i.e., the constants may be regarded as elements of Hermitian matrices D_a and F_a. We note from Eqs. (22a) and (22b) that $X_s^{(\pm)}$ cannot be negative if the initial and final states are the same.

2. Exotic-state condition. The quantity $X_{\nu\omega}^i$ must vanish if the state i is exotic in either the ν or ω channel. If i is exotic in either the t or u channel, it is seen from Eq. (20) that $|X_s^{(+)i}|=|X_s^{(-)i}|$. This is the Veneziano form of the parity-cancellation conditions of Eqs. (2a) and (2b).

3. Crossing condition. This is $X_{\nu\omega}^i=X_{\omega\nu}^i$, as shown in Eqs. (14) and (16). In the case of X_{su}, this condition is in the form of Eq. (1), and implies that $\mathbf{X}_{su}$, the vector whose components are X_{su}^i, is an eigenvector of the $s\leftrightarrows u$ crossing matrix, with eigenvalue 1.

4. Inclusion of external particles in internal set. Since the external particles are spinless, the lowest angular momentum j_0 on the leading trajectories must be zero, so that the set of external particles is the set of lowest states on the even-parity trajectories. Completion of the model would require treating the other particles on the trajectories as external. In a complete theory, with the set of external particles equal to the set of internal particles, condition (2) would be redundant.

IV. GROUP PROPERTIES

We take a self-conjugate set of meson states in the amplitudes of Eq. (3), so that $\bar{a}=a$ and $\bar{c}=c$. Then s-u crossing is simply the interchange $a\leftrightarrows c$. According to Eq. (14), $X_{su}^i-X_{us}^i=0$. We write X_{su} in terms of $X_s^{(\pm)}$ by using Eq. (20), then use Eqs. (22a), (22b), and (23) to write this in terms of D and F coefficients, and then subtract X_{us} by interchanging a and c and subtracting. The result is

$$\sum_r(D_{cdr}D_{arb}-F_{cdr}F_{arb}-D_{adr}D_{crb}+F_{adr}F_{crb})=0. \quad (24)$$

In the case of MM scattering, we take the states b, d, and r to be self-conjugate also. The Bose statistics of the MM states implies $F_{ijk}=-F_{jik}$ and $D_{ijk}=D_{jik}$. These relations, combined with the Hermiticity properties of Eq. (23), imply that the D coefficients are real and the F are imaginary. Similar equations for MM scattering may be obtained by considering the s-t and u-t pairs of channels. The algebraic structure of the resulting set of equations is not new. Previously, the author obtained an equivalent set from a superconvergence assumption applied to collinear amplitudes, together with the assumption that scattering lengths may be neglected.[7] In Ref. 7, the interactions refer only to virtual particles of the lowest angular momentum for each parity, rather than to Regge trajectories. On the other hand, the spin components of the particles may be treated rigorously with this superconvergence method.

Using the superconvergence method, it is possible to enlarge the set of external meson states to include all the internal meson states.[8] It is shown in Ref. 8 that the consistency conditions imply that the mesons of both parities correspond to the direct sum of the regular and identity representations of a Lie group, that the interactions are invariant to the transformations of the group, and that the groups $SU(n)$ (but not all Lie groups) satisfy the requirements. In the present case, we cannot prove quite so much because the virtual particles involved in the F-type interactions all possess spin, and thus are not included as external particles. Nevertheless, we will assume these results—that the meson trajectories of both parities correspond to the regular and identity representations of $SU(n)$, and that

[6] Frequently, but not always, one uses a representation of the particles for which the coupling constants F and D are real.

[7] R. H. Capps, Phys. Rev. **168**, 1731 (1968). Our Eq. (24) may be put in the form of Eq. (8) of this reference if we make the replacements $D_{abc}\to\gamma_{abc}^e$ and $F_{abc}\to i\gamma_{abc}^o$, and use the Hermiticity properties of the D and F and the reality of the γ's.

[8] R. H. Capps, Phys. Rev. **171**, 1591 (1968). This paper is an extension of Ref. 7.

the meson-meson-meson interactions are invariant under the group transformations.

On the other hand, the MB scattering conditions of this paper and Ref. 7 are different. The Veneziano approach leads to MB conditions that are obtainable from the superconvergence approach only if $\mu=m$. The requirement of equal masses in the superconvergence model is replaced by the requirement that the trajectory slopes be equal. This point is discussed further in Sec. VI.

