Photon-Photon Scattering Contribution to the Sixth-Order Magnetic Moments of the Muon and Electron*

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(Received 2 January 1970)

We have calculated the three-photon exchange contribution to the sixth-order anomalous magnetic moment of the leptons. Our result for the electron-loop contribution to the muon moment $(18.4\pm1.1)(\alpha/\pi)^3$ brings the theoretical prediction into agreement with the CERN measurements within the 1-standard-deviation experimental accuracy. The result for the electron-loop contribution to the electron moment is $(0.36\pm0.04)(\alpha/\pi)^3$. The theoretical errors represent the accuracy of the required seven-dimensional numerical integrations.

1. INTRODUCTION AND SUMMARY

THE anomalous magnetic moments of the electron and muon have played central roles in the testing of the validity of quantum electrodynamics and the search for possible differences in the basic properties of the leptons. The increasing precision of present and projected measurements of the g factor now promises a confrontation with the predictions of theory through sixth order in perturbation theory. In addition, the muon moment can provide a fundamental sum-rule limit on the electromagnetic coupling to the entire spectrum of hadrons as well as a limit on the influence of weak interactions on the lepton field.[1]

Unfortunately, the complete calculation of the sixth-order radiative corrections to the lepton vertex—especially those graphs which cannot be obtained from insertions of second- or fourth-order corrections to the photon and fermion propagators—is horrendous. There are two central problems: (1) the reduction of matrix elements with three loop integrations to Feynman parametric form, and (2) the multidimensional integration of the resulting integrand.

In this paper we present a computation of the photon-photon scattering subdiagram contribution to the sixth-order magnetic moment of the electron and muon. In order to avoid computational errors in the reduction to parametric form we have carried out our calculation in two different ways: One follows the standard Landau techniques outlined in the book of Bjorken and Drell,[2] and the other is based on the method developed by Nakanishi[3] and Kinoshita.[4] We have calculated most integrands, including all those that contribute to the $\ln(m_\mu/m_e)$ term, by hand. In the end, all of the trace algebra and substitutions were performed automatically using REDUCE, an algebraic computation program developed by Hearn.[5] For the practical solution to the second problem we have resorted to numerical integration using a novel program [originally developed by G. Sheppey at CERN[6] and improved by one of us (AJD)] which on successive iterations improves the Riemann integration grid through a random-variable sampling technique. In the rest of this section we present a comparison of theory and experiment and outline the remainder of the paper.

The most recent CERN measurement of the anomalous part of the muon g factor gives[7]

$$a_{\rm expt}=(116\,616\pm31)\times10^{-8}. \qquad (1.1)$$

The experimental error is about 7% of the $(\alpha/\pi)^2$ term in the theoretical prediction. Thus, for a serious confrontation of theory and experiment, the theoretical result must be improved to an accuracy of order 10^{-7} or better, which requires knowledge of the α^3 radiative

* Work supported in part by the U. S. Atomic Energy Commission and the U. S. Office of Naval Research.
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[1] For a review and references, see F. J. M. Farley, in Proceedings of the First Meeting of the European Physical Society at Florence, 1969 (unpublished); S. J. Brodsky, in Proceedings of the International Conference on Electron and Photon Interactions at High Energies, Daresbury, 1969 (unpublished). See also Ref. 18.

[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Sec. 18.4.
[3] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **17**, 401 (1957).
[4] T. Kinoshita, J. Math. Phys. **3**, 650 (1962).
[5] A. C. Hearn, Stanford University Report No. ITP-247 (unpublished); A. C. Hearn, in *Interactive Systems for Experimental Applied Mathematics*, edited by M. Klerer and J. Reinfelds (Academic, New York, 1968).
[6] We wish to thank Dr. G. R. Henry for bringing this program to our attention.
[7] J. Bailey, W. Bartl, G. Von Bochmann, R. C. A. Brown, F. J. M. Farley, H. Jöstlein, E. Picasso, and R. W. Williams, Phys. Letters **28B**, 287 (1968).

corrections, hadronic corrections, and possibly corrections due to weak intermediate bosons.

The theoretical result for the muon g factor which has been calculated previous to this work from standard quantum electrodynamics is

$$\tfrac{1}{2}(\alpha/\pi)+0.76578(\alpha/\pi)^2+3.00(\alpha/\pi)^3. \quad (1.2)$$

The fourth-order term has been evaluated analytically.[8] The last term consists of two parts: One is the contribution to the lepton vertex which involves only one type of lepton, and the other in which both leptons appear. An estimate of the first contribution based on the technique of sidewise dispersion relations gives $0.13(\alpha/\pi)^3$.[9] A term $[0.055(\alpha/\pi)^3]$ not included in the above estimate was obtained recently by an analytic calculation of diagrams containing fourth-order vacuum polarization due to muon pairs.[10] (These mass-independent contributions are of course common to the electron g factor.) The second part is obtained by insertion of electron loops of fourth and second order into the virtual photon lines of the second- and four-order electromagnetic vertices of the muon.[11–14] This contribution can be written in the form

$$\left\{\frac{2}{9}\left[\ln\left(\frac{m_\mu}{m_e}\right)\right]^2-1.114\ln\left(\frac{m_\mu}{m_e}\right)+2.44\right\}\left(\frac{\alpha}{\pi}\right)^3 =2.82\left(\frac{\alpha}{\pi}\right)^3. \quad (1.3)$$

It was found that the coefficients of the logarithmic terms can be obtained simply by algebraic manipulation of the renormalization constant Z_3 and the muon magnetic moment of the second and fourth orders.[11] Several terms contributing to the nonlogarithmic terms in (1.3) have been calculated directly.[11,14] Although some nonlogarithmic terms are still to be evaluated, they are at least estimated in Ref. 13.[15] The error of this estimate will probably not exceed $\pm0.5(\alpha/\pi)^3$.[16]

The latest estimate of the contribution from strong interactions (vacuum polarization due to hadrons) to the muon g factor, based on the Orsay colliding-beam data for $e^+ + e^- \to \rho$, ω, and ϕ resonances, is[17]

$$a_{\text{hadrons}}=(6.5\pm0.5)\times10^{-8}. \quad (1.4)$$

If one uses the value[18]

$$\alpha^{-1}=137.03608\pm0.00026 \quad (1.5)$$

for the fine-structure constant, one obtains from (1.2) and (1.4) the theoretical prediction

$$a=(116\,564\pm2)\times10^{-8}. \quad (1.6)$$

which disagrees slightly (1.7 standard deviations) with the experimental value (1.1). The error interval in (1.6) reflects the uncertainty in the strong-interaction contribution (0.5×10^{-8}), in the value of $\alpha/2\pi$ (0.2×10^{-8}), and in the sixth-order correction (1.3) (0.6×10^{-8}). It does not take into account the uncertainty in the magnitude of the vacuum-polarization contribution of higher-mass hadrons.[19] We have also not included possible weak-interaction corrections to the muon moment,[20] which could be expected to be of order 1×10^{-8}.

Also not included in the above error estimate is the contribution from the sixth-order diagrams containing photon-photon scattering subdiagrams (Fig. 1). Of course, this is because it has not been successfully calculated or estimated thus far. Earlier attempts[11,13,21] have been directed at finding out whether this contribution contains $\ln(m_\mu/m_e)$ terms or not. Unfortunately, it is not easy to detect the presence or absence of logarithmic terms without extensive calculations. In fact, on the basis of general consideration of the mass singularity,[4] it can be shown that the individual diagrams of Fig. 1 may contribute to the logarithmic terms. On the other hand, these terms might cancel each other when contributions from all six diagrams are put together. Indeed, several arguments have been put forward indicating such a cancellation.[11] However, since none of these arguments has been free from loopholes, we have been convinced that this question cannot be settled short of an all-out effort. Once we decided to

[8] H. H. Elend, Phys. Letters **20**, 682 (1966); **21**, 720(E) (1966); and private communication. See also H. Suura and E. H. Wichmann, Phys. Rev. **105**, 1930 (1957); A. Petermann, *ibid.* **105**, 1931 (1957); Fortschr. Physik **6**, 505 (1958).

[9] S. D. Drell and H. R. Pagels, Phys. Rev. **140**, B397 (1965); R. G. Parsons, *ibid.* **168**, 1562 (1968).

[10] J. A. Mignaco and E. Remiddi, Nuovo Cimento **60A**, 519 (1969).

[11] T. Kinoshita, Nuovo Cimento **51B**, 140 (1967).

[12] S. D. Drell and J. S. Trefil (unpublished). For a discussion of their preliminary result, see S. D. Drell, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1966), p. 93; S. D. Drell, in *Particle Interactions at High Energies*, edited by T. W. Preist and L. L. J. Vick (Oliver and Boyd, Edinburgh, 1966).

[13] T. Kinoshita, in *Cargèse Lectures in Physics* (Gordon and Breach, New York, 1968), Vol. 2, p. 209.

[14] B. E. Lautrup and E. deRafael, Phys. Rev. **174**, 1835 (1968).

[15] For more discussion on this point, see Ref. 7 of our preliminary report: J. Aldins, T. Kinoshita, S. J. Brodsky, and A. J. Dufner, Phys. Rev. Letters **23**, 441 (1969).

[16] An additional support of this estimate is given by the calculation of B. E. Lautrup and E. deRafael, CERN Report No. TH 1042, 1969 (unpublished).

[17] M. Gourdin and E. deRafael, Nucl. Phys. **B10**, 667 (1969). See also T. Kinoshita and R. J. Oakes, Phys. Letters **25B**, 143 (1967); J. E. Bowcock, Z. Physik **211**, 400 (1968).

[18] This is the value of α^{-1} derived by the adjustment of fundamental constants using no quantum-electrodynamics data, given by B. N. Taylor, W. H. Parker, and D. N. Langenberg, Rev. Mod. Phys. **41**, 375 (1969). See also W. H. Parker, B. N. Taylor, and D. N. Langenberg, Phys. Rev. Letters **18**, 287 (1967). Recent fine-structure measurements in H and D and the hyperfine splitting in H yield values of α^{-1} consistent with (1.5).

[19] H. Terazawa, Progr. Theoret. Phys. (Kyoto) **39**, 1326 (1968); J. S. Bell and E. deRafael, Nucl. Phys. **B11**, 611 (1969).

[20] R. A. Shaffer, Phys. Rev. **135**, B187 (1964); S. J. Brodsky and J. D. Sullivan, *ibid.* **156**, 1644 (1967); T. Burnett and M. J. Levine, Phys. Letters **24B**, 467 (1967).

[21] H. Terazawa, Progr. Theoret. Phys. (Kyoto) **38**, 863 (1967).

settle the question of logarithmic terms by an extensive calculation, it was not much harder to evaluate the Feynman integrals for the graphs of Fig. 1 exactly.

The result of our calculation of the contribution from the three-photon exchange diagrams turns out to be surprisingly large:

$$\Delta a_{\text{ph-ph}} = (18.4\pm 1.1)(\alpha/\pi)^3 = (23.0\pm 1.4)\times 10^{-8}. \quad (1.7)$$

This leads us to a revised theoretical prediction

$$a_{\text{theor}} = (116\,587\pm 3)\times 10^{-8} \quad (1.8)$$

and

$$a_{\text{expt}} - a_{\text{theor}} = (29\pm 34)\times 10^{-8} = 250\pm 290 \text{ ppm}. \quad (1.9)$$

Thus the addition of the photon-photon scattering contribution essentially eliminates the discrepancy mentioned above. The theoretical error in (1.8) includes the uncertainty due to the numerical integration of the contribution (1.7) (1.4×10^{-8}). This error could be reduced if necessary. We wish to emphasize that, with the inclusion of the photon-photon scattering contribution (1.7), all of the Feynman diagrams from quantum electrodynamics which contribute to the difference of the muon and electron magnetic moments through sixth order have been calculated or estimated.[13]

The largeness of the contribution (1.7) is closely related to a logarithmic dependence on the muon and electron mass ratio. In fact, in the limit of large m_μ/m_e the result (1.7) can be expressed in the form

$$\Delta a_{\text{ph-ph}} = [(6.4\pm 0.1)\ln(m_\mu/m_e) + \text{const}](\alpha/\pi)^3. \quad (1.10)$$

Thus earlier arguments[11] indicating a cancellation among the diagrams of Fig. 1 for the logarithmic terms are disproved.

Since no approximations are made in the reduction of the Feynman integrals to parametric form, we can also obtain the photon-photon scattering contribution to the sixth-order anomalous magnetic moment of the electron. Our result is

$$(\Delta a_e)_{\text{ph-ph}} = (0.36\pm 0.04)(\alpha/\pi)^3 = (0.45\pm 0.05)\times 10^{-8}, \quad (1.11)$$

where the error limits represent the uncertainty in the required numerical integrations in seven dimensions. For completeness, the mass-independent contribution (1.11) must be added into the muon result (1.8).

