Remarks on the Lee-Zumino Summation Method*

STEPHEN FELS
Department of Physics, University of California, Los Angeles, California 90024
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A method proposed recently by Lee and Zumino for calculating higher-order corrections to the S matrix in nonpolynomial field theories is investigated. When applied to a perturbed harmonic oscillator, grave difficulties are encountered. The reason is found to lie not in the specific summation method, but in the behavior of the perturbing potential at infinity. In the more realistic field-theoretical case, it is shown that perturbation theory gives anomalous values for the vacuum self-energy and the wave-function renormalization constant under certain circumstances.

I. INTRODUCTION

INVESTIGATION of the renormalizability of theories with nonpolynomial interaction Lagrangians[1–3] has taken on new importance in light of the successes of chiral symmetry. It is tempting to hope that the same theories which give experimentally correct results for lowest-order matrix elements might also be used to calculate radiative corrections to them.[4]

It is characteristic of many such theories that there exists, in addition to the usual problems connected with ultraviolet divergences, a conceptually different question involving the summation of formally divergent series. These series arise because, to a given process in a particular order of perturbation theory, there corresponds an infinite number of Feynman diagrams. It is possible to sum such series in a variety of ways; a particularly simple model has been investigated by Lee and Zumino (LZ).[5] We shall not recapitulate their work in detail, but rather point out those parts of it which are of particular interest to us here.

Let us, with LZ, consider the interaction

$$\mathcal{L}_I=\lambda :\phi/(1-g\phi^2):.$$

They show that the correct analyticity properties for the second-order regulated propagator are ensured if

$$\begin{aligned}F_{00}(x)&\equiv\langle 0|T[\mathcal{L}_I(x)\mathcal{L}_I(0)]|0\rangle\\&=(\tfrac{1}{2}+ib)\int_0^\infty\frac{ie^{-z}\Delta_F{}^R(x)z\,dz}{1+(g+i\epsilon)^2[\Delta_F{}^R(x)]^2z^2}\\&\quad+(\tfrac{1}{2}-ib)\int_0^\infty\frac{ie^{-z}\Delta_F{}^R(x)z\,dz}{1+(g-i\epsilon)^2[\Delta_F{}^R(x)]^2z^2},\qquad(1.1)\end{aligned}$$

where $\Delta_F{}^R(x)\equiv[\Delta_F(x,m)-\Delta_F(x,M)]$, and b is an arbitrary real number.

In particular,

$$F(s)=\lim_{M\to\infty} i\int e^{ip\cdot x}F_{00}(x)\,d^4x$$

is real below threshold, and has the cut structure implied by second-order unitarity. Since b is arbitrary, however, the real part of $F(s)$ is not well defined; it does not affect unitarity in this order, but obviously does enter in third and higher orders. In theory, then, one should go on to calculate higher-order processes to verify or disprove the unitarity of the formulation. In practice, however, this seems to be forbiddingly difficult.

To investigate the meaning of the LZ procedure, therefore, we look either for simpler models on which it can be tested, or further properties (aside from unitarity), which the second-order amplitudes must possess. In Sec. II, we adopt the first strategy, and investigate a perturbed harmonic oscillator. We find that in this model, the LZ procedure fails miserably. In addition, we note in Sec. III that it is possible to see exactly why this happens and to conclude that it is not just a failure of one particular summation method, but of perturbation theory itself. In Sec. IV, we examine the sign of the vacuum energy, and the size of the wave-function renormalization constant Z. We find that they are anomalous for certain values of the coupling constants.

II. PERTURBED OSCILLATOR MODEL

The LZ summation method is evidently in part a prescription for calculating matrix elements as functions of $\Delta_F(x)$. We can therefore explore this part of their program in the context of the simplest of all field theories—the perturbed harmonic oscillator. Because such models have propagators which are well defined even in the limit where $t_1=t_2$, we can investigate the behavior of matrix elements of equal-time products of the interaction Hamiltonian; such matrix elements must have certain properties if the Hamiltonian is Hermitian.

The investigation of such "toy" models may provide some insight into the structure of the far more complex field theories in which we are ultimately interested.