We now write the $MB\rightarrow MB$ consistency equations of the Veneziano model in terms of coupling constants. One condition is Eq. (24), where b, d, and r now refer to baryons. This equation may be written in terms of the D and F matrices defined in Sec. III C, i.e.,

$$[D_c,D_a]_{db}-[F_c,F_a]_{db}=0. \quad (25)$$

Another condition may be obtained by considering the quantity $X_{st}{}^i$. It is seen from Eqs. (20) and (21) that this quantity is given by

$$\tfrac{1}{2}[X_s{}^{(+)i}+X_s{}^{(-)i}]=\tfrac{1}{2}[X_t{}^{(+)i}+X_t{}^{(-)i}].$$

If this equation is mutliplied by 2, and the $X^{(\pm)}$ are written in the form of Eqs. (22a) and (22b), the result is

$$\sum_r (D_{cdr}D_{arb}+F_{cdr}F_{arb}) =\sum_n (D_{\bar{b}dn}{}'D_{anc}{}'+F_{\bar{b}dn}{}'F_{anc}{}'), \quad (26)$$

where the sum r is over s-channel baryon trajectories, the sum n is over t-channel meson trajectories, and the primes are included to emphasize that the t-channel coupling constants are different from the s-channel constants.

We next interchange a and c in Eq. (26) and subtract the results, writing the left-hand side in terms of the D and F matrices. The constant $D_{anc}{}'$ is symmetric in this interchange, so the D' term drops out on the right, and the result becomes

$$[D_c,D_a]_{db}+[F_c,F_a]_{db}=2\sum_n F_{\bar{b}dn}{}'F_{anc}{}'. \quad (27)$$

The right-hand side of Eq. (27) refers to interactions of odd-parity meson trajectories. We take n as well as a and c to refer to self-conjugate meson states. The properties of meson-meson-meson interactions discussed earlier in this section (those taken from Ref. 8) imply a group symmetry and that F_{anc} is proportional to $i\gamma_{can}$, where the γ are structure constants. Furthermore, $F_{\bar{b}dn}$ is proportional to the db matrix element of a Hermitian matrix $F_n{}'$ that corresponds to odd-parity meson interactions. If we use these proportionality relations, and combine Eqs. (25) and (27), the result is

$$[D_c,D_a]_{db}=[F_c,F_a]_{db}=i\kappa\sum_n \gamma_{can}(F_n{}')_{db}, \quad (28)$$

where κ is a real constant.

Application of the parity-cancellation principle to baryon-baryon states leads to the result that the interactions with baryons of corresponding meson trajectories of opposite signatures must be proportional.[9] This implies that $(F_n{}')_{db}$ is proportional to D_{db}, so that Eq. (28) implies that the D interactions transform like a representation of the symmetry group, and that the F interactions have similar properties.

V. SOLUTION INVOLVING $SU(3)$

In Sec. IV it was shown that the particles involved in any solution to the consistency equations must correspond to representations of a Lie group. Only certain sets of representations lead to solutions. In Ref. 8 it was shown that mesons satisfying an analogous set of equations must correspond to the identity and regular representations of the group. The purpose of this section is to find a solution for the baryons. The most physical solution corresponds to the group $SU(6)$. However, we consider here $SU(3)$, because it is simpler and the relevant crossing matrices are all in the literature. A brief discussion of the extension to $SU(6)$ is given in Sec. VI.

In order to illustrate the procedure, we write and solve the consistency equations for the scattering of octet M particles from octet B particles. Only singlet and octet meson trajectories and singlet, octet, and decuplet baryon trajectories are allowed. We consider the quantities $X_{st}{}^i$ and $X_{su}{}^i$ of Eq. (18), using X^i to refer to either of these quantities.

It is convenient to choose the amplitudes i to be either symmetric or antisymmetric under the interchange $s\leftrightarrows u$. This is equivalent to definite symmetry under interchange of the mesons a and c [see Eq. (3)], and thus the amplitudes crorespond to t-channel MM representations of definite symmetry. We use the octet-octet crossing matrix of $SU(3)$ to relate these amplitudes X^i to amplitudes corresponding to a definite representation in the s channel.[10] The crossing relations are