Combining (1.11) with the previously calculated or estimated sixth-order contributions given in Refs. 9 and 10, the theoretical prediction for the electron moment is

$$a_e = \tfrac{1}{2}(\alpha/\pi) - 0.32848(\alpha/\pi)^2 + 0.55(\alpha/\pi)^3. \quad (1.12)$$

The last term is by no means the entire theoretical result for the sixth-order coefficient, since second-order vacuum polarization insertions into the fourth-order vertex have not been calculated and, in addition, the reliability of the estimate of Ref. 9 is not certain. Note that the calculation of Mignaco and Remiddi[10] corresponds to the contribution of three- and four-particle intermediate states in the sixth-order Feynman diagrams containing fourth-order vacuum polarization. The fact that this contribution is not so small casts some doubt on the validity of the two-particle approximation used in the dispersion-theoretical calculations.

The experimental value of the electron moment from the Michigan group is[22]

$$(a_e)_{\text{expt}} = (1\,159\,549\pm 30)\times 10^{-9} = \tfrac{1}{2}(\alpha/\pi) - 0.32848(\alpha/\pi)^2 - (7.0\pm 2.4)(\alpha/\pi)^3, \quad (1.13)$$

where we have used the value of α from (1.5) and the fourth-order theoretical prediction to obtain an experimental determination of the sixth-order coefficient. It will be interesting to see whether future experiments and further development of the theoretical result will confirm the indicated discrepancy of sign and magnitude of the sixth-order coefficient.

In the next sections we discuss the calculation of the results (1.7) and (1.11). In Sec. 2 we introduce a method which enables us to extract the magnetic-moment contribution of the diagrams of Fig. 1 automatically. This leads us to the introduction of the set of four modified Feynman diagrams shown in Fig. 2. There are, of course, many ways of introducing the Feynman parameters, and it is important to choose a method which gives as simple a result as possible, as well as exposing all the identities implicit in the formulas. Because of its simplicity and versatility, we shall use the double parametric representation of Feynman amplitudes introduced a few years ago.[4] Its application to the diagrams of Fig. 2 is given in Sec. 3. In Sec. 4 we carry out the trace calculations and other simplifying operations and present the exact form of the Feynman integrals using "currents" as auxiliary variables, which is perhaps the most transparent and economical way of writing down these integrals. In Sec. 5 we discuss an alternative, more standard method which we have also used to derive the Feynman parametric integrals. The connections between the two reduction methods is discussed, and an important identity, readily utilized by REDUCE, to simplify numerator expressions with high powers of loop momenta is

[22] D. T. Wilkinson and H. R. Crane, Phys. Rev. **130**, 852 (1963); A. Rich, Phys. Rev. Letters **20**, 967 (1967); **20**, 1221(E) (1968); G. R. Henry and J. E. Silver, Phys. Rev. **180**, 1262 (1969). [*Note added in proof.* The preliminary result of a new electron anomalous magnetic-moment measurement by J. C. Wesley and A. Rich (unpublished) is $a_e = (1\,159\,644\pm 7)\times 10^{-9} = \frac{1}{2}(\alpha/\pi) - 0.32848(\alpha/\pi)^2 + (0.54\pm 0.58)(\alpha/\pi)^3$.]

given. In Sec. 6 we study the behavior of the Feynman integrals in the limit where $\rho=(m_e/m_\mu)^2$ tends to zero. The method of numerical integration used to evaluate the integrals as well as the results of computation are discussed in Sec. 7. Some properties of the functions A_i and B_{ij} are described in Appendix A. In Appendix B we give the unsimplified output of REDUCE for graph IV. Some formulas needed in Sec. 6 are given in Appendix C.

2. EXTRACTION OF MAGNETIC-MOMENT TERM

According to the Feynman-Dyson rules, we can write the contribution of the graphs of Fig. 1 in the form[23]

$$\langle p'|S|p,\Delta\rangle=-i(2\pi)^4\delta^4(p'-p-\Delta)\frac{1}{(2\pi)^{9/2}}\times\frac{m_\mu}{(2\Delta_0p_0p_0')^{1/2}}eM\,, \quad (2.1)$$

where

$$M=\frac{e^2}{(2\pi)^8}\int d^4p_1d^4p_3\,p_1^{-2}p_2^{-2}p_3^{-2}\times\epsilon^\mu\Pi_{\kappa\rho\sigma\mu}(-p_1,p_2,p_3,-\Delta)\bar{u}(p')\gamma^\kappa(\not{p}_4-m_\mu)^{-1}\times\gamma^\rho(\not{p}_5-m_\mu)^{-1}\gamma^\sigma u(p)\,, \quad (2.2)$$

and $\Pi_{\kappa\rho\sigma\mu}$ is the polarization tensor of fourth rank representing the photon-photon scattering

$$\Pi_{\kappa\rho\sigma\mu}(-p_1,p_2,p_3,-\Delta)=\frac{-ie^4}{(2\pi)^4}\int d^4p_6\,\mathrm{Tr}[\gamma_\kappa(\not{p}_6-m_e)^{-1}\gamma_\rho(\not{p}_7-m_e)^{-1}\times\gamma_\sigma(\not{p}_8-m_e)^{-1}\gamma_\mu(\not{p}_9-m_e)^{-1}+(\text{five other terms})-(\text{regularization terms})]. \quad (2.3)$$

As usual, all momenta are restricted by the energy-momentum conservation law at each vertex. As is well known, individual terms of $\Pi_{\kappa\rho\sigma\mu}$ are logarithmically divergent for large p_6, but the sum of all six terms is convergent and well defined if it is properly regularized. In the integral (2.2), each term may again diverge because of the photon-photon scattering subdiagrams. In addition, each term may diverge logarithmically when all three momenta p_1, p_3, and p_6 go to infinity simultaneously. Nevertheless, it is expected that cancellation of ultraviolet divergences takes place, as in photon-photon scattering, and that there will be no real divergence problem as far as the magnetic-moment term of (2.2) is concerned.

Although it is not difficult to show by direct calculation that this is in fact the case, it would be convenient if the formula (2.2) could be rewritten so that the cancellation of ultraviolet divergences is manifestly evident from the beginning. This can be achieved by making use of the identity

$$\Pi_{\kappa\rho\sigma\mu}(-p_1,p_2,p_3,-\Delta)=-\Delta^\nu\frac{\partial}{\partial\Delta^\mu}\Pi_{\kappa\rho\sigma\nu}(-p_1,p_2,p_3,-\Delta)\,, \quad (2.4)$$

which is easily obtained by differentiating the condition of gauge invariance

$$\Delta^\nu\Pi_{\kappa\rho\sigma\nu}(-p_1,p_2,p_3,-\Delta)=0 \quad (2.5)$$

with respect to Δ^μ, regarding, e.g., Δ, p_1, and p_3 as independent variables.

Substituting (2.4) into (2.3), we obtain

$$M=\epsilon^\mu\Delta^\nu\bar{u}(p')M_{\mu\nu}u(p)\,, \quad (2.6)$$

where

$$M_{\mu\nu}=-\frac{e^2}{(2\pi)^8}\int d^4p_1d^4p_3\,p_1^{-2}p_2^{-2}p_3^{-2}\times\left[\frac{\partial}{\partial\Delta^\mu}\Pi_{\kappa\rho\sigma\nu}(-p_1,p_2,p_3,-\Delta)\right]\times\gamma^\kappa(\not{p}_4-m_\mu)^{-1}\gamma^\rho(\not{p}_5-m_\mu)^{-1}\gamma^\sigma. \quad (2.7)$$

Now, when the differentiation with respect to Δ^μ is carried out explicitly in (2.7), M can be regarded as a

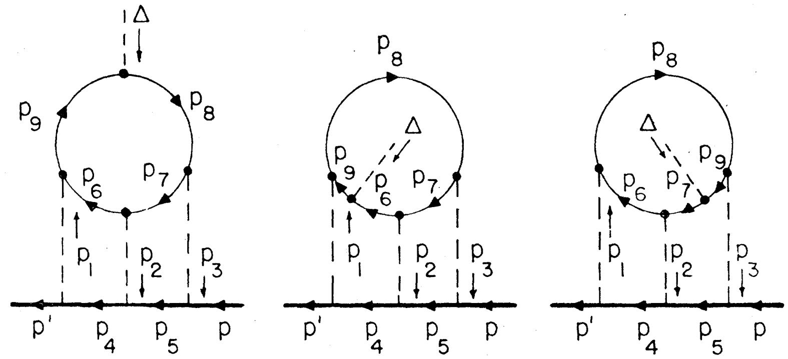

FIG. 1. Feynman diagrams containing subdiagrams of photon-photon scattering type. The heavy, thin, and dotted lines represent the muon, electron, and photon, respectively. There are three more diagrams obtained by reversing the direction of the electron loop.

[23] Our metric and conventions are the same as that of Ref. 2. The Born current corresponds to $M=\bar{u}(p')\epsilon^\mu\gamma_\mu u(p)$, $p'=p+\Delta$.

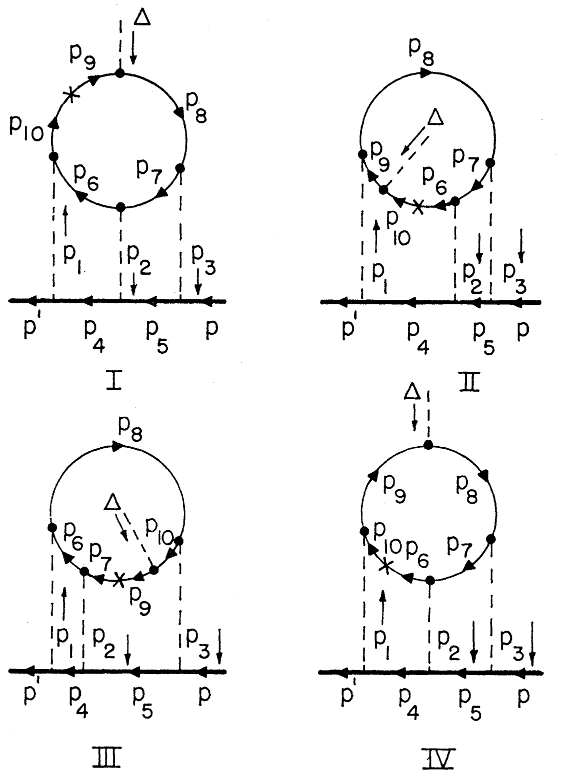

FIG. 2. Feynman diagrams obtained from those of Fig. 1 by differentiation with respect to Δ. The crosses represent differentiation vertices. The external momenta are routed so that Δ always flows through the middle photon line.

sum of the modified Feynman diagrams shown in Fig. 2. Since each diagram of Fig. 2 contains an electron loop with five vertices, it is clear that no ultraviolet divergences arise from integrations over internal momenta. Thus no diagram of Fig. 2 requires subtraction or regularization any longer and each gives a well-defined convergent contribution to the muon magnetic moment. This means that each term can be evaluated separately by a straightforward application of the technique of Feynman parametrization.

Also, since M in (2.6) is already proportional to Δ^ν, we can put $\Delta=0$ in $M_{\mu\nu}$ after differentiation to obtain the static magnetic moment.[24] This simplifies the calculation considerably.

In order to extract the magnetic-moment term from the second-rank tensor $\bar{u}M_{\mu\nu}u$, we note that, because of covariance under Lorentz transformations, it can be expressed in the form

$$\bar{u}(p')M_{\mu\nu}u(p)=\bar{u}(p')[Ag_{\mu\nu}+B(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)+CP_\mu\gamma_\nu+DP_\nu\gamma_\mu+EP_\mu P_\nu]u(p)\,, \quad (2.8)$$

where $P=\frac{1}{2}(p+p')$ and we have omitted terms containing Δ in (2.8) according to our remark in the preceding paragraph.[24] Since $\Delta^\nu P_\nu=0$, the D and E terms do not contribute to the magnetic moment. The C term does not contribute either since $\Delta^\nu\bar{u}\gamma_\nu u=0$ by current conservation. The coefficient A must be equal to zero in order that (2.8) satisfy gauge invariance. Thus the only contribution to the magnetic moment arises from the B term and is equal to $\Delta a=-4m_\mu B$. In order to project out the magnetic-moment term in (2.6), we have only to multiply both sides of (2.8) by $\bar{u}(p)\times(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)u(p')$ and sum over initial and final positive-energy spin states of the muon. Thus we obtain[25]

$$\Delta a=-4m_\mu B=\frac{1}{48m_\mu}\lim_{p'\to p}\mathrm{Tr}((\not{p}+m_\mu)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\times(\not{p}'+m_\mu)M_{\mu\nu})\,. \quad (2.9)$$

3. DOUBLE PARAMETRIC REPRESENTATION

In introducing Feynman parameters in (2.9), it is important to choose them so that the result can be expressed in as simple a form as possible. Otherwise problems of this complexity easily become unmanageable. We shall use the double parametric representation of Feynman amplitudes,[4] probably one of the simplest systematic methods. Further simplification is achieved by a judicious choice of Feynman parameters common to all graphs of Fig. 2, enabling us to express the denominators of all integrands in identical form, and by introducing (as is shown in Sec. 4) auxiliary parameters, called "currents," which simplify the form of numerators enormously.