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Let us consider, therefore, a perturbed harmonic oscillator (HO) whose Hamiltonian is given formally by

$$H=H_0+\lambda U\,,$$
$$H_0=\tfrac{1}{2}(p^2+\omega^2q^2)\,,$$

and

$$U=:\frac{q}{1-gq^2}:=\sum_{n=0}^{\infty}:q^{2n+1}:g^n\,. \tag{2.1}$$

Taking $\{|n\rangle\}$ to be the usual HO basis states, it is clear that the matrix elements $\langle m|U|n\rangle$ are all well defined. Moreover, since $\langle n|U|m\rangle=(\langle m|U|n\rangle)^*$, U as defined is Hermitian. Let us therefore attempt to calculate the diagonal matrix elements of $U(0)U(0)$, i.e., $\langle n|U^2|n\rangle$. Since U is a Hermitian operator, we know that $\langle n|U^2|n\rangle$ must be positive semidefinite. We shall now show, however, that a calculation via Wick's theorem supplemented by the LZ summation procedure leads to values for these matrix elements which are negative for certain ranges of the parameters ω and g.

We begin by considering the matrix element $\langle 0|U^2|0\rangle$:

$$\langle 0|U(0)U(0)|0\rangle=\langle 0|\sum_{m,n=0}^{\infty}:q^{2m+1}::q^{2n+1}:g^{m+n}|0\rangle\,. \tag{2.2}$$

A formal application of Wick's theorem gives

$$\langle 0|U^2|0\rangle=\sum_{n=0}^{\infty}g^{2n}(2n+1)!\Delta^{2n+1}(0)\,, \tag{2.3}$$

where

$$\Delta(0)\equiv\langle 0|q(0)q(0)|0\rangle=1/2\omega\,.$$

Of course, the above sum is precisely that which we find in the corresponding field theory, but now $\Delta(0)$ is well defined. Let us therefore sum the series *à la* Borel, as prescribed by Lee and Zumino:

$$\langle 0|U^2|0\rangle=\int_0^\infty e^{-z}\,dz\sum_{n=0}^{\infty}g^{2n}z^{2n+1}\Delta^{2n+1}(0)\,; \tag{2.4}$$

for $(g^2z^2\Delta^2)<1$, the series is summed to give $\Delta z/(1-g^2\Delta^2z^2)$, which we then continue to all values of $g\Delta z$. At this stage, then,

$$\langle 0|U^2|0\rangle=\int_0^\infty e^{-z}\frac{\Delta z}{1-g^2\Delta^2z^2}dz\,. \tag{2.5}$$

At it stands, the integral evidently diverges because of the singularity on the real positive axis, if g is real. The heart of the LZ procedure is to define the integral for complex g, and then continue in a special way to real g. In particular, we are told that the correct prescription is to let

$$\langle 0|U^2|0\rangle=(\tfrac{1}{2}+ib)\int_0^\infty\frac{e^{-z}z\Delta}{1-(g+i\epsilon)^2\Delta^2z^2}dz$$
$$+(\tfrac{1}{2}-ib)\int_0^\infty\frac{e^{-z}z\Delta}{1-(g-i\epsilon)^2\Delta^2z^2}dz\,, \tag{2.6}$$

where b is an arbitrary real parameter.

It can be shown that with this definition, U has an asymptotic series given by the right-hand side of Eq. (2.3). The prescription $1/(x+i\epsilon)=i\pi\delta(x)+P$ tells us that

$$(\tfrac{1}{2}+ib)\int_0^\infty f(z)\frac{1}{1-(g+i\epsilon)\Delta z}dz$$
$$+(\tfrac{1}{2}-ib)\int_0^\infty f(z)\frac{1}{1-(g-i\epsilon)\Delta z}dz$$
$$=P\int_0^\infty f(z)\frac{1}{1-g\Delta z}dz-\frac{2\pi b}{\Delta g}f(1/\Delta g)\,. \tag{2.7}$$