$$\begin{aligned}
X^1&=\tfrac{1}{8}X_1+X_{dd}+X_{ff}+(5/4)X_{10},\\
X^{dd}&=\tfrac{1}{8}X_1-\tfrac{3}{10}X_{dd}+\tfrac{1}{2}X_{ff}-\tfrac{1}{2}X_{10},\\
X^{df}&=X_{df}+\tfrac{1}{4}(5)^{1/2}X_{10},\\
X^{27}&=\tfrac{1}{8}X_1+\tfrac{1}{5}X_{dd}-\tfrac{1}{3}X_{ff}-\tfrac{1}{12}X_{10},\\
X^{ff}&=\tfrac{1}{8}X_1+\tfrac{1}{2}X_{dd}+\tfrac{1}{2}X_{ff},\\
X^{fd}&=X_{df}-\tfrac{1}{4}(5)^{1/2}X_{10},\\
X^{10}&=X^{10*}=\tfrac{1}{8}X_1-\tfrac{2}{5}X_{dd}+\tfrac{1}{4}X_{10},
\end{aligned} \quad (29)$$

[9] This follows from the fact that all baryon-baryon states are exotic.

[10] The octet-octet crossing matrix is given by J. J. de Swart, Nuovo Cimento **31**, 420 (1964). The signs of our t-channel (fd and df) amplitudes are opposite to those of de Swart. This sign change is made because the f-type interaction for three multiplets depends on the order of the multiplets. With the de Swart convention, the f-type interaction of a meson with the $\bar{B}B$ states is opposite to that usually assumed.

TABLE I. Relative values of $X_s^{(+)}{}_j$ and $X_s^{(-)}{}_j$ for the three solutions of Eqs. (30)–(33). The parentheses and subscript s are suppressed.

Solution	$X^+{}_{10}$	$X^+{}_{dd}$	$X^+{}_{ff}$	$X^+{}_{df}$	$X^-{}_1$	$X^-{}_{dd}$	$X^-{}_{ff}$	$X^-{}_{df}$
I	0	0	9	0	16	5	0	0
II	24	15	3	$3\sqrt{5}$	16	5	9	$-3\sqrt{5}$
III	8	5	25	$5\sqrt{5}$	48	15	3	$3\sqrt{5}$

where the superscripts and subscripts refer to t-channel and s-channel representations, respectively. The d and f denote symmetric and antisymmetric octet states, and the first index of X^{fd} and X^{df} refers to the MM state in the t channel. The amplitudes antisymmetric in the interchange $s \leftrightarrows u$ are X^{ff}, X^{fd}, and $X^{10}=X^{10*}$. Since the initial and final s-channel states are MB states, $X_{df}=X_{fd}$. We apply the exotic-state condition in the s and u channels by setting X_{10^*} and X_{27} equal to zero; these terms have been omitted from the right-hand side of Eq. (29).

The t-channel exotic-state condition implies that $X_{st}{}^i$ vanishes when i is exotic in the t channel, i.e.,

$$X_{st}{}^{27}=X_{st}{}^{10}=0. \tag{30}$$

The crossing condition of Sec. III C implies that $X_{su}{}^i$ vanishes for antisymmetric states i, i.e.,

$$X_{su}{}^{ff}=X_{su}{}^{fd}=X_{su}{}^{10}=0. \tag{31}$$

Some consistency conditions are expressed most simply in terms of $X_s^{(\pm)}=X_{st}\pm X_{su}$. Condition (1) of Sec. III C implies that all the $X_s^{(\pm)}{}_j$ except $X_s^{(\pm)}{}_{df}$ must be non-negative (where the subscript is the s-channel amplitude).

We now assume a specific set of baryon representations, requiring that the even- and odd-parity baryon trajectories correspond to **8**⊕**10** and **8**⊕**1**, respectively. Since there is only one octet of each parity, the factorizability condition of Sec. III C implies that

$$X_s^{(\pm)}{}_{dd}X_s^{(\pm)}{}_{ff}=X_s^{(\pm)}{}_{df}{}^2. \tag{32}$$

Furthermore, the absence of an even-parity singlet and odd-parity decuplet implies that

$$X_s^{(+)}{}_1=X_s^{(-)}{}_{10}=0. \tag{33}$$

We have yet to consider the t-channel residues and to use the crossing symmetry of X_{st} and X_{ut}. However, it is straightforward to show that there are only three discrete nontrivial solutions to Eqs. (30)–(33), so we list these solutions first and then test consistency with the t-channel conditions. The $X_s^{(\pm)}$ of the three solutions are listed in Table I. The over-all normalization of each solution is arbitrary, but there are no other variable parameters.[11] The constants are related to the conventional F/D ratio by