Let us first parametrize graph I, whose lines are labeled as shown in Fig. 2. We shall write the propagator for the ith internal boson line of mass m_i as

$$[(r_i+q_i)^2-m_i^2+i\epsilon]^{-1}, \quad (3.1)$$

where we have put $p_i=r_i+q_i$, r_i and q_i representing variable and fixed momenta. We choose r_i and q_i in such a way that they satisfy the separate "momentum conservation laws"

$$\sum\pm r_i=0\,,\quad \sum\pm q_i+(\text{external momenta})=0 \quad (3.2)$$

for each vertex, where + or − is chosen according as r_i+q_i is incoming or outgoing. Other than that, they are left indeterminate for the moment. If the ith line is a fermion line, the corresponding propagator is obtained by applying the operator

$$\pm D_i+m_i\,,\quad \text{with}\quad D_i{}^\mu=\frac{1}{2}\int_{m_i^2}^\infty dm_i^2\frac{\partial}{\partial q_{i\mu}}\,, \quad (3.3)$$

on (3.1), where the sign + or − should be chosen ac-

[24] From Eq. (2.4) alone it is not possible to exclude the possibility that $(\partial/\partial\Delta^\mu)\Pi_{\kappa\rho\sigma\nu}$ and hence $M_{\mu\nu}$ have a mild singularity (less singular than Δ^{-1}) in the neighborhood of $\Delta=0$. However, the analysis of mass singularity discussed in Ref. 4 shows that no such singularity is possible insofar as $m_e\neq0$. This is why we can put $\Delta=0$ in $M_{\mu\nu}$.

[25] Alternatively, one can use general projection operators for the muon form factors $F_1(q^2)$, $F_2(q^2)$ and obtain (2.9) as a special case for $F_2(0)$. See S. J. Brodsky and J. D. Sullivan, Ref. 20.

cording as q_i is in the direction of the arrow of the fermion line or not.

Noting that the D operator (3.3) can be interchanged with the integration over the momenta r_1, r_3, and r_6 in (2.7) because this integral is absolutely convergent, we can express the contribution of graph I to the anomalous magnetic moment of the muon (2.9) as follows[26]:

$$\Delta a_{\rm I}=\frac{1}{48m_\mu}\frac{2ie^6}{(2\pi)^{12}}F_0F_{\rm I}\int\frac{d^4r_1d^4r_3d^4r_6}{\prod_{i=1,\cdots,10}((r_i+q_i)^2-m_i^2+i\epsilon)}, \tag{3.4}$$

where

$$F_0=\mathrm{Tr}[(p\!\!\!/+m_\mu)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)(p\!\!\!/'+m_\mu)\times\gamma^\kappa(D_4+m_\mu)\gamma^\rho(D_5+m_\mu)\gamma^\sigma] \tag{3.5}$$

and

$$F_{\rm I}=\mathrm{Tr}[\gamma_\kappa(D_6+m_e)\gamma_\rho(D_7+m_e)\gamma_\sigma(D_8+m_e)\times\gamma_\nu(D_9+m_e)\gamma_\mu(D_{10}+m_e)]. \tag{3.6}$$

Before we carry out the r_i integration, let us first collect all propagators whose integration momenta r_i are identical and can be expressed by a common variable r_α. The set of all such propagators will be called a chain α. For instance, the lines 1 and 4 have the same integration momentum $r_1=r_4$, and will constitute the chain α. Making use of the Feynman parameters x_1 and x_4 with $x_1+x_4=1$, we shall combine the corresponding propagators into the form[27]

$$(n_\alpha-1)!\int\frac{dx(\alpha)}{[(r_\alpha+q_\alpha)^2-V_\alpha(x)+i\epsilon]^{n_\alpha}}, \tag{3.7}$$

where $n_\alpha=2$ is the number of lines in the chain α and

$$dx(\alpha)=\delta(1-x_1-x_4)dx_1dx_4,\quad q_\alpha=x_1q_1+x_4q_4,\quad V_\alpha(x)=x_1m_1^2+x_4m_4^2-x_1x_4(q_1-q_4)^2. \tag{3.8}$$

Expressions of the form (3.7) can be written down in the same fashion for the chains $\beta=(3,5)$ and $\gamma=(8,9,10)$. Remaining lines form chains by themselves: $\lambda=(7)$, $\mu=(6)$, $\nu=(2)$.

Now the integrand of (3.4) is a product of factors of the form (3.7), which can be combined into a single denominator using the formula

$$\prod_{i=1}^m\frac{(n_i-1)!}{a_i^{n_i}}=(n-1)!\times\int\frac{\delta(1-z_1-\cdots-z_m)z_1^{n_1-1}dz_1\cdots z_m^{n_m-1}dz_m}{(z_1a_1+z_2a_2+\cdots+z_ma_m)^n}, \tag{3.9}$$

where $n=n_1+n_2+\cdots+n_m$. The resulting expression can be integrated easily with respect to the momenta r_α, etc., and the integral in (3.4) can be expressed in the double parametric form

$$3!i^3\pi^6\int\frac{dz}{U^2(z)[V(x,z)-i\epsilon]^4}, \tag{3.10}$$

where the discriminant $U(z)$ is a homogeneous polynomial of order 3 in z,[28]

$$V(x,z)=z_\alpha V_\alpha(x)+\cdots+z_\nu V_\nu(x)+v(x,z), \tag{3.11}$$

$$\begin{aligned}-v(x,z)U(z)=&z_\beta z_\gamma z_\lambda(z_\alpha+z_\mu+z_\nu)(q_\beta-q_\gamma+q_\lambda)^2\\&+z_\alpha z_\gamma z_\mu(z_\beta+z_\lambda+z_\nu)(q_\alpha-q_\gamma+q_\mu)^2\\&+z_\alpha z_\beta z_\nu(z_\gamma+z_\lambda+z_\mu)(q_\alpha-q_\beta-q_\nu)^2\\&+z_\lambda z_\mu z_\nu(z_\alpha+z_\beta+z_\gamma)(-q_\lambda+q_\mu+q_\nu)^2\\&+z_\alpha z_\beta z_\lambda z_\mu(q_\alpha-q_\beta-q_\lambda+q_\mu)^2\\&+z_\alpha z_\gamma z_\lambda z_\nu(q_\alpha-q_\gamma+q_\lambda-q_\nu)^2\\&+z_\beta z_\gamma z_\mu z_\nu(q_\beta-q_\gamma+q_\mu+q_\nu)^2,\end{aligned} \tag{3.12}$$

and

$$dz\equiv\delta(1-z_\alpha-\cdots-z_\nu)dz(\alpha)\cdots dz(\nu),\quad dz(\alpha)=z_\alpha^{n_\alpha-1}dz_\alpha dx(\alpha),\quad \text{etc.} \tag{3.13}$$

Substituting (3.10) into (3.4), we finally obtain

$$\Delta a_{\rm I}=\frac{1}{256m_\mu}\left(\frac{\alpha}{\pi}\right)^3F_0F_{\rm I}\int\frac{dz}{U^2(z)[V(x,z)-i\epsilon]^4}. \tag{3.14}$$

Advantages in adopting this parametrization are twofold: (1) The discriminant $U(z)$ is determined completely by the topological structure of the chain diagram and not by individual lines. Since all six graphs have the same chain structure, $U(z)$ is common to all graphs if we name the chains in an appropriate manner. (2) The denominator function $V(x,z)$ takes the most compact and explicit form for this parametrization. The formula for $v(x,z)$ will be much more lengthy than (3.12) for any other way of parametrization. In addition, if we introduce chains in the other graphs so that they have the same chain structure as graph I—for instance, $\alpha=(1,4)$, $\beta=(3,5)$, $\gamma=(8,9)$, $\lambda=(7)$, $\mu=(6,10)$, $\nu=(2)$ for graph IV—we find that not only $U(z)$ but also $v(x,z)$ given by (3.12) [and hence $V(x,z)$] are identical with those of graph I, the only differences between different graphs being contained in the explicit expressions for q_α, $V_\alpha(x)$, and $dz(\alpha)$ given by (3.8) and (3.13). As is shown later, even these differences disappear in the end.

Thus, at least formally, the contributions $\Delta a_{\rm II}$, $\Delta a_{\rm III}$, and $\Delta a_{\rm IV}$ to the muon magnetic moment from the remaining graphs can be expressed by the formula

[26] This is actually the sum of contributions from two diagrams; one is that of graph I of Fig. 2 and the other is that of a graph in which the direction of the arrow of the electron loop is reversed. Both contribute equally to (3.4).

[27] For a general treatment of double parametric representation, see Ref. 4.

[28] See Ref. 4, formula (2.18), for an explicit form of this $U(z)$. A form more convenient for our purpose is given later by formula (4.18).

(3.14) if only we replace F_{I} by

$$F_{\mathrm{II}}=\mathrm{Tr}[\gamma_\mu(D_6+m_e)\gamma_\rho(D_7+m_e)\gamma_\sigma(D_8+m_e)\times\gamma_\kappa(D_9+m_e)\gamma_\nu(D_{10}+m_e)], \quad (3.15)$$

$$F_{\mathrm{III}}=-\mathrm{Tr}[\gamma_\kappa(D_6+m_e)\gamma_\rho(D_7+m_e)\gamma_\mu(D_9+m_e)\times\gamma_\nu(D_{10}+m_e)\gamma_\sigma(D_8+m_e)], \quad (3.16)$$

$$F_{\mathrm{IV}}=\mathrm{Tr}[\gamma_\kappa(D_{10}+m_e)\gamma_\mu(D_6+m_e)\gamma_\rho(D_7+m_e)\times\gamma_\sigma(D_8+m_e)\gamma_\nu(D_9+m_e)]. \quad (3.17)$$

The factor -1 in F_{III} arises because Δ^ν flows in the opposite direction around the electron loop relative to the other three graphs.

4. TRACES AND D OPERATIONS

Our next task is to perform the trace calculation and determine the effect of D operations explicitly.

The trace calculation for graph I is simplified considerably if one notes that F_{I} can be written as

$$F_{\mathrm{I}}=(-D_9D_{10}+m_e^2)F_{\mathrm{I}}'+\text{remainder}, \quad (4.1)$$

with

$$F_{\mathrm{I}}'=\mathrm{Tr}[\gamma_\kappa(D_6+m_e)\gamma_\rho(D_7+m_e)\gamma_\sigma\gamma_\mu(D_8-m_e)\gamma_\nu], \quad (4.2)$$

where the remainder consists of terms which are either symmetric in μ and ν or proportional to D_8-D_9 and D_9-D_{10} and thus give vanishing contribution to Δa_{I}. Furthermore, we have[29]

$$(-D_9D_{10}+m_e^2)\int\frac{dz}{U^2V^4}=\frac{1}{3}\int\frac{dz'}{(U^2V^3)_{x_{10}=0}}, \quad (4.3)$$

where dz' has the same form as dz defined by (3.13) except that $dz(\gamma)$ is replaced by

$$dz'(\gamma)=z_\gamma dz_\gamma dx_8dx_9\delta(1-x_8-x_9). \quad (4.4)$$

Using (4.1) and (4.3), we can therefore simplify (3.14) to

$$\Delta a_{\mathrm{I}}=\frac{1}{768m_\mu}\left(\frac{\alpha}{\pi}\right)^3F_0F_{\mathrm{I}}'\int\frac{dz'}{(U^2V^3)_{x_{10}=0}}. \quad (4.5)$$

In the same fashion, Δa_{II} and Δa_{III} can be expressed in the form (4.5) if we replace F_{I}' by

$$F_{\mathrm{II}}'=\mathrm{Tr}[\gamma_\kappa\gamma_\mu(D_6-m_e)\gamma_\nu\gamma_\rho(D_7+m_e)\gamma_\sigma(D_8+m_e)] \quad (4.6)$$

and

$$F_{\mathrm{III}}'=-\mathrm{Tr}[\gamma_\kappa(D_6+m_e)\gamma_\rho\gamma_\nu(D_7-m_e)\times\gamma_\mu\gamma_\sigma(D_8+m_e)], \quad (4.7)$$

and interpret dz' somewhat differently. In the case of Δa_{IV}, we do not obtain too much simplification. But we write F_{IV} as

$$\begin{aligned}F_{\mathrm{IV}}=F_{\mathrm{IV}}'=&\,\mathrm{Tr}[\gamma_\kappa\gamma_\mu\gamma_\rho D_7\gamma_\sigma\gamma_\nu]\\&\times(-D_6D_{10}+m_e^2)(-D_8D_9+m_e^2)\\&+2D_{8\nu}\,\mathrm{Tr}[\gamma_\kappa\gamma_\mu\gamma_\rho(D_7+m_e)\gamma_\sigma(D_9+m_e)]\\&\times(-D_6D_{10}+m_e^2)+2D_{6\mu}\,\mathrm{Tr}[\gamma_\kappa(D_{10}+m_e)\\&\times\gamma_\rho(D_7+m_e)\gamma_\sigma\gamma_\nu](-D_8D_9+m_e^2)+4D_{6\mu}D_{8\nu}\\&\times\mathrm{Tr}[\gamma_\kappa(D_{10}+m_e)\gamma_\rho(D_7+m_e)\gamma_\sigma(D_9+m_e)]\\&+(\text{vanishing terms}), \quad (4.8)\end{aligned}$$

and apply formulas similar to (4.3).