We therefore write $\langle 0|U^2|0\rangle=\Delta(A_{00}-bB_{00})$, with

$$A_{00}=P\int_0^\infty\frac{e^{-z}z}{1-g^2\Delta^2z^2}dz\,, \tag{2.8a}$$

$$B_{00}=\frac{\pi}{(\Delta g)^2}e^{-(1/\Delta g)}\,. \tag{2.8b}$$

A similar decomposition is made for all matrix elements; in general, we will write[6]

$$\langle n|U^2|n\rangle=\Delta(A_{nn}-bB_{nn})\,. \tag{2.8c}$$

Our investigation begins by considering the simplest case, that for which $b=0$. We shall later relax this condition. We then have

$$A_{00}(x)=\frac{1}{x^2}P\int_0^\infty\frac{ze^{-z/x}}{1-z^2}dz\,, \tag{2.9}$$

where $x\equiv\Delta g$. A moment's thought shows that while the integral is indeed positive for sufficiently small x, it becomes negative for large x; a simple numerical integration shows that the sign changes at $x_0=1.05$. We conclude, then, that this simplest of all prescriptions (i.e., with $b=0$) gives nonsensical results for $x_0<x$.

[6] In what follows, we shall restrict ourselves to the case in which b is the same for all matrix elements. Relaxing this requirement introduces an infinite number of parameters into the model, completely destroying its simplicity. We have in mind here the applicability of the summation method to realistic field theories, in which the necessity of introducing a new parameter each time we add an external line to a diagram would severely curtail its predictive ability.

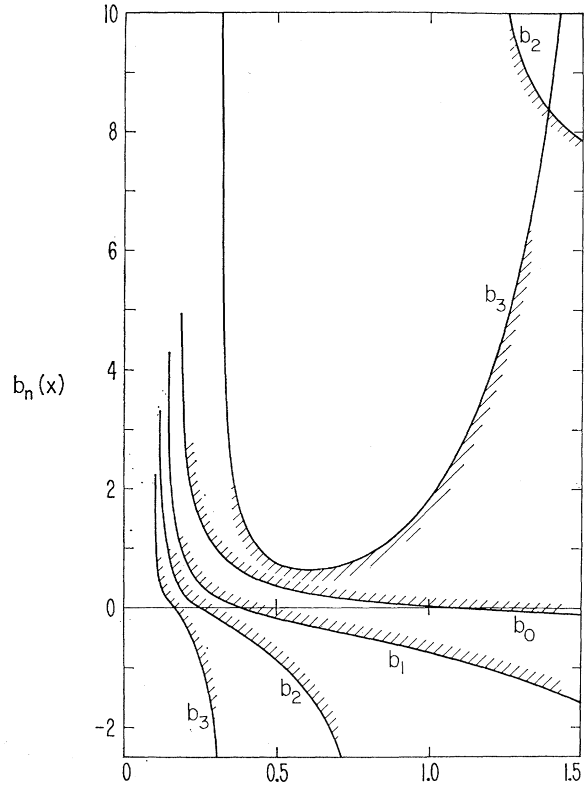

FIG. 1. Impossible values of $b(x)$. Shading indicates values excluded by positivity requirements.

It is interesting to observe that precisely this problem would have arisen in a slightly different guise had we attempted to calculate the second-order correction to the ground-state energy in this model. The latter should, simply by the Hermiticity of H and the vanishing of the first-order correction, be negative. A direct calculation, however, supplemented by the LZ summation technique, gives

$$\Delta E^{(2)} = -\lambda^2 \sum_{l=0}^{\infty} \frac{g^{2l}(2l+1)!}{(2\omega)^{2l+1}(2l+1)} = -\frac{\lambda^2}{2\omega}\int_0^\infty e^{-z}\frac{z}{1-(g/2\omega)^2z^2}dz; \quad (2.10)$$

this is, of course, the integral encountered above.