$$(9/5)^{1/2}F/D=X_{ff}/X_{df}=X_{df}/X_{dd}.$$

We are also interested in the solutions that occur when the trajectory parity assignments are reversed, i.e., when the even- and odd-parity baryons correspond to the representations **8**⊕**1** and **8**⊕**10**, respectively. These solutions, distinguished by bars over the X's, may be obtained from the unbarred solutions by the prescription

$$\bar{X}_s^{(\pm)}{}_j=X_s^{(\mp)}{}_j. \tag{34}$$

We next consider the t-channel residues. For our amplitudes i, $X_{ut}{}^i=\pm X_{st}{}^i$, where the sign corresponds to the symmetry under s-u crossing and also to the orbital parity of the MM state in the t channel. For either parity, $X_t^{(\pm)}=2X_{st}$, and we need consider only X_{st}. This may be written in terms of coupling constants g and G:

$$2X_{st}{}^i=\sum_j g_{j,MM}G_{j,B\bar{B}}, \tag{35}$$

where the sum is over meson trajectories, the MM and $B\bar{B}$ states are those of the amplitude i, and the normalization is similar to that used for the s-channel residues, Eqs. (22a) and (22b).

We now find a solution to all the conditions for all $MM\to MM$ and $MB\to MB$ amplitudes, using the results of Table I as a guide. The conditions of Eqs. (30)–(33) apply to octet-octet MM scattering, as well as to MB scattering. Solution I of Table I must apply to MM scattering, since the other solutions involve decuplets. However, since a singlet meson trajectory couples to MM states only if it is of even parity, we must take the barred modification of this solution, given by Eq. (34). The resulting solution is identical to that of Refs. 7 and 8, since our $MM\to MM$ conditions are algebraically the same as those of these references. It is shown in Ref. 8 that the solution may be extended to amplitudes involving external singlet mesons in a way that satisfies the conditions in all three channels.

We now turn to the $MB\to MB$ octet-octet amplitudes, assuming as before that the even- and odd-parity trajectories correspond to the multiplets **8**⊕**10** and **8**⊕**1**, respectively. Only solutions II and III of Table I are eligible, since there is no decuplet in solution I. We test these possible solutions with the t-channel consistency condition, Eq. (35). Since the B and M are the lowest states on the even-parity baryon and meson trajectories, the MBB coupling constants occur in s-channel and t-channel residues. This leads to the consistency condition

$$X_{st}{}^{dd}/X_{st}{}^{df}=X_s^{(+)}{}_{dd}/X_s^{(+)}{}_{df}, \tag{36}$$

since both these ratios are the d/f coupling ratio of the MBB interaction. One may use Eq. (29) to show that this condition is satisfied in the case of solution III of

[11] Recently, the parity-cancellation form of the duality principle has been applied to the three channels of the $MB\to MB$ process by V. Barger and C. Michael, Phys. Rev. **186**, 1592 (1969). The self-consistency conditions of this reference are equivalent to conditions (1) and (2) of our Sec. III C, applied to the baryon residues. The solutions of Barger and Michael for octet-octet MB scattering contain one extra continuous parameter.

Table I, but not in the case of solution II. Thus, solution III is the consistent solution. [If the assumed parities of the singlet and decuplet trajectories were reversed, so that the residues corresponded to the barred $X^{(\pm)}$ of Eq. (34), then solution II would be the one that satisfied the t-channel conditions.]

Application of the parity-cancellation principle to baryon-baryon scattering implies that the couplings to baryons of odd- and even-parity meson trajectories are proportional.[9] This implies another t-channel consistency condition, similar to Eq. (36), but involving the odd-parity meson residues. The condition is

$$X_{st}{}^{fd}/X_{st}{}^{ff}=X_s{}^{(+)}{}_{dd}/X_s{}^{(+)}{}_{df}. \tag{37}$$

This condition is also satisfied by solution III.