It is also convenient to write F_0 coming from the muon lines as

$$\begin{aligned}F_0=&\,2m_\mu\,\mathrm{Tr}[(p^\kappa\gamma^\rho D_3\gamma^\sigma+p^\sigma\gamma^\kappa D_1\gamma^\rho)\{p(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\\&+(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)p\}]+m_\mu\,\mathrm{Tr}[\gamma^\kappa D_1\gamma^\rho D_3\gamma^\sigma\\&\times\{p(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)+(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)p\}]. \quad (4.9)\end{aligned}$$

This is obtained using the identities resulting from the mass-shell condition $p^2=m_\mu^2$:

$$\begin{aligned}(p+m_\mu)\gamma^\kappa(D_4+m_\mu)&=(p+m_\mu)(\gamma^\kappa D_1+2p^\kappa),\\(D_5+m_\mu)\gamma^\sigma(p+m_\mu)&=(D_3\gamma^\sigma+2p^\sigma)(p+m_\mu),\end{aligned} \quad (4.10)$$

where we have made use of the equations

$$D_1=D_4-p,\quad D_3=D_5-p, \quad (4.11)$$

which follow from (A3).

In order to carry out the D operation explicitly, it is convenient to introduce the functions

$$Q_i^\mu=-\frac{1}{2x_iz_\alpha}\frac{\partial V}{\partial q_{i\mu}}, \quad (4.12)$$

$$g_{\mu\nu}B_{ij}=\frac{U}{2x_iz_\alpha x_jz_\beta}\frac{\partial^2V}{\partial q_i^\mu\partial q_j^\nu}, \quad (4.13)$$

where the lines i and j belong to the chains α and β, respectively. Then it is easy to see that the result of applying D operators on $1/V^n$, n being a sufficiently large positive integer, can be expressed in terms of Q's and B's as follows:

$$\begin{aligned}D_{i\mu}\frac{1}{V^n}&=\frac{Q_{i\mu}}{V^n},\\D_{i\mu}D_{j\nu}\frac{1}{V^n}&=\frac{Q_{i\mu}Q_{j\nu}}{V^n}-\frac{1}{2(n-1)}\frac{g_{\mu\nu}B_{ij}}{UV^{n-1}},\\D_{i\mu}D_{j\nu}D_{k\lambda}\frac{1}{V^n}&=\frac{Q_{i\mu}Q_{j\nu}Q_{k\lambda}}{V^n}\\&-\frac{1}{2(n-1)}\frac{Q_{i\mu}g_{\nu\lambda}B_{jk}+Q_{j\nu}g_{\mu\lambda}B_{ik}+Q_{k\lambda}g_{\mu\nu}B_{ij}}{UV^{n-1}},\\&\cdots.\end{aligned} \quad (4.14)$$

[29] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) Suppl. 18, 1 (1961).

As is shown in Appendix A, the quantity Q_i satisfies the "Kirchhoff's laws."[2] Thus it can be regarded as the "current" running through the ith line for given "external currents" p,p' and given "resistances" (i.e., given values of Feynman parameters) of internal lines.

In our problem, in which we put $p'=p$ eventually, all internal currents $Q_{i\mu}$ become proportional to p_μ and thus the proportionality coefficient A_i defined by

$$Q_{i\mu}=(A_i/U)p_\mu \qquad (4.15)$$

itself may be regarded as a current satisfying the Kirchhoff's laws. The functions A_i and B_{ij} are homogeneous polynomials of $z_\alpha, z_\beta, \ldots, z_\nu$. Their explicit form and properties are discussed in Appendix A.

We can now carry out the trace calculations and D operations and write down the integrals for $\Delta a_{\rm I}, \ldots, \Delta a_{\rm IV}$ explicitly in terms of A_i's and B_{ij}'s. We shall write them as follows:

$$\Delta a_{\rm I'}=(\alpha/\pi)^3[M_{\rm I'a}+M_{\rm I'b}+M_{\rm I'c}+M_{\rm I'd}],\quad {\rm I'=I,II,III,IV} \qquad (4.16)$$

where $M_{\rm I'a}$ and $M_{\rm I'c}$ are obtained from the first term of (4.9) and $M_{\rm I'b}$ and $M_{\rm I'd}$ from the second term of (4.9). Also $M_{\rm I'a}$ and $M_{\rm I'b}$ are terms which arise from the m_e-independent part of the electron trace $F_{\rm I'}'$, and $M_{\rm I'c}$ and $M_{\rm I'd}$ represent the remainders.

In these integrals it is trivial to carry out the x_9 and x_{10} integrations. After this is done, differences among $V(x,z)$ of different graphs disappear completely, as was mentioned in Sec. 3. We shall now introduce the new z variables as follows:

$$\begin{aligned} &z_1=z_\alpha x_1,\quad z_2=z_\nu,\quad z_3=z_\beta x_3,\quad z_4=z_\alpha x_4,\\ &z_5=z_\beta x_5,\quad z_6=z_\mu,\quad z_7=z_\lambda,\quad z_8=z_\gamma. \end{aligned} \qquad (4.17)$$

In terms of these variables $U(z)$ can be written as

$$\begin{aligned} U(z)=&(z_1+z_4)(z_3+z_5)(z_6+z_7+z_8)\\ &+(z_1+z_4)[z_2(z_6+z_7+z_8)+z_7(z_6+z_8)]\\ &+(z_3+z_5)[z_2(z_6+z_7+z_8)+z_6(z_7+z_8)]\\ &+z_8(z_2z_6+z_2z_7+z_6z_7). \end{aligned} \qquad (4.18)$$

We shall also introduce the notation

$$dz_0\equiv\delta(1-\sum_{i=1}^{8}z_i)\prod_{i=1}^{8}dz_i \qquad (4.19)$$

and

$$W=m_\mu^{-2}UV, \qquad (4.20)$$

where V is given by (3.11) and (3.12).

Since we have carried out all D operations, we can now put $p'=p$ and $\Delta=0$ in the denominator V. Then, after one puts the photon mass equal to zero, the function W takes the very simple form

$$W=az_4^2+bz_5^2+c(z_4+z_5)^2+\rho(z_6+z_7+z_8)U, \qquad (4.21)$$

where

$$\rho=(m_e/m_\mu)^2 \qquad (4.22)$$

and

$$\begin{aligned} a&=B_{47}-B_{46}=(z_3+z_5)(z_6+z_7+z_8)+z_7z_8,\\ b&=B_{56}-B_{57}=(z_1+z_4)(z_6+z_7+z_8)+z_6z_8,\\ c&=B_{48}-B_{47}=B_{58}-B_{56}=z_2(z_6+z_7+z_8)+z_6z_7, \end{aligned} \qquad (4.23)$$

B_{ij} being given in Appendix A.

We are now ready to write down our integrals:

$$\begin{aligned} M_{\rm Ia}=&-4\int\frac{dz_0}{U^3W^3}z_8A_1A_6A_7A_8+\int\frac{dz_0}{U^3W^2}z_8[3A_7A_8B_{46}+A_6(A_8B_{47}+A_7B_{48})+2A_1(A_7B_{68}+A_8B_{67})]\\ &-\int\frac{dz_0}{U^3W}z_8(2B_{47}B_{68}+B_{48}B_{67}), \end{aligned} \qquad (4.24)$$

$$\begin{aligned} M_{\rm Ib}=&-3\int\frac{dz_0}{U^4W^3}z_8A_1A_3A_6A_7A_8+\frac{1}{2}\int\frac{dz_0}{U^4W^2}z_8[A_6A_7A_8B_{45}+2A_1A_6(A_7B_{58}+8A_8B_{57})+A_1A_3(A_8B_{67}+4A_7B_{68})]\\ &-\frac{1}{2}\int\frac{dz_0}{U^4W}z_8[2A_1(B_{56}B_{78}+5B_{57}B_{68})+3A_8(B_{45}B_{67}+5B_{46}B_{57})-2A_6(B_{47}B_{58}-4B_{48}B_{57})], \end{aligned} \qquad (4.25)$$

$$M_{\rm Ic}=4\rho\int\frac{dz_0}{UW^3}z_8A_1(A_1+A_7)-\rho\int\frac{dz_0}{UW^2}z_8(B_{48}+3B_{47}-3B_{46}), \qquad (4.26)$$

$$M_{\rm Id}=\rho\int\frac{dz_0}{U^2W^3}z_8A_1A_3(A_6+A_7+A_8)+\frac{\rho}{2}\int\frac{dz_0}{U^2W^2}z_8[(2A_7-3A_8)B_{45}-2A_1(5B_{57}-B_{56})], \qquad (4.27)$$

$$\begin{aligned} M_{\rm IIa}=&2\int\frac{dz_0}{U^3W^3}z_6(A_7-A_6)A_6A_7A_8+\frac{1}{2}\int\frac{dz_0}{U^3W^2}z_6[A_7A_8(B_{56}-B_{46})+3A_6A_8(B_{57}-B_{47})+A_6A_7(B_{58}-3B_{48})\\ &-2A_1(A_8B_{67}+A_7B_{68}-A_6B_{78})+2A_3(A_8B_{67}+A_6B_{78})]\\ &-\frac{1}{2}\int\frac{dz_0}{U^3W}z_6[B_{56}B_{78}+2B_{58}B_{67}+B_{46}B_{78}-2B_{47}B_{68}-2B_{48}B_{67}], \end{aligned} \qquad (4.28)$$

$$M_{\mathrm{IIb}}=\int \frac{dz_0}{U^4W^3}z_6A_1A_3A_6A_7A_8+\frac{1}{2}\int \frac{dz_0}{U^4W^2}z_6[A_6A_7A_8B_{45}-A_3A_8(2A_6B_{47}+A_7B_{46})$$

$$-A_1(A_7A_8B_{56}+2A_6A_8B_{57}+2A_6A_7B_{58})+A_1A_3(A_6B_{78}-2A_7B_{68})]$$

$$+\frac{1}{2}\int \frac{dz_0}{U^4W}z_6[3A_6(B_{47}B_{58}-B_{45}B_{78})+A_7(2B_{46}B_{58}+B_{48}B_{56})+A_8(B_{46}B_{57}+2B_{47}B_{56})$$

$$+A_1(5B_{57}B_{68}+B_{58}B_{67})+A_3(B_{47}B_{68}-B_{48}B_{67})], \quad (4.29)$$

$$M_{\mathrm{IIc}}=2\rho\int \frac{dz_0}{UW^3}z_6[A_1(A_6-A_7-A_8)+A_3(A_6-A_7+A_8)]$$

$$-\frac{\rho}{2}\int \frac{dz_0}{UW^2}z_6[B_{46}-B_{47}-B_{48}+B_{56}-3B_{57}+3B_{58}], \quad (4.30)$$

$$M_{\mathrm{IId}}=\rho\int \frac{dz_0}{U^2W^3}z_6A_1A_3(A_6-3A_7+A_8)$$

$$-\frac{\rho}{2}\int \frac{dz_0}{U^2W^2}z_6[(3A_6-A_7-A_8)B_{45}-A_1(5B_{57}-B_{58})-3A_3(B_{47}-B_{48})], \quad (4.31)$$

$$\Delta a_{\mathrm{III}}=\Delta a_{\mathrm{II}}, \quad (4.32)$$

$$M_{\mathrm{IVa}}=\int \frac{dz_0}{U^3W^2}(B_{67}-z_2z_8)(z_6A_1A_6+z_7A_3A_7)-3\int \frac{dz_0}{U^3W}(B_{67}-z_2z_8)(z_6B_{46}+z_7B_{57}), \quad (4.33)$$

$$M_{\mathrm{IVb}}=-\int \frac{dz_0}{U^4W^2}[z_8A_1A_8(B_{67}A_3+z_2z_8A_7)+(z_7A_7)^2\{(z_1+z_4)A_1+z_6A_6\}]$$

$$+3\int \frac{dz_0}{U^4W}[z_8B_{48}(A_8B_{67}-A_7B_{68})-z_6z_7B_{67}\{(z_1+z_4)A_1-z_6A_6\}], \quad (4.34)$$

$$M_{\mathrm{IVc}}=-2\rho\int \frac{dz_0}{UW^2}z_6(B_{46}+z_7z_8), \quad (4.35)$$

$$M_{\mathrm{IVd}}=\rho\int \frac{dz_0}{UW^2}z_4z_7(z_6-z_8). \quad (4.36)$$

We have obtained all terms except M_{IVb} by hand calculation. The complete integrand was obtained with the help of REDUCE.[5] Also some of the formulas given in Appendix A have been used to simplify the integrands. For the benefit of those readers who wish to check our calculations, we shall give in Appendix B the integrals $M_{\mathrm{IVa}}, \ldots, M_{\mathrm{IVd}}$ in unsimplified form.