We expect that examination of the other diagonal matrix elements will lead to other conditions on the allowed values of x, and this is indeed the case; unfortunately, however, we have been able to find no simple way of calculating the general diagonal elements and have, therefore, contented ourselves with the calculation of $\langle n|U^2|n\rangle$ for $n=0$, 1, 2, and 3. Explicit expressions for these matrix elements are given in Appendix A. The reader will notice that their algebraic complexity grows rapidly with U. Given these expressions, however, it is not difficult to evaluate them numerically. We then find that in every case there is an x_n such that if $x_n<x$, $\langle n|U|n\rangle<0$. We list x_n:

$$x_0=1.05\,,\quad x_1=0.39\,,\quad x_2=0.24,\quad x_3=0.14\,.$$

Observe that x_n decreases rather rapidly with n. Since the summation method with $b=0$ only makes sense if $0<x<x_n$ for *all* n, it is not an unreasonable conjecture that the only allowed value for x is 0; the trivial case in which $U=\,:q:$ is the only one which does not violate the positive-semidefinite requirements on the diagonal matrix elements.

When we go on to consider the more general case in which $b\neq 0$, the situation is necessarily more complicated. We shall see, however, that in this case, too, certain large ranges of b can be shown to lead to negative matrix elements for $n=0$, 1, 2, 3.

Recalling Eq. (2.8c), let us define

$$b_n(x)\equiv A_{nn}(x)/B_{nn}(x)\,.$$

If, for a given x, $B_{nn}(x)$ is positive, then $b>b_n$ makes $U_{nn}{}^2(x)$ negative, while if $B_{nn}(x)$ is negative, $b<b_n$ implies that $U_{nn}{}^2(x)$ is negative. By considering a *single* matrix element, therefore, one can no longer rule out any values of x since, by suitable choice of b, one can always make that element positive. All that we can do for a single matrix element is rule out certain ranges of b as a function of x. But the fact that the *same* b must give positive results for *all* n means that for a given x, there will be an infinite number of conditions on b, which may be mutually inconsistent. In Fig. 1 we show the forbidden values of b as a function of x which arise from $n=0$, 1, 2, and 3.[7] It will be noticed that for $x>0.3$ there is no value of b allowed by all four matrix elements. The additional freedom provided by allowing nonzero b has really not changed the situation at all—the summation method has been shown to fail if $x>0.3$. Once again, it is not unreasonable to conjecture that as we go to larger n, the range of x which gives a sensible result will contract to the point $x=0$. As before, the complexity of matrix elements for large U has prevented us from verifying that this is indeed the case. Nonetheless, there evidently are difficulties which arise in the LZ summation method, and it is important to gain a deeper understanding of precisely where the problem lies. It is to this task that we now turn.

III. INTERACTION POTENTIAL

It has already been pointed out that the matrix elements U_{mn} are all well defined. It should be possible, therefore, to find an explicit expression $U(q,p)$ which gives rise to U_{mn}; by so doing, we might gain a more profound understanding of the structure of the theory.

[7] For the sake of clarity, we have only shown the region $0\leq x\leq 1.5$, but examination of the numerical results show that there is no allowed value of b for $x>1.5$.

We begin by observing that there exists an extremely simple method for normal ordering the operator q^n:

$$:q^n:=H_n(\omega^{1/2}q)/(2\omega^{1/2})^n, \quad (3.1)$$

where H_n is the nth Hermite polynomial. This fact, which is perhaps not as widely known as one might expect, may be proved easily.

By Wick's theorem,

$$:q::q^n:= :q^{n+1}:+n:q^{n-1}:\Delta. \quad (3.2)$$

Thus, if the statement is true for $l\leq n$,

$$:q^{n+1}:=-\frac{n}{2\omega}\frac{H_{n-1}(\omega^{1/2}q)}{(2\omega^{1/2})^{n-1}}+q\frac{H_n(\omega^{1/2}q)}{(2\omega^{1/2})^n}, \quad (3.3)$$

since $:q:=q$. Because

$$-2nH_{n-1}(\omega^{1/2}q)+2\omega^{1/2}qH_n(\omega^{1/2}q)=H_{n+1}(\omega^{1/2}q),$$

we find

$$:q^{n+1}:=H_{n+1}(\omega^{1/2}q)/(2\omega^{1/2})^{n+1}. \quad (3.4)$$

Since the theorem is trivially true for $n=0$ and $n=1$, it is true for all n.