One may apply the consistency conditions to all MB amplitudes, where the M refer to the singlet as well as the octet, and the B refer to the decuplet as well as the octet. It can be shown that there is a unique extension of our solution (solutions I and III for MM and MB scattering, respectively) that satisfies all the consistency conditions. The coupling constants of the solution are[12]

$$G_{8,d}=5^{1/2},\quad G_{8,f}=5,\quad G_{8,10,8}=(10)^{1/2},\quad G_{8,8,1}=8^{1/2},$$
$$G_{10,8,8}=8^{1/2},\quad G_{10,10,1}=8^{1/2},\quad G_{10,10,8}=(32)^{1/2},\quad \bar{G}_{8,d}=(15)^{1/2},$$
$$\bar{G}_{8,f}=3^{1/2},\quad \bar{G}_{8,10,8}=(30)^{1/2},\quad \bar{G}_{8,8,1}=0,\quad \bar{G}_{1,8,8}=(48)^{1/2},$$
$$g_{8,d}=(10)^{1/2},\quad g_{8,8,1}=(8)^{1/2},\quad g_{1,8,8}=(32)^{1/2},\quad g_{1,1,1}=2,$$
$$\bar{g}_{8,f}=(18)^{1/2},\quad \bar{g}_{8,8,1}=0.$$

Here G_{ijk} refers to the coupling of an even-parity baryon trajectory of multiplicity i with a BM state of B multiplicity j and M multiplicity k, while g_{ijk} refers to the coupling of an even-parity meson trajectory of multiplicity i with MM states of the multiplets j and k. The barred constants refer to the couplings of odd-parity trajectories. The $8d$ and $8f$ refer to d- and f-type octet-octet-octet coupling. The normalization is such that $G_{ijk}{}^2$ is the sum over the BM states of the squares of the couplings to one member of the trajectory multiplet i. The relative MMM/MBB normalization is chosen to satisfy Eq. (35) and the s-channel conditions $X_s{}^{(+)}{}_j=G_j{}^2$ and $X_s{}^{(-)}{}_j=\bar{G}_j{}^2$. The couplings of meson trajectories to $B\bar{B}$ states may all be obtained from the G's and the condition that the couplings of even- and odd-parity meson trajectories to $B\bar{B}$ states are the same. The signs of the constants depend on convention; the only relative signs that we have computed are the positive F/D ratios of the couplings of baryon trajectories to BM states.

There is no simpler solution involving an octet, i.e., there is no solution (except for the meson solution) in which there are fewer than four basic multiplets. It is interesting that each of the even- and odd-parity meson and baryon multiplets of the solution corresponds to either the representation $(3,\bar{3})$ or $(3,6)$ of $SU(3)\otimes SU(3)$.

VI. CONCLUDING REMARKS

The self-consistency conditions of Sec. III C do not depend on the details of the Veneziano model. It is possible that when an improved dynamical model is developed, the consistency conditions (or modifications of them) will remain. In a sense, the Veneziano model was used here as a method of bridging the gap between the two types of bootstrap conditions discussed in Sec. II, the parity-cancellation conditions for exotic states, and the crossing-matrix condition of the meson-nucleon static model. We note that if the static-model condition of Eq. (1) is applied to all meson-nucleon exotic states, a solution with a finite number of internal quantum numbers is impossible.[13] Thus, the inclusion of virtual particles of both parities is indispensible to a model with a finite number of Regge trajectories. In our model, the quantity satisfying the crossing-matrix condition is X_{su}, which is the difference of the contributions of trajectories of opposite parities, as is seen from the inverse of Eq. (20). It happens that in the case of πN scattering, the contribution of odd-parity baryon trajectories is relatively small.[14] This is one reason that the static model is so useful in the πN case, and a little less useful when extended to $SU(3)$ multiplets.

The Veneziano model has one definite advantage over most previous models: One can use it to obtain simple consistency conditions for the leading trajectories without neglecting the baryon-meson mass difference. This is mentioned briefly in Sec. IV, before Eq. (25). An example of this advantage occurs in the treatment of MB states [such as $SU(3)$ singlets] for which resonances occur of one parity only. If one treated this phenomenon by applying duality to conventional

[12] When we computed these coupling constants, we made use of the octet-octet and octet-decuplet crossing matrices of Ref. 10 and the $8+8\rightarrow 8+10$ crossing matrix given by K. Y. Lin and R. E. Cutkosky, Phys. Rev. **140**, B205 (1965).

[13] This results from the fact that in the static model, the baryon exchange force in the meson-baryon state of largest total charge is positive in all cases. This effect is discussed by R. H. Capps, Phys. Rev. Letters **13**, 536 (1964).

[14] This contribution is estimated by R. H. Capps, Phys. Rev. **185**, 2008 (1969). It is emphasized in this reference that the smallness of the odd-parity contribution is not a violation of the combination of $SU(3)$ with exchange-degeneracy requirements on exotic states.