5. ALTERNATIVE METHOD

As an alternative and check of the calculation presented in Secs. 3 and 4, we have also performed the reduction of the formula (2.2) to the parametric form [formulas (4.24)–(4.36)] using the standard techniques outlined in Sec. 18.4 of Ref. 2. The required extension of these techniques to the graphs of Fig. 1 is discussed here.

After the traces and index contractions are performed to project out Δa as in (2.9), and after some simplification with respect to the lines 9 and 10, our integrals can be reduced to the basic form

$$I=\int d^4l_1d^4l_2d^4l_3\, F(p_1,\ldots,p_8)/\prod_{j=1}^{8}(p_j^2-m_j^2), \quad (5.1)$$

where F is a polynomial in $p_1, \ldots, p_8$ and the denominator may be multiplied by another factor of $p_j^2-m_j^2$, $j=6$, 7, or 8, depending on which graph of Fig. 2 we are considering. Our labeling of loop momenta l_1, l_2, l_3 is shown in Fig. 3. In accordance with the prescription of Ref. 2, we shall write p_j as

$$p_j=k_j+l_j=k_j+\sum_{r=1}^{3}\eta_{jr}l_r, \quad (5.2)$$

where η_{jr} is the projection $(\pm1,0)$ of p_j along l_r. The k_j can be any choice of fixed momenta (independent of l_r) such that four-momentum conservation is satisfied at the six vertices of Fig. 3.

Next we introduce Feynman parameters $z_1, \ldots, z_8$, and rewrite (5.1) as

$$I=7!\int\delta(1-\sum_{k=1}^{8} z_k)dz_1\cdots dz_8\int d^4l_1d^4l_2d^4l_3\, F(p)/\Big[\sum_{j=1}^{8} z_j(p_j^2-m_j^2)\Big]^8. \quad (5.3)$$

If we choose the k_j such that

$$\sum_{j=1}^{8} z_jk_j\eta_{jr}=0,\quad r=1,2,3 \quad (5.4)$$

then the denominator in (5.3) has no $k\cdot l$ cross terms:

$$\sum_{j=1}^{8} z_j(p_j^2-m_j^2)=-D+\sum_{r,r'=1}^{3} U_{rr'}l_r\cdot l_{r'}, \quad (5.5)$$

where

$$D=\sum_{j=1}^{8} z_j(m_j^2-k_j^2) \quad (5.6)$$

and

$$U_{rr'}=\sum_{j=1}^{8} z_j\eta_{jr}\eta_{jr'}. \quad (5.7)$$

The fixed momenta $k_1, \ldots, k_8$ are subject to $(6-1)$ linearly independent equations (momentum conservation at each vertex, or Kirchhoff's first law) and three equations (5.4) (Kirchhoff's second law). Thus these momenta are completely and uniquely determined as functions of external momenta and Feynman parameters. Since Q_j's defined by (4.12) also satisfy the same set of equations as is shown in Appendix A, and since the solution is unique, k_j must be identical with Q_j. Note that, although q_j defined by (3.1) and k_j defined by (5.2) look quite similar, they are in fact entirely different. The former does not satisfy Kirchhoff's second law, while the latter does. The former is a constant vector independent of Feynman parameter z, while the latter is a function of z.

Although k_j is identical with Q_j and is thus given explicitly in Appendix A, it will be instructive to see how they may be determined directly by the Kirchhoff's laws. Let us first write down the second law (5.4) explicitly:

$$\begin{aligned} &z_1k_1+z_2k_2+z_4k_4-z_6k_6=0,\\ &z_2k_2-z_3k_3-z_5k_5+z_7k_7=0,\\ &z_6k_6+z_7k_7+z_8k_8=0. \end{aligned} \quad (5.8)$$

Making use of the first law (four-momentum conservation),

$$\begin{aligned} &k_4=k_1+p, && k_5=k_3+p,\\ &k_2=k_4-k_5=k_1-k_3, &&\\ &k_6=k_8-k_1, && k_7=k_8-k_3, \end{aligned} \quad (5.9)$$

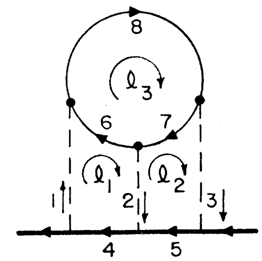

FIG. 3. Diagram indicating the labeling of loop momenta l_1, l_2, and l_3.

we may rewrite (5.8) as

$$\begin{aligned} (z_1+z_2+z_4+z_6)k_1-z_2k_3-z_6k_8&=-z_4p,\\ -z_2k_1+(z_2+z_3+z_5+z_7)k_3-z_7k_8&=-z_5p,\\ -z_6k_1-z_7k_3+(z_6+z_7+z_8)k_8&=0. \end{aligned} \quad (5.10)$$

The solution k_1 is given by

$$k_1=(p/U)\begin{vmatrix} -z_4 & -z_2 & -z_6\\ -z_5 & z_2+z_3+z_5+z_7 & -z_7\\ 0 & -z_7 & z_6+z_7+z_8 \end{vmatrix} =(p/U)A_1=Q_1, \quad (5.11)$$

where A_1 is given by (A7) and U by

$$U(z)=\det(U_{rr'}), \quad (5.12)$$

or, more explicitly, by (4.18). Other k_j's can be determined in the same fashion, confirming

$$k_j=(A_j/U)p=Q_j,\quad j=1,2,\ldots,8. \quad (5.13)$$

Substituting these k_j's in D of (5.6), we also obtain

$$D=V=m_\mu^2W/U, \quad (5.14)$$

where V is given by (3.11) and W is defined by (4.21).

Now, when the substitution (5.2) is made in the integral (5.1), and averaging over the direction of p is made, the integrand $F(p_1,\ldots,p_8)$ becomes a polynomial in l_r. We are now ready to carry out the integration over l_1, l_2, and l_3. The basic integration over loop momenta is

$$7!\int d^4l_1d^4l_2d^4l_3/\Big[-D+\sum_{r,r'=1}^{3} U_{rr'}l_r\cdot l_{r'}\Big]^8 =i^3\pi^6\frac{1}{U^2D^2}. \quad (5.15)$$

Integrands of (5.1) containing extra denominator factors $p_k^2-m_k^2$ can be integrated using parametric differentiation of (5.15) with respect to m_k^2. Similarly, integrands of (5.1) which contain numerator factors $l_j\cdot l_k$ can be integrated[30] using parametric differentiation

[30] Note that explicit dependence on the direction of the external four-vectors can always be removed by tensor methods as in (2.8) and (2.9).

with respect to the $U_{rr'}$. For example,

$$8!\int (l_j\cdot l_k)d^4l_1d^4l_2d^4l_3/[-D+\sum_{r,r'} U_{rr'}l_r\cdot l_{r'}]^9$$

$$=-\sum_{s,s'=1}^{3}\eta_{js}\eta_{ks'}\frac{\partial}{\partial\bar{U}_{ss'}}7!\int d^4l_1d^4l_2d^4l_3/[-D+\sum_{r,r'} U_{rr'}l_r\cdot l_{r'}]^8$$

$$=-\frac{i^3\pi^6}{D^2}\sum_{s,s'}\eta_{js}\eta_{ks'}\frac{\partial}{\partial\bar{U}_{ss'}}\left(\frac{1}{U^2}\right)=2i^3\pi^6\frac{B_{jk}}{U^3D^2}. \quad (5.16)$$

We have defined

$$\bar{U}_{ss'}=U_{ss'} \quad \text{for} \quad s=s'$$
$$=2U_{ss'} \quad \text{for} \quad s\neq s' \quad (5.17)$$

and

$$B_{jk}=\sum_{s,s'}\eta_{js}\eta_{ks'}B_{ss'}=\sum_{s,s'}\eta_{js}\eta_{ks'}\frac{\partial U}{\partial\bar{U}_{ss'}}. \quad (5.18)$$

Notice that $B_{ss'}$ is the signed cofactor of $U_{ss'}$ in U. Consequently $B_{ss'}/U$ is the inverse of the matrix $U_{ss'}$. Again the calculation of B_{jk} given here agrees with that of Appendix A in terms of $z_1, \ldots, z_8$.

In addition to quadratic terms, numerators with up to six powers of loop momenta l_r appear in the computation of graph IV. An important identity for reducing the required higher-order derivatives is

$$U\frac{\partial^2 U}{\partial\bar{U}_{ab}\partial\bar{U}_{cd}}=B_{ab}B_{cd}-\tfrac{1}{2}B_{ac}B_{bd}-\tfrac{1}{2}B_{ad}B_{bc}, \quad (5.19)$$

which holds for symmetric matrices $U_{rr'}=U_{r'r}$. To prove this, let us first assume that $U_{rr'}$ is not symmetric and all its elements are independent, and show that

$$U\frac{\partial^2 U}{\partial U_{ab}\partial U_{cd}}=B_{ab}B_{cd}-B_{ad}B_{cb}, \quad B_{ab}\equiv\frac{\partial U}{\partial U_{ab}} \quad (5.20)$$

holds for such a U. We start from the identity

$$\sum_{j=1}^{3}U_{ij}B_{cj}=\delta_{ic}U. \quad (5.21)$$

Differentiating both sides with respect to U_{ab}, we obtain

$$\delta_{ai}B_{cb}+\sum_j U_{ij}\frac{\partial^2 U}{\partial U_{ab}\partial U_{cj}}=\delta_{ic}B_{ab}. \quad (5.22)$$

Multiplication by B_{id} on both sides and summation over i then yields (5.20). The proof of (5.19) for the symmetric case is the same except that

$$\frac{\partial U_{ij}}{\partial\bar{U}_{ab}}=\tfrac{1}{2}\delta_{ia}\delta_{jb}+\tfrac{1}{2}\delta_{ib}\delta_{ja}. \quad (5.23)$$

We note that Eqs. (5.19) and (5.20) hold for matrices of any finite dimension $n\geq 2$. We also note that (5.19) is equivalent to (A26).

As a consequence of (5.19), we readily find

$$9!\int\frac{d^4l_1d^4l_2d^4l_3\,(l_a\cdot l_b)(l_c\cdot l_d)}{[-D+\sum U_{rr'}l_r\cdot l_{r'}]^{10}}=\frac{i^3\pi^6}{D^2}\sum_{ss'}\sum_{rr'}\eta_{as}\eta_{bs'}\eta_{cr}\eta_{dr'}\frac{\partial^2}{\partial\bar{U}_{ss'}\partial\bar{U}_{rr'}}\left(\frac{1}{U^2}\right)=\frac{i^3\pi^6}{D^2U^4}(4B_{ab}B_{cd}+B_{ad}B_{bc}+B_{ac}B_{bd}) \quad (5.24)$$

and

$$9!\int d^4l_1d^4l_2d^4l_3\,(l_a\cdot l_b)(l_c\cdot l_d)(l_e\cdot l_f)/[-D+\sum U_{rr'}l_r\cdot l_{r'}]^{10}$$

$$=\frac{i^3\pi^6}{D}\sum_{ss'}\sum_{rr'}\sum_{tt'}\eta_{as}\eta_{bs'}\eta_{cr}\eta_{dr'}\eta_{et}\eta_{ft'}\frac{\partial^3}{\partial\bar{U}_{ss'}\partial\bar{U}_{rr'}\partial\bar{U}_{tt'}}\left(\frac{1}{U^2}\right)$$

$$=\frac{-i^3\pi^6}{U^5D}(8B_{ab}B_{cd}B_{ef}+2B_{ab}B_{cf}B_{ed}+2B_{ab}B_{ce}B_{df}+2B_{ac}B_{bd}B_{ef}+2B_{ad}B_{bc}B_{ef}+2B_{ae}B_{bf}B_{cd}+2B_{af}B_{be}B_{cd}$$
$$+\tfrac{1}{2}B_{ad}B_{be}B_{cf}+\tfrac{1}{2}B_{ae}B_{bd}B_{cf}+\tfrac{1}{2}B_{ac}B_{bf}B_{de}+\tfrac{1}{2}B_{af}B_{bc}B_{de}+\tfrac{1}{2}B_{ad}B_{bf}B_{ce}+\tfrac{1}{2}B_{af}B_{bd}B_{ce}$$
$$+\tfrac{1}{2}B_{ac}B_{be}B_{df}+\tfrac{1}{2}B_{ae}B_{bc}B_{df}). \quad (5.25)$$

In our calculation of Δa for the graphs of Fig. 2, REDUCE, after it performed the traces and index contractions, made substitutions including (5.24) and (5.25) to complete the reduction to parametric form. The result agrees exactly with Eqs. (4.24)–(4.36). The final form given for M_{IV} is obtained after algebraic reduction using the Kirchhoff's laws given in Appendix A.