Let us therefore consider

$$V(q)=\sum_{l=0}^{\infty} g^l\frac{H_{2l+1}(\omega^{1/2}q)}{(2\omega^{1/2})^{2l+1}}. \quad (3.5)$$

It is by no means obvious that this series converges, but we may once again apply the Borel trick, and redefine $V(q)$ as

$$V(q)=\int_0^\infty e^{-t}dt\sum_{l=0}^{\infty} g^l t^{2l+1}\frac{H_{2l+1}(\omega^{1/2}q)}{(2l+1)!(2\omega^{1/2})^{2l+1}}. \quad (3.6)$$

The inner sum is now well defined, and not difficult to evaluate. Letting

$$f(q,t)=\sum_{l=0}^{\infty} g^l\left(\frac{t}{2\omega^{1/2}}\right)^{2l+1}\frac{H_{2l+1}(\omega^{1/2}q)}{(2l+1)!}, \quad (3.7)$$

it is easy to show that

$$\frac{1}{t}\frac{df}{dt}+g^{1/2}f=\sum_{l=0}^{\infty}\frac{1}{l!}\left[\tfrac{1}{2}t\left(\frac{g}{\omega}\right)^{1/2}\right]^l H_l(\omega^{1/2}q)$$
$$=\exp[-(gt^2/4\omega)+g^{1/2}tq], \quad (3.8)$$

from which it follows that

$$f(q,t)=\frac{1}{2g^{1/2}}e^{-gt^2/4\omega}(e^{g^{1/2}tq}-e^{-g^{1/2}tq}) \quad (3.9)$$

and

$$V(q)=\frac{1}{2g^{1/2}}\int_0^\infty e^{-gt^2/4\omega-t}(e^{g^{1/2}tq}-e^{-g^{1/2}tq})dt. \quad (3.10)$$

Because of the cavalier fashion in which we have exchanged the order in divergent sums and integrals, the above should be viewed with some scepticism, until we have verified that V does indeed have the correct matrix elements. This turns out to be the case, however; we have relegated our proof of this assertion to Appendix B.

The behavior of V at infinity is extremely interesting. Evidently,

$$V(q)\sim\pm[(\pi\omega)^{1/2}/g]\exp[\omega(1/g^{1/2}\mp q)^2] \quad (3.11)$$

as $q\to\pm\infty$.

Up to normalization factors,

$$\langle m|A|n\rangle=\int e^{-\omega q^2}H_m(\omega^{1/2}q)AH_n(\omega^{1/2})\,dq, \quad (3.12)$$

and we see that the matrix elements of V are finite. Since V^2 goes like $e^{2\omega q^2}$ for large q, on the other hand, its matrix elements *do not exist*. Hence conventional perturbation theory, i.e., the expansion of various quantities in powers of the coupling constant λ has no validity at all in this case, because the perturbing potential is too singular at infinity. The situation is not unsimilar to what happens when perturbation theory is applied to r^{-2} potentials.

It is amusing to observe that we can recover Eq. (2.5) by using our representation of V, and making a patently erroneous change in the order of integration.

Write

$$\langle 0|V^2|0\rangle=\frac{1}{4g}\left(\frac{\omega}{\pi}\right)^{1/2}\int_0^\infty dsdt$$
$$\times\exp\left[-s-t-\frac{g}{4\omega}(s^2+t^2)\right]\int_{-\infty}^{\infty}dq\,e^{-\omega q^2}$$
$$\times\{e^{g^{1/2}q(s+t)}-e^{g^{1/2}q(s-t)}-e^{-g^{1/2}q(s-t)}+e^{-g^{1/2}q(s+t)}\}. \quad (3.13)$$

Then

$$\langle 0|V^2|0\rangle=\int_0^\infty e^{-t}\frac{t\Delta}{1-g^2\Delta^2t^2}dt. \quad (3.14)$$

The case at hand is unfortunate in one respect: Because V goes to $-e^{\omega q^2}$ as $q\to-\infty$, the discrete nature of the spectrum is completely destroyed. We wish to stress, however, that this is not the crux of the matter, as one sees when the interaction $:\lambda/(1-gq^2):$ is analyzed. This latter form is equivalent to a potential $V'(q)$ which goes like $e^{\omega q^2}$ as $q\to\pm\infty$, and does, of course, preserve the character of the simple harmonic-oscillator spectrum. Nonetheless, one encounters here precisely the same difficulties which were found previously.