Regge amplitudes,[15] a relation between the masses of the baryons and mesons of the two crossed channels would be predicted. In the Veneziano formulation, the mass condition is replaced by that of trajectory-slope equality, which corresponds better to experiment.

Since the spins of the external particles have been neglected, the specific results would be more relevant physically if the group considered were $SU(6)$ rather than $SU(3)$. The model would then apply in the approximation that spin crossing matrices can be treated like those for internal quantum numbers. The most physical solution of the $SU(6)$ model would involve even- and odd-parity baryons corresponding to the representations $\mathbf{70}\oplus\mathbf{56}$ and $\mathbf{70}\oplus\mathbf{20}$, respectively.[16] This solution is analogous to the $SU(3)$ solution given in Sec. V.

If one applies these self-consistency conditions to physical amplitudes, some of the conditions are expected to be more accurate than others. For example, in the case of MB scattering, it is likely that the s-channel exotic condition is more accurate than the t-channel exotic condition.[17] However, one expects amplitudes corresponding to t-channel exotic states to be smaller than those corresponding to t-channel nonexotic states. Therefore, finding exact solutions to the set of consistency equations may be useful as a way of finding a first approximation to the physical hadron-hadron scattering amplitudes.

In a future publication, the exact treatment of the spins of external particles will be discussed.

[15] A. Schwimmer, Phys. Rev. **184**, 1508 (1969). Schwimmer uses the conventional Regge representation to analyze $\pi\eta$ scattering, where the resonances are all of even parity. The resulting predicted degeneracy of the trajectories in the two crossed channels (A_2 and f_0 trajectories) is accurate, in this case.

[16] The fact that the 20-fold baryon multiplet should be present in a complete model involving the $SU(6)$ group has been emphasized by P. G. O. Freund and R. Waltz, Phys. Rev. **188**, 2270 (1969).

[17] A plausible reason for this effect is given by J. Mandula, J. Weyers, and G. Zweig, Phys. Rev. Letters **23**, 266 (1969). See also Ref. 14.

PHYSICAL REVIEW D VOLUME 1, NUMBER 8 15 APRIL 1970

Theory of Deep-Inelastic Lepton-Nucleon Scattering and Lepton Pair Annihilation Processes. IV. Deep-Inelastic Neutrino Scattering*

Tung-Mow Yan and Sidney D. Drell
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 13 November 1969)

This is the last in a series of four papers devoted to a theoretical study based on canonical field theory of the deep-inelastic lepton processes. In the present paper we present the detailed calculations leading to the limiting behavior—or the "parton model"— for deep-inelastic neutrino scattering, i.e., $\nu+p\rightarrow e^-+$"anything," $\bar{\nu}+p\rightarrow e^++$"anything," where "anything" refers to all possible hadrons. In particular, we show that the structure functions depend only on the ratio of energy to momentum transfer $2M\nu/q^2$ as conjectured by Bjorken on general grounds. Experimental implications, including sum rules and the relation of ν and $\bar{\nu}$ cross sections to each other as well as to deep-inelastic electron scattering cross sections, are derived and discussed.

I. INTRODUCTION

In this fourth and final article of a series of papers[1] on lepton-hadron interactions we study neutrino and antineutrino scatterings in the deep-inelastic region.

The smallness of the fine-structure constant for lepton electromagnetic interactions and of the Fermi coupling constant for their weak interactions permits the lowest-order perturbation expansion in these parameters. We assume the weak currents of the leptons to be well described by the universal $V-A$ theory. The conserved-vector-current hypothesis of Feynman and Gell-Mann[2] and the Cabibbo theory of the weak currents for the hadrons[3] are also generally accepted as working assumptions.

Apart from the question of whether the weak interaction is really of current-current type or is mediated by intermediate vector bosons, neutrinos as well as antineutrinos, like electrons and muons in electromagnetic interactions, also probe the structure of hadrons via scatterings from hadron targets. The parton model derived in previous papers of this series[1] for deep-inelastic electron scattering can be generalized to a form appropriate for neutrino and antineutrino scattering. Accomplishing this generalization is the task of

* Work supported by the U. S. Atomic Energy Commission.

[1] S. D. Drell, D. J. Levy, and T. M. Yan, Phys. Rev. Letters **22**, 744 (1969); Phys. Rev. **187**, 2159 (1969); Phys. Rev. D **1**, 1035 (1970); **1**, 1617 (1970). The last three papers will be referred to as Papers I, II, and III, respectively.

[2] R. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[3] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).