6. LOGARITHMIC TERMS

We shall now study the behavior of Δa in the limit where $\rho=(m_e/m_\mu)^2$ tends to zero. For this purpose we note that the denominator function W is positive everywhere within the domain of integration as is seen from (4.21) and (4.23). Therefore any singularity which the integrals (4.24)–(4.36) may have at $\rho=0$ can come only from the domain of integration in the neighborhood of the boundary defined by

$$z_4=0,\quad z_5=0. \tag{6.1}$$

According to the general analysis of mass singularity,[4] this singularity at $z_4=z_5=0$ is associated with the vanishing of photon and electron masses. The formula (4.21) also shows that W vanishes at

$$z_6=z_7=z_8=0. \tag{6.2}$$

However, this takes place because U vanishes there and not because V vanishes. Thus it is associated with the singularity at large virtual momentum of the electron loop[29] and does not lead to any singularity at $\rho=0$.

We shall therefore examine the behavior of our integrals in the neighborhood of $z_4=z_5=0$. It is then easy to see by counting the power of z_4 and z_5 in the numerators and denominators that the integrals $M_{\mathrm{I'b}}$, $M_{\mathrm{I'c}}$, $M_{\mathrm{I'd}}$, $\mathrm{I'=I, II, III, IV}$, are all convergent as $\rho\to 0$, and only the integrals $M_{\mathrm{I'a}}$, $\mathrm{I'=I, II, III, IV}$, may have a logarithmic singularity in ρ. In order to determine the coefficients of $\ln\rho$ in these integrals, we may carry out the integration with respect to z_4 and z_5 over a small domain in the neighborhood of $z_4=z_5=0$. For this purpose let us consider the integral

$$\int_{0\leq z_4+z_5\leq K}\frac{dz_4dz_5}{W}, \tag{6.3}$$

where W is given by (4.21) and K is a small fixed positive number satisfying $\rho\ll K\ll 1$ such that the terms of order $(z_4+z_5)^3$ in W can be ignored. The integration in (6.3) can be easily performed, giving

$$\int_{0\leq z_4+z_5\leq K}\frac{dz_4dz_5}{W} =-\tfrac{1}{2}(\ln\rho)G+(\text{nonlogarithmic terms}), \tag{6.4}$$

where

$$G=G(a_0,b_0,c_0) =\frac{1}{\sqrt{\Delta_0}}\left[\tan^{-1}\left(\frac{a_0}{\sqrt{\Delta_0}}\right)+\tan^{-1}\left(\frac{b_0}{\sqrt{\Delta_0}}\right)\right], \tag{6.5}$$

$$\Delta_0=a_0b_0+a_0c_0+b_0c_0=(z_6+z_7+z_8)U_0, \tag{6.6}$$

and a_0, b_0, c_0 are a, b, c defined by (4.23) evaluated at $z_4=z_5=0$. Similarly, U_0 is U of (4.18) evaluated at $z_4=z_5=0$. Differentiating both sides of (6.4) with respect to a_0, b_0, and c_0, we can obtain further relations of the type (6.4). They are given in Appendix C.

We may now express the A_i's in terms of z_4, z_5 and the B_{ij}'s as given by (A16) in the integrals (4.24), (4.28), and (4.33), and carry out the z_4 and z_5 integrations with the help of formulas (6.4) and (C1)–(C5). After a straightforward but lengthy calculation, we find that

$$M_{\mathrm{Ia}}+M_{\mathrm{IIa}}+M_{\mathrm{IIIa}} =\tfrac{1}{2}\pi(\ln\rho)\int dz_0''\frac{z_8[z_1z_7-z_8(z_2+z_7)]}{U_0\Delta_0^{3/2}}+\cdots \tag{6.7}$$

and

$$M_{\mathrm{IVa}}=\tfrac{5}{2}\pi(\ln\rho)\int dz_0''\frac{z_1z_3z_6B_{46}}{U_0^3\Delta_0^{1/2}}+\cdots, \tag{6.8}$$

where

$$dz_0''\equiv\delta(1-z_1-z_2-z_3-z_6-z_7-z_8)\times dz_1dz_2dz_3dz_6dz_7dz_8. \tag{6.9}$$

In deriving (6.7) and (6.8), we have made extensive use of the "Kirchhoff's laws" discussed in Appendix A, the identity

$$\tan^{-1}\left(\frac{a_0}{\sqrt{\Delta_0}}\right)+\tan^{-1}\left(\frac{b_0}{\sqrt{\Delta_0}}\right)+\tan^{-1}\left(\frac{c_0}{\sqrt{\Delta_0}}\right)=\tfrac{1}{2}\pi, \tag{6.10}$$

as well as the symmetry of the integrals in (6.7) and (6.8) under the transformations

(i) $z_1\leftrightarrow z_3$, $z_6\leftrightarrow z_7$, z_2, z_8 unchanged,

(ii) $z_2\leftrightarrow z_3$, $z_6\leftrightarrow z_8$, z_1, z_7 unchanged,

(iii) $z_1\leftrightarrow z_2$, $z_7\leftrightarrow z_8$, z_3, z_6 unchanged.

Clearly the leading term of (6.8) is positive definite since B_{46} is negative everywhere. On the other hand, the leading term of (6.7) has both positive and negative contributions and its sign cannot be determined by inspection. However, in view of the fact that the pairs (z_1,z_7) and (z_2,z_8) are more or less equivalent according to the symmetry (iii), it is plausible that (6.7) is also positive. This is in fact confirmed in Sec. 7 by numerical integration.

7. NUMERICAL INTEGRATION

It is obviously beyond our capability to evaluate the sevenfold integrals (4.24)–(4.36) analytically. We have therefore resorted to the method of numerical integration. This was greatly facilitated by the availability of a multiple integration program written by G. Sheppey,[6] which could be readily modified to suit our need.

We are primarily interested in the values of Δa at $\rho=(1/207)^2$ (muon moment) and $\rho=1$ (electron moment). However, in view of the results of Sec. 6, we are also interested in examining numerically the functional dependence of Δa on ρ.

Sheppey's program is essentially a simple Riemann summation combined with a sampling technique which produces an efficient grid by successive approximations. At the start of the iteration process the domain of integration is divided up into a number of hypercubes by the user's specification of the number and size of the integration intervals along each axis. (Initial specifica-

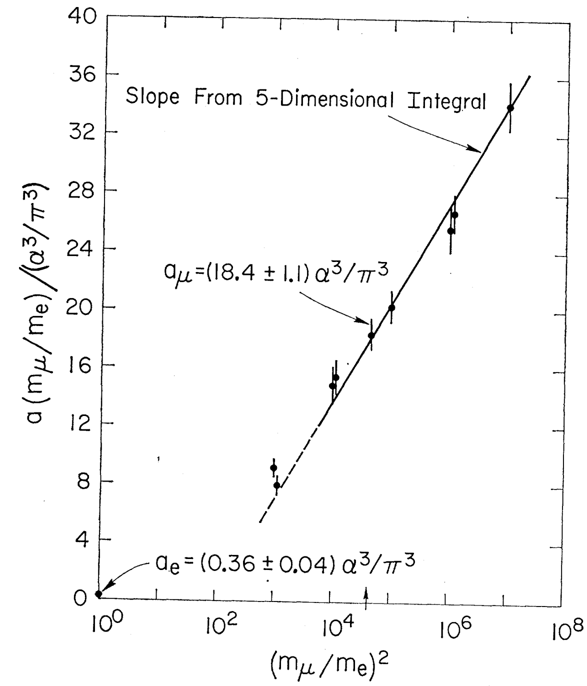

FIG. 4. Anomalous magnetic-moment contribution Δa as a function of $X=\log_{10}(\rho^{-1})=\log_{10}(m_\mu/m_e)^2$.

tion can be somewhat arbitrary because on successive iterations the program will automatically readjust all interval sizes based on the relative errors it associates with each.) Two points $x_n{}^1$ and $x_n{}^2$ are selected at random within each hypercube n as points for the evaluation of the integrands—rather than choosing the central value. The arithmetic average of the two values is used for the Riemann sum estimate

$$I=\sum_n \tfrac{1}{2}[f(x_n{}^1)+f(x_n{}^2)]\Delta V_n\,, \qquad (7.1)$$

where ΔV_n is the nth hypercube volume. A variance for each cube is defined as the square of one-half the difference between the random estimates of the integrand value. The associated error for the Riemann sum is the square root of the sum of all variances multiplied by 1.82 to give a 91% confidence level:

$$\sigma^2=(1.82)^2\sum_n \left|\frac{f(x_n{}^1)-f(x_n{}^2)}{2}\right|^2 (\Delta V_n)^2\,. \qquad (7.2)$$

Upon completion of such an iteration, those cubes which are found to contain the greatest relative variance are reduced in size along each edge in proportion to that dimension's contribution to the error, and the process then cycles through another iteration based on this new set of intervals.

The successive iterated values of the integral I_i and error σ_i, $i=1, 2, \ldots, N$, are accumulated under the assumption that they are normally distributed (verified independently by histograms). Weights (W_i) for each iteration are calculated as

$$W_i=(I_i/\sigma_i)^2\,, \qquad (7.3)$$

which gives the most probable (weighted mean) value of the integral:

$$\bar{I}=\sum_{i=1}^{N} I_i W_i \Big/ \sum_{i=1}^{N} W_i\,, \qquad (7.4)$$

with a standard deviation

$$\bar{\sigma}=\bar{I}\Big/\Big(\sum_{i=1}^{N} W_i\Big)^{1/2}. \qquad (7.5)$$

The result of each iteration I_i was found in practice to overlap with $\bar{I}$ within the error σ_i more than 90% of the time.

The integration package was tested on many multidimensional integrals, some of which were five-dimensional parametric forms similar to the function analyzed in this paper, but with known analytic solution. Full confidence in the utility of the program was obtained before it was applied to the problem at hand.

In applying this program to our problem, we instructed REDUCE to punch out the result of trace calculation and D operations described in Sec. 4 in a FORTRAN-compatible form so that it can be directly fed into the integration program. The integrand takes the form of a ratio of polynomials times a θ function and is well behaved everywhere in the domain of integration for $\rho>0$.

In Fig. 4 we show the anomalous magnetic-moment contribution Δa arising from the graphs of Fig. 1 divided by $(\alpha/\pi)^3$ as a function of

$$X=\log_{10}(\rho^{-1})=\log_{10}(m_\mu/m_e)^2\,.$$

The error bars indicate a better than 91% confidence interval. Typical points required 10 min of computation time on the SLAC IBM 360/91, after an initial 30 min had been used to obtain a distribution of the 50 000 hypercubes which would be approximately valid for all X.

The result (1.7) for the special case $\rho=(1/207)^2$ represents the result of more extensive effort and was obtained after about 30 iterations (about one per minute) with up to 90 000 hypercubes. Results consistent with (1.7) were also obtained with grids constrained to have a minimum of five points per axis. But results with smallest σ_i were those on which no such constraints were imposed. A typical result for an iteration is [apart from the factor $(\alpha/\pi)^3$]

$$I_i=17.7\,, \quad \sigma_i=2.1\,, \qquad (7.6)$$

with 2, 4, 2, 22, 25, 2, and 6 intervals along the seven axes, respectively.[31] We have found that largest num-

[31] The variable z_8 was eliminated beforehand using the δ function.

ber of points are concentrated along the z_4 and z_5 axes, which is not surprising in view of consideration of Sec. 6.

A convergent value for $\rho=1$ proved much more difficult to obtain, partly because the integrand is not peaked in any particular regions of the variable space and partly because the θ-function constraint on the variables occurs where the integrand is not small. The latter problem could be avoided and eliminated by a change of variables

$$\begin{aligned} z_7 &= (1-z_1-z_2-\cdots-z_6)\alpha_7\,, \\ z_6 &= (1-z_1-z_2-\cdots-z_5)\alpha_6\,, \\ &\cdots \\ z_1 &= \alpha_1\,, \end{aligned} \tag{7.7}$$

which turns the integral into the form

$$\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \cdots \int_0^1 d\alpha_7\ f(z)(1-z_1)(1-z_1-z_2)\cdots \times(1-z_1-z_2-\cdots-z_6)\,. \tag{7.8}$$

It was also found convenient to switch z_2 and z_8. Using this form and 600 000 hypercubes, the integral gave consistent results with small error in a 1-h run (five iterations) on the IBM 360/91. The two best individual iterations gave

$$\begin{aligned} I_i &= 0.34\,, \quad \sigma_i=0.06\,, \\ I_i &= 0.37\,, \quad \sigma_i=0.09\,. \end{aligned} \tag{7.9}$$

The cumulative result is given by (1.11). All other runs including those without the change of variables (7.7) overlapped with this result. On the best run the grid chosen had 583 200 cubes consisting of 12, 45, 4, 6, 3, 3, and 5 intervals along each of the axes α_1–α_7.[32]

As was shown in Sec. 6, the analytic dependence of the photon-photon scattering contribution on $\rho=(m_e/m_\mu)^2$ for small ρ is of the form

$$\Delta a(\rho)=(\alpha/\pi)^3[C_1\ln\rho+C_2]\,, \quad \rho\ll 1\,. \tag{7.10}$$

The coefficient C_1 as given by (6.7) and (6.8) was numerically integrated over a five-dimensional space, giving

$$C_1=-3.19\pm0.04\,, \tag{7.11}$$

and the result (1.10). As a consistency check we have also integrated (6.7) and (6.8) separately. They gave approximately equal contributions and their sum agreed with the above result. The result (7.11) for C_1 is not inconsistent with a linear fit to the points of Fig. 4 for small ρ.