We are thus led to the conclusion that in these models, at least, it is not summation methods which are wrong, so much as the fact that perturbation theory itself is completely useless. The infinities which Lee and

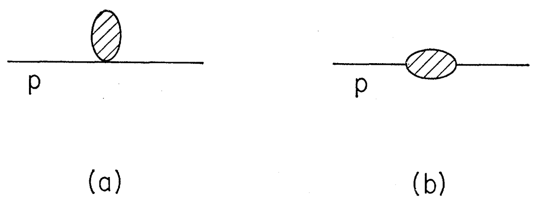

FIG. 2. Contributions to $\Sigma(p)$.

Zumino get rid of are indeed spurious, since one can construct interactions which lead to perfectly reasonable, finite level shifts, but they are intrinsic to any attempt to misapply perturbation theory.[8]

It might seem that the bad behavior at infinity is merely a reflection of the fact that $1/(1-gq^2)$ has a nonintegrable singularity on the real q axis. That this is not the case may be readily seen by analytically continuing our expansion for V to negative g; the behavior at infinity is still the same.

Since the formal existence of higher-order terms in perturbation theory depends crucially on the behavior of the normal-ordered potential at infinity, it is interesting to investigate several others. We list the results in tabular form below:

$:U(q):$	Dominant term at ∞
(1) $:(1-q)^{-\nu}:$ $(\nu\neq -n)$	$[1/\Gamma(\nu)](\pi/\omega)(2\omega q)^{\nu}e^{\omega(1-q)^2}$
(2) $:\ln(1-q):$	$-e^{\omega(1-q)^2}/2\omega q$
(3) $:e^{\pm q}:$	$\exp(-1/4\omega\pm q)$
(4) $:e^{-q^2}:$	$2(1-1/\omega)^{-1/2}e^{-q^2/[4(1-1/\omega)]}$ $(\omega>1)$

Interestingly, the last two forms are sufficiently well behaved at infinity so that matrix elements of the form $\langle n|(:U:)^l|m\rangle$ will always exist. Field theories based on interactions such as (3) have been investigated by Volkov.[9]

We conclude this section by remarking that the difficulties encountered above are evidently due to the fact that our interactions are normal ordered. Although it carries polynomials into polynomials of the same degree, normal ordering drastically alters functions with any singularities in the finite plane. This suggests that nonpolynomial models which are *not* normal ordered might be of interest in the full field-theoretical case. In such models, however, there exist problems owing to the necessity of contracting fields at the same point in space-time, and questions relating to the fashion in which the theory is regulated become very delicate.[10]

IV. FIELD-THEORETICAL RESULTS

We shall now discuss several anomalies which exist in the field-theoretical model. The first concerns the sign of the vacuum energy and the second the magnitude of the wave-function renormalization constant Z.

It is easy to show that if the *bare* vacuum expectation value of the interaction Hamiltonian vanishes, then the energy density shift of the vacuum must be negative; let us therefore investigate the second-order perturbation-theory level shift predicted by Lee and Zumino. We utilize the expression of Gell-Mann and Low[11]:

$$\Delta E=\lim_{\alpha\to 0}\frac{i\alpha\lambda}{2}\frac{(\partial/\partial\lambda)\langle 0|U_\alpha(\infty,-\infty)|0\rangle}{\langle 0|S_\alpha|0\rangle}, \quad (4.1)$$

where α is defined via $H=H_0+\lambda e^{-\alpha|t|}H_I$. Assuming that perturbation theory is applicable, we discover $\Delta E=-\frac{1}{2}\lambda^2F(0)$, where

$$F(p^2)\equiv i\int d^4x\, e^{ip\cdot x}\langle 0|T(\mathfrak{L}_I(x)\mathfrak{L}_I(0))|0\rangle. \quad (4.2)$$

$F(0)$ in this model has the ambiguous real part alluded to earlier; let us evaluate it with $b=0$. Using Eq. (12) of LZ, we have, with a *regulated* propagator,

$$F(0)=2\pi^2\int_0^\infty r^3dr\, P\int_0^\infty e^{-z}\frac{z\Delta^R(r)}{1-g^2z^2[\Delta^R(r)]^2}dz. \quad (4.3)$$

$\Delta^R(r)$ is positive definite, and behaves like $-(1/8\pi^2)\times(M^2-m^2)\ln r$ for small r, where M is the regulator mass. It is clear then that for sufficiently large g, $F(0)$ will be negative, and the vacuum energy density will be positive. This result may be shown to persist as $M\to\infty$.

While it is clear that this result in itself is not particularly serious (since by choosing an appropriate nonzero b, we can restore the correct sign of the vacuum energy density), it *does* place some restriction on the allowed values of the "arbitrary" parameter b. More importantly, however, we recall that anomalous signs in low-order matrix elements of the harmonic-oscillator model portended the collapse of the entire theory. The same may be true here.

The second anomaly which we shall explore concerns the magnitude of Z—we shall see that under certain circumstances, this can exceed unity.

To second order in λ, $Z=1+\lambda^2\Sigma'(m^2)$, where $\Sigma(p^2)$ is the self-mass insertion in second order. There are two types of terms which contribute to $\Sigma(p^2)$ [cf. Figs. 2(a) and 2(b)]. The first class, however, does not depend on p^2, and we need only calculate those from Fig. 2(b):

$$\Sigma(p^2)=-i\int e^{ip\cdot x}d^4x\, P\int_0^\infty e^{-z}\frac{z^2-z}{1+g^2\Delta^2z^2}dz \quad (4.4)$$

[8] In the light of this remark, the reader may wonder why such pains were taken in Sec. II to demonstrate the lack of positive definiteness implied by the LZ method. This misses the point, however, for the following reason. The summation method can be regarded as a set of rules for calculating matrix elements of the square of a perturbation potential. We could, in theory, find that potential, given the matrix elements of its square. The results of Sec. III tell us that what we would find in that case could not possibly be $V(q)$. The results of Sec. II, however, tell us that what we would find would be non-Hermitian, and therefore of no physical interest.

[9] M. K. Volkov, Commun. Math. Phys. **7**, 289 (1968).

[10] H. M. Fried, Phys. Rev. **174**, 1725 (1968).

[11] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

[cf. Eq. (A6)]. Regulating and making the usual contour rotation, we find

$$\Sigma_{\text{reg}}(s) = -\frac{4\pi^2}{s^{1/2}} \int_0^\infty r^2 dr\, I_1(s^{1/2} r) \times P \int_0^\infty e^{-z} \frac{z^2 - z}{1 - g^2 z^2 \Delta_R{}^2(r)} dz. \quad (4.5)$$

Because $I_\nu(u) \sim (2\pi u)^{-1/2} e^u$ for large u, our integral diverges as it stands. We can, however, evaluate it by returning to Eq. (4.4), which we rewrite as

$$\Sigma(p) = -i \int e^{ip\cdot x} d^4x \left[P \int_0^\infty e^{-z} \frac{(z - z^2)\Delta^2 g^2 z^2}{1 + \Delta^2 g^2 z^2} dz + 1 \right]$$

$$= -i \int e^{ip\cdot x} d^4x\, P \int e^{-z} \times \frac{\Delta^2 g^2 (z^3 - z^4)}{1 + \Delta^2 g^2 z^2} dz - (2\pi)^4 i \delta^4(p); \quad (4.6)$$

we find, then, that for $s \neq 0$

$$\Sigma_{\text{reg}}(s) = -\frac{4\pi^2}{s^{1/2}} \int_0^\infty r^2 dr\, I_1(s^{1/2} r) \times P \int_0^\infty e^{-z} \frac{\Delta_R{}^2(r) g^2 (z^4 - z^3)}{1 - g^2 z^2 \Delta_R{}^2(r)} dz \quad (4.7)$$

and

$$Z_{\text{reg}} \cong 1 - \frac{2\pi^2 \lambda^2 g^2}{\mu^2} \int_0^\infty r^3 dr\, I_2(\mu r) \Delta_R{}^2(r) \times P \int_0^\infty \frac{e^{-z}(z^4 - z^3)}{1 - g^2 z^2 \Delta_R{}^2(r)} dz. \quad (4.8)$$

Once again, we observe that for sufficiently large g, the iterated integral becomes negative, giving us a Z which exceeds unity. Now it is well known that for local field theories, $0 \leq Z \leq 1$; on the other hand, models such as the one under consideration have been shown to be nonlocal,[5] so that our result is perhaps not completely surprising. Nonetheless, it is rather disturbing, in light of the conventional interpretation of Z as the probability for finding the bare particle in the physical one. As before, by admitting nonzero values of b, we can always arrange things so that $0 \leq Z \leq 1$; thus we again have only succeeded in placing restrictions on the allowed values of the arbitrary parameter in the model.