ACKNOWLEDGMENTS

We are greatly in debt to Dr. A. C. Hearn for providing his algebraic computation program REDUCE and his assistance in its use. One of us (TK) would like to thank Dr. K. C. Wali for the kind hospitality at Argonne National Laboratory, Dr. S. D. Drell for the kind hospitality at the Stanford Linear Accelerator Center, and Dr. R. R. Rau for the kind hospitality at Brookhaven National Laboratory, where part of this work was done. Two of us (SJB and AJD) wish to thank Dr. S. D. Drell for his suggestions and encouragement.

APPENDIX A: KIRCHHOFF'S LAWS FOR A_i AND B_{ij}

The quantity Q_i^μ defined by (4.12) satisfies the "Kirchhoff's laws," namely, the sum of "currents Q_i^μ" entering any vertex v is conserved:

$$\sum_v \pm Q_i^\mu = -\sum_v \pm(\text{external currents})\,, \tag{A1}$$

where + or − is chosen according as Q_i is incoming or outgoing, and the sum of "voltage drops" around any closed loop C is zero

$$\sum_C \eta_{iC} x_i z_\alpha Q_i^\mu = 0\,, \tag{A2}$$

where the Feynman parameter $x_i z_\alpha$ is regarded as the "resistance" of the line i of chain α, and η_{iC} is the projection (+1, −1) of q_i along C.

The first law (A1) follows from

$$\Big(\sum_v \pm D_i^\mu\Big)\int d^4r_1 d^4r_3 d^4r_6 / \prod\,[(r_j+q_j)^2-m_j^2] = \int [\sum_v \pm(r_i^\mu+q_i^\mu)] d^4r_1 d^4r_3 d^4r_6 / \prod[(r_j+q_j)^2-m_j^2]\,, \tag{A3}$$

where q_i are fixed momenta satisfying (3.2). The second law (A2) is a consequence of the fact that $V(x,z)$ is invariant under the simultaneous transformation of all $q_{i(C)}$

$$q_{i(C)} \to q_{i(C)} + \eta_{iC} q^C\,, \tag{A4}$$

where $q_{i(C)}$ represents q_i belonging to the closed loop C and q^C is an arbitrary constant 4-vector common to all lines of the loop C.[4] This invariance leads us to

$$\Big(\sum_C \eta_{iC}\frac{\partial V}{\partial q_{i(C)}}\Big) q^C = 0\,, \tag{A5}$$

which is equivalent to (A2) as is easily seen from the definition (4.12).

In our problem, in which the only external current is p^μ (since $\Delta=0$), all internal currents are proportional to p^μ. Thus the proportionality coefficients A_i defined by

$$Q_i^\mu = (A_i/U)p^\mu \tag{A6}$$

may themselves be regarded as currents satisfying the Kirchhoff's laws. In this appendix we shall write down

[32] The variable z_2 was eliminated using the δ function, and the variables $\alpha_1, \ldots, \alpha_7$ were defined by (7.7), in which z_2 was replaced by z_8.

the explicit forms of A_i and B_{ij} and show that these functions of z in fact satisfy the Kirchhoff's laws.

Let us first calculate A_i's for graph I from the definitions (4.12) and (A6). Although they are functions of $x_1z_\alpha, \ldots, x_{10}z_\gamma$, it is sufficient for our purpose to write them down for the case

$$x_9=x_{10}=0. \tag{A7}$$

Then, in terms of the new z variables defined by (4.17), they can be written down as follows:

$$\begin{aligned}
A_1&=-z_4[(z_3+z_5)(z_6+z_7+z_8)+z_7z_8]\\
&\quad-(z_4+z_5)[z_2(z_6+z_7+z_8)+z_6z_7],\\
A_2&=-z_4[(z_3+z_5)(z_6+z_7+z_8)+z_7z_8]\\
&\quad+z_5[(z_1+z_4)(z_6+z_7+z_8)+z_6z_8],\\
A_3&=-z_5[(z_1+z_4)(z_6+z_7+z_8)+z_6z_8]\\
&\quad-(z_4+z_5)[z_2(z_6+z_7+z_8)+z_6z_7],\\
A_6&=z_4[z_8(z_2+z_7)+(z_7+z_8)(z_3+z_5)]\\
&\quad+z_5[z_2z_8-z_7(z_1+z_4)],\\
A_7&=z_5[z_8(z_2+z_6)+(z_6+z_8)(z_1+z_4)]\\
&\quad+z_4[z_2z_8-z_6(z_3+z_5)],\\
A_8&=-z_4[z_6(z_2+z_3+z_5+z_7)+z_2z_7]\\
&\quad-z_5[z_7(z_1+z_2+z_4+z_6)+z_2z_6].
\end{aligned} \tag{A8}$$

We have not written down A_4, A_5, A_9, and A_{10} explicitly since they are easily obtained from others making use of the current conservation at external vertices (i.e., vertices to which at least one external line is attached):

$$\begin{aligned}
A_4-A_1&=U,\\
A_5-A_3&=U,\\
A_9=A_{10}&=A_8\;.
\end{aligned} \tag{A9}$$

It is now easy to check the current conservation at internal vertices:

$$\begin{aligned}
A_1+A_6-A_{10}&=0,\\
A_2+A_6-A_7&=0,\\
A_2-A_4+A_5&=0,\\
A_3+A_7-A_8&=0.
\end{aligned} \tag{A10}$$

The second Kirchhoff's law can be similarly checked:

$$\begin{aligned}
z_6A_6+z_7A_7+z_8A_8&=0,\\
z_1A_1+z_2A_2+z_4A_4-z_6A_6&=0,\\
z_2A_2-z_3A_3-z_5A_5+z_7A_7&=0.
\end{aligned} \tag{A11}$$

Next we shall examine B_{ij} defined by (4.13). Since B_{ij} is symmetric in i and j, there are 55 B_{ij}'s altogether. However, they are related to each other by various identities. For instance, it is obvious from the definition (4.13) that

$$\begin{aligned}
&B_{1i}=B_{4i} && \text{if } i\neq 1 \text{ or } 4,\\
&B_{3i}=B_{5i} && \text{if } i\neq 3 \text{ or } 5,\\
&B_{8i}=B_{9i}=B_{10i} && \text{if } i\neq 8,\ 9, \text{ or } 10,\\
&B_{89}=B_{8,10}=B_{9,10}.
\end{aligned} \tag{A12}$$

From (4.13), (3.11), and (3.12) we see that B_{ij} are second derivatives of V, which is quadratic in the q_j, with respect to the q_j. Since the derivatives are taken before the q_j are fixed, B_{ij} cannot depend on the external momenta or their routing, but only on the topological structure of the graph. In order to find further properties of B_{ij} let us note that Q_i^μ can be expressed as a linear combination of B_{ij}:

$$UQ_i^\mu=-\sum_{j=1}^{10} q_j^\mu x_j z_{\alpha_j} B_{ij}, \quad i=1, 2, \ldots, 10 \tag{A13}$$

where α_j is the chain to which the line j belongs. The q_j are arbitrary constant momenta subject only to the 4-momentum conservation law (3.2); we need not restrict ourselves to Q_i^μ proportional to p^μ. This equation follows from definitions (4.12), (4.13), and the fact that $V(x,z)$ is quadratic in q_j. If we choose

$$q_4=q_5=p, \quad \text{all other } q\text{'s}=0 \tag{A14}$$

in (A13), which is consistent with (3.2), we obtain

$$UQ_i^\mu=(-z_4B_{4i}-z_5B_{5i})p^\mu. \tag{A15}$$

Since $\Delta=0$ for this choice of q_j, we may use (A6) to get

$$A_i=-z_4B_{4i}-z_5B_{5i}. \tag{A16}$$

Substituting (A16) into (A9) or (A15) into (A1) for external vertices, and taking (A12) into account, we get equations involving diagonal B_{ij}:

$$\begin{aligned}
B_{14}-B_{44}&=U/z_4,\\
B_{35}-B_{55}&=U/z_5.
\end{aligned} \tag{A17}$$

Relaxing condition (A7) and choosing different q_i and external momenta consistent with (3.2), we get equations similar to (A15). Substituting these into (A1) for external vertices and using (A12), we get the general relation

$$B_{ii}-B_{ij}=U/z_{\alpha i}x_i, \tag{A18}$$

where lines i and j belong to the same chain α_i.

Similarly, substitution of (A16) into (A10) and of expressions like (A15) into (A1) for internal vertices yields

$$\begin{aligned}
B_{1i}+B_{6i}-B_{10i}&=0,\\
B_{2i}+B_{6i}-B_{7i}&=0,\\
B_{2i}-B_{4i}+B_{5i}&=0,\\
B_{3i}+B_{7i}-B_{8i}&=0,
\end{aligned} \tag{A19}$$

if none of the B_{ij} are diagonal. Otherwise we need a slight modification—for instance,

$$B_{22}+B_{26}-B_{27}=-U/z_2. \tag{A20}$$

These relations may be regarded as Kirchhoff's first law for B_{ij}.

In order to obtain the second law for B_{ij}, we substitute (A16) into (A11) and expressions like (A15) into (A2). Again using $x_9=x_{10}=0$, we find

$$\begin{aligned} z_6B_{6i}+z_7B_{7i}+z_8B_{8i}&=0\,,\\ z_1B_{1i}+z_2B_{2i}+z_4B_{4i}-z_6B_{6i}&=0\,,\\ z_2B_{2i}-z_3B_{3i}-z_5B_{5i}+z_7B_{7i}&=0\,. \end{aligned}\tag{A21}$$

Finally, we give some B_{ij} explicitly in terms of new z variables defined by (4.17) which corresponds to putting $x_9=x_{10}=0$:

$$\begin{aligned} B_{45}&=z_2(z_6+z_7+z_8)+z_6z_7\,,\\ B_{46}&=-z_8(z_2+z_3+z_5+z_7)-z_7(z_3+z_5)\,,\\ B_{47}&=z_6(z_3+z_5)-z_2z_8\,,\\ B_{48}&=z_6(z_2+z_3+z_5+z_7)+z_2z_7\,,\\ B_{56}&=z_7(z_1+z_4)-z_2z_8\,,\\ B_{57}&=-z_8(z_1+z_2+z_4+z_6)-z_6(z_1+z_4)\,,\\ B_{58}&=z_7(z_1+z_2+z_4+z_6)+z_2z_6\,,\\ B_{67}&=z_2(z_1+z_3+z_4+z_5+z_8)+(z_1+z_4)(z_3+z_5)\,,\\ B_{68}&=(z_1+z_4)(z_2+z_3+z_5+z_7)+z_2(z_3+z_5)\,,\\ B_{78}&=(z_3+z_5)(z_1+z_2+z_4+z_6)+z_2(z_1+z_4)\,. \end{aligned}\tag{A22}$$

Other formulas of great use can be derived from the observation that, aside from the term

$$\sum_\alpha z_\alpha\Big(\sum_{i\in\alpha} x_im_i^2\Big)\,,$$

the denominator function $V(x,z)$ can be regarded as the "power" burned up in the network.[2] This leads us to a set of equations[29]

$$-Q_i^2+m_i^2=\partial V/\partial z_i\,,\quad i=1,2,\ldots,8 \tag{A23}$$

where Q_i is defined by (4.12). With the help of (A6) and (4.20) this can be transformed into

$$A_i^2=W(\partial U/\partial z_i)-U(\partial W/\partial z_i)+\rho_iU^2\,, \tag{A24}$$

where

$$\begin{aligned} \rho_i&=0 \quad \text{for } i=1,2,3\\ &=1 \quad \text{for } i=4,5\\ &=\rho \quad \text{for } i=6,7,8\,. \end{aligned}\tag{A25}$$

Suppose we express W and A_i in (A24) in terms of B_{ij} and z_4 and z_5. Then, comparing the coefficients of z_4 and z_5 in the resulting expression and using the Kirchhoff's law (A19) repeatedly, we can write down an enormous number of formulas quadratic in B_{ij}'s. Some of them are shown below:

$$\begin{aligned} &B_{46}B_{57}-B_{47}B_{56}=z_8U\,,\\ &B_{48}B_{56}-B_{46}B_{58}=z_7U\,,\\ &B_{47}B_{58}-B_{48}B_{57}=z_6U\,,\\ &B_{45}B_{67}-B_{47}B_{56}=B_{48}B_{67}-B_{47}B_{68}\\ &\qquad=B_{58}B_{67}-B_{56}B_{78}=B_{67}B_{88}'-B_{68}B_{78}\\ &\qquad=B_{48}B_{78}-B_{47}B_{88}'=B_{58}B_{68}-B_{56}B_{88}'\\ &\qquad=B_{45}B_{88}'-B_{48}B_{58}=z_2U\,, \qquad\qquad (A26)\\ &B_{66}'B_{88}'-B_{68}B_{68}=B_{48}B_{68}-B_{46}B_{88}'\\ &\qquad=(z_2+z_3+z_5+z_7)U\,,\\ &B_{77}'B_{88}'-B_{78}B_{78}=B_{58}B_{78}-B_{57}B_{88}'\\ &\qquad=(z_1+z_2+z_4+z_6)U\,,\\ &B_{66}'B_{78}-B_{67}B_{68}=(z_3+z_5)U\,,\\ &B_{66}'B_{58}-B_{56}B_{68}=(z_2+z_7)U\,, \end{aligned}$$

where B_{ii}' is the polynomial part of B_{ii} which may also be written as $\partial U/\partial z_i$. We note that (A26) is equivalent to the result (5.19).

Formulas such as (A11), (A21), and (A26) are extremely useful in simplifying the numerators of Feynman integrals. Although the usefulness of Kirchhoff's laws for the study of analytic properties of Feynman integrals (which are derived from the properties of denominators of the integrands) is well known, it appears that their use in the simplification of Feynman integrals has not been emphasized thus far.

It will be obvious that the above results are also applicable to other graphs of Fig. 2 if we interpret A_9, A_{10}, B_{9i}, and $B_{10,i}$ in an appropriate fashion.

APPENDIX B: UNSIMPLIFIED INTEGRANDS FOR GRAPH IV

For the benefit of readers who wish to check our calculations, we give here the unsimplified output of REDUCE for graph IV. Simplification of these integrands with the help of the identities given in Appendix A leads us to the formulas (4.33)–(4.36). As is defined in (A26), B_{ii}' is the polynomial part of B_{ii} and is equal to $\partial U/\partial z_i$.

$$\begin{aligned} M_{\mathrm{IVa}}=&\int\frac{dz_0}{U^4W^2}z_6z_8[A_1A_6(B_{68}B_{78}-B_{67}B_{88}')+A_3A_7(B_{66}'B_{88}'-B_{68}^2)]\\ &+\int\frac{dz_0}{U^3W^2}[z_6(A_1A_6B_{67}-A_3A_7B_{66}')-z_8A_3A_7B_{88}']+\int\frac{dz_0}{U^2W^2}A_3A_7\\ &+\frac{1}{2}\int\frac{dz_0}{U^4W}z_6z_8[B_{48}(B_{67}B_{68}-B_{66}'B_{78})+B_{47}(B_{66}'B_{88}'-B_{68}^2)+5B_{46}(B_{67}B_{88}'-B_{68}B_{78})+3B_{57}(B_{68}^2-B_{66}'B_{88}')\\ &+3B_{56}(B_{68}B_{78}-B_{67}B_{88}')+3B_{58}(B_{67}B_{68}-B_{66}'B_{78})]+\frac{1}{2}\int\frac{dz_0}{U^3W}[z_6(3B_{57}B_{66}'+3B_{56}B_{67}-B_{47}B_{66}'-5B_{46}B_{67})\\ &+z_8(3B_{57}B_{88}'+3B_{58}B_{78}-B_{47}B_{88}'+B_{48}B_{78})]+\frac{1}{2}\int\frac{dz_0}{U^2W}(B_{47}-3B_{57})\,. \end{aligned}\tag{B1}$$

$$M_{\mathrm{IVb}}=\frac{1}{2}\int\frac{dz_0}{U^5W^2}z_6z_8\{A_1A_3[A_6(B_{68}B_{78}-B_{67}B_{88}')+A_7(B_{66}'B_{88}'-B_{68}^2)+A_8(B_{78}B_{66}'-B_{67}B_{68}]$$

$$+A_1[A_6A_7(B_{58}B_{68}-B_{56}B_{88}')+A_6A_8(B_{58}B_{67}-B_{56}B_{78})+A_7A_8(B_{56}B_{68}-B_{58}B_{66}')]$$

$$+A_3[A_6A_7(B_{46}B_{88}'-B_{48}B_{68})+A_6A_8(B_{48}B_{67}-B_{46}B_{78})+A_7A_8(B_{46}B_{68}-B_{48}B_{66}')]+A_6A_7A_8(B_{48}B_{56}-B_{46}B_{58})\}$$

$$+\frac{1}{2}\int\frac{dz_0}{U^4W^2}z_6[A_1A_3(A_6B_{67}-A_7B_{66}')+A_6A_7(A_1B_{56}-A_3B_{46})]$$

$$+\frac{1}{2}\int\frac{dz_0}{U^4W^2}z_8[-A_1A_3(A_7B_{88}'+A_8B_{78})+A_7A_8(A_3B_{48}+A_1B_{58})]$$

$$+\frac{1}{2}\int\frac{dz_0}{U^3W^2}A_1A_3A_7+\frac{1}{2}\int\frac{dz_0}{U^5W}z_6z_8\{3A_1(B_{56}B_{67}B_{88}'-B_{58}B_{66}'B_{78})$$

$$+A_3[2B_{47}(B_{68}^2-B_{66}'B_{88}')-B_{46}(B_{67}B_{88}'+2B_{68}B_{78})+B_{48}(4B_{67}B_{68}-B_{66}'B_{78})]$$

$$+A_6[B_{45}(B_{67}B_{88}'-B_{68}B_{78})+3B_{46}B_{58}B_{78}+2B_{47}(B_{56}B_{88}'-B_{58}B_{68})+B_{48}(B_{56}B_{78}-4B_{58}B_{67})]$$

$$+A_7[B_{45}(B_{68}^2-B_{66}'B_{88}')-B_{46}(2B_{56}B_{88}'+B_{58}B_{68})+B_{48}(4B_{58}B_{66}'-B_{56}B_{68})]$$

$$+A_8[B_{45}(B_{66}'B_{78}-B_{67}B_{68})+B_{46}(2B_{56}B_{78}+B_{58}B_{67})+2B_{47}(B_{66}'B_{58}-B_{56}B_{68})-3B_{48}B_{56}B_{67}]\}$$

$$+\frac{1}{2}\int\frac{dz_0}{U^4W}z_6[-3A_1B_{56}B_{67}+A_3(2B_{47}B_{66}'+B_{46}B_{67})-A_6(2B_{47}B_{56}+B_{45}B_{67})+A_7(2B_{46}B_{56}+B_{45}B_{66}')]$$

$$+\frac{1}{2}\int\frac{dz_0}{U^4W}z_8[3A_1B_{58}B_{78}+A_3(2B_{47}B_{88}'+B_{48}B_{78})+A_7(B_{45}B_{88}'-4B_{48}B_{58})-A_8(B_{45}B_{78}+2B_{47}B_{58})]$$

$$+\frac{1}{2}\int\frac{dz_0}{U^3W}(-2A_3B_{47}-A_7B_{45}), \quad \text{(B2)}$$

$$M_{\mathrm{IVc}}=\rho\int\frac{dz_0}{U^2W^2}z_6z_8[(B_{48}B_{67}-B_{46}B_{78})+(B_{56}B_{68}-B_{58}B_{66}')+(B_{46}B_{88}'-B_{48}B_{68})+(B_{58}B_{68}-B_{56}B_{88}')]$$

$$+\rho\int\frac{dz_0}{UW^2}[z_8B_{58}+z_6(B_{56}-B_{46})], \quad \text{(B3)}$$

$$M_{\mathrm{IVd}}=\tfrac{1}{2}\rho\int\frac{dz_0}{U^3W^2}z_6z_8\{(A_6+A_7+A_8)(B_{46}B_{58}-B_{48}B_{56})$$

$$+A_1[(B_{56}B_{88}'-B_{58}B_{68})+(B_{56}B_{68}-B_{58}B_{66}')+(B_{56}B_{78}-B_{58}B_{67})]$$

$$+A_3[(B_{48}B_{68}-B_{46}B_{88}')+(B_{48}B_{66}'-B_{46}B_{68})+(B_{48}B_{67}-B_{46}B_{78})]\}$$

$$+\tfrac{1}{2}\rho\int\frac{dz_0}{U^2W^2}[z_6(A_3B_{46}-A_1B_{56})+z_8(A_1B_{58}-A_3B_{48})]. \quad \text{(B4)}$$

APPENDIX C: FORMULAS DERIVED FROM (6.4)

Differentiating both sides of (6.4) with respect to a, b, and c, where we have omitted the suffix 0 in a_0, etc., we obtain further relations:

$$\int\frac{z_4^2dz_4dz_5}{W^2}=\frac{\ln\rho}{4\Delta_0}\left(\frac{c}{a+c}-(b+c)G\right)+\cdots, \quad \text{(C1)}$$

$$\int\frac{z_4z_5dz_4dz_5}{W^2}=\frac{\ln\rho}{4\Delta_0}(-1+cG)+\cdots, \quad \text{(C2)}$$

$$\int \frac{z_4{}^4 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0{}^2}\left(\frac{c(5\Delta_0+3c^2)}{(a+c)^2} - 3(b+c)^2 G\right) + \cdots, \quad \text{(C3)}$$

$$\int \frac{z_4{}^3 z_5 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0{}^2}\left(-\frac{2\Delta_0+3c^2}{a+c} + 3c(b+c)G\right) + \cdots, \quad \text{(C4)}$$

$$\int \frac{z_4{}^2 z_5{}^2 dz_4 dz_5}{W^3} = \frac{\ln\rho}{16\Delta_0{}^2}[3c-(\Delta_0+3c^2)G] + \cdots. \quad \text{(C5)}$$

The domain of integration is $0 \leq z_4+z_5 \leq K$ in all cases. Similar formulas for the integrals

$$\int \frac{z_5{}^2 dz_4 dz_5}{W^2}, \quad \int \frac{z_4 z_5{}^3 dz_4 dz_5}{W^3}, \quad \int \frac{z_5{}^4 dz_4 dz_5}{W^3}$$

are obtained from (C1), (C4), and (C3) by interchanging a and b.

PHYSICAL REVIEW D VOLUME 1, NUMBER 8 15 APRIL 1970

Bootstrap Conditions from the Veneziano Representation*

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A set of self-consistency conditions for ratios of hadron-hadron-hadron coupling constants is derived from the Veneziano model for scattering amplitudes and some simple assumptions. This set includes the crossing-matrix condition of the static meson-baryon model and the condition that the absence of resonances in exotic states implies a cancellation from Regge trajectories of opposite signatures. For simplicity, the spins of the external mesons and baryons are neglected. The conditions require that the particles correspond to representations of a Lie group, and that only certain sets of representations are allowed. The relation to previous bootstrap models is discussed. An illustrative solution corresponding to $SU(3)$ is obtained.

I. INTRODUCTION

In recent years, several dynamical models of hadrons have been based on various analyticity and crossing properties of hadron-hadron scattering amplitudes. Examples are the meson-nucleon static model, superconvergence relations, Regge-pole models, and the Veneziano model. The addition of the bootstrap hypothesis to the first three of these models has led to additional predictions, many of which are in approximate agreement with experiment. The main purpose of this paper is to obtain a set of bootstrap conditions from the Veneziano model, a set that includes the more successful conditions of earlier models.

Our definition of bootstrap includes two parts: (i) The subtraction constants or background terms are sufficiently small in number so that the dynamical equations determine the physical masses and interaction constants in terms of a very small set of external parameters; (ii) a complete set of two-hadron → two-hadron amplitudes should be considered, and the set of internal particles should be the same as the set of external particles. Two comments must be made concerning these points. First, statement (i) is an assumption that certain conceivable terms are negligible. Thus, at least in present models, the bootstrap is not implied by crossing and general analyticity requirements. It must be judged by its theoretical simplicity and the success of its predictions. Second, in any bootstrap model, it is reasonable to begin by considering amplitudes in which the lightest hadrons are external. In such a first approximation to a "complete" model, statement (ii) requires only that the set of virtual particles consists of the external particles plus heavier particles.

In Sec. II we review briefly the most successful bootstrap conditions of the static meson-nucleon model and of the combination of Regge theory with duality. Consistency conditions are derived from an idealized Veneziano model (with the spins of external particles treated like internal quantum numbers) in Sec. III. The conditions are summarized in Sec. III C. The fact that a solution must involve a Lie-group symmetry is discussed in Sec. IV, while Sec. V contains an illustrative solution involving $SU(3)$. The exact treatment of particle spin leads to a more complicated model, which will be discussed in a future paper.

II. BOOTSTRAP CONDITIONS OF PREVIOUS MODELS

The simplest type of conceivable hadron spectrum in a Reggeized model contains only a finite number of Regge trajectories. The experimental hadron data are consistent with such a spectrum. In such a model,

* Supported in part by the U. S. Atomic Energy Commission.