V. CONCLUSION AND OUTLOOK

While our results suggest that the Lee-Zumino procedure is somewhat suspect, they do not leave completely groundless the hope that nonpolynomial theories may provide a complete description of hadronic physics. The harmonic-oscillator model suggests two possible ways of bypassing the difficulties encountered. We may either try to work with interactions which are not normal ordered, or retain the normal ordering and look for methods of approximation which do not involve an expansion in the coupling constant. Perhaps such approaches might lead to further progress. It would be pleasant to find that nonpolynomial Lagrangians are more than concise mnemonics for approximate calculation of strong-interaction processes.

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APPENDIX A

The evaluation of the diagonal matrix element $\langle n | U^2 | n \rangle$ is simple in principle but algebraically complicated in practice. We will illustrate how things work for $\langle 1 | U^2 | 1 \rangle$, and simply list the results for the other matrix elements. First we write

$$\langle 1 | U^2 | 1 \rangle = \sum_{m,n}^{\infty} \langle 1 | :q^{2m+1}::q^{2n+1}: | 1 \rangle g^{m+n}. \quad (\text{A1})$$

Wick's theorem may be used to write $:q^i::q^j:$ as a sum of terms of the form $:q^{i+j-2l}:\Delta^l$. It is clear that when this is done, the above matrix element will have contributions from two sets of terms: (1) those in which all q's have been contracted, and (2) those which have all but two q's contracted.

Looking at the first type of term, we see that we have simply

$$\sum_{n=0}^{\infty} \langle 1 | 1 \rangle g^{2n} \Delta^{2n+1} (2n+1)!,$$

which of course is just what we found for $\langle 0 | U^2 | 0 \rangle$.

Turning to the terms with two uncontracted q's, we observe that they arise in two different ways: (a) Both uncontracted q's come from the *same* normal product, and (b) one q comes from each.

The contribution from (a) is

$$2 \sum_{m,n}^{\infty} \langle 1 | :q^2: | 1 \rangle \delta_{m,n+1} \Delta^{2n+1} g^{m+n} \frac{(2m+1)!}{2}. \quad (\text{A2})$$

The over-all factor of 2 comes from the fact that we may select the uncontracted q's from either of the normal products.

This term, therefore, gives

$$\langle 1 | :q^2: | 1 \rangle \sum_{n=0}^{\infty} \Delta^{2n+1} g^{2n+1} (2n+3)!. \quad (\text{A3})$$

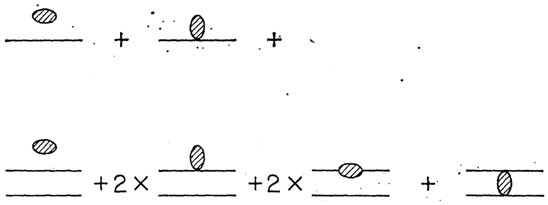

FIG. 3. Contributions to $\langle 1|U^2|1\rangle$ (top line) and to $\langle 2|U^2|2\rangle$ (bottom line).

Using the Borel method, this becomes

$$\langle 1|:q^2:|1\rangle\int_0^\infty e^{-z}z^2dz\sum_{n=0}^{\infty}(\Delta gz)^{2n+1} \tag{A4}$$

or

$$\langle 1|:q^2:|1\rangle\int_0^\infty e^{-z}\frac{\Delta gz^3}{1-\Delta^2g^2z^2}dz. \tag{A5}$$

We next look at term (b), which gives

$$\langle 1|:q^2:|1\rangle\sum_{n=0}^{\infty}\Delta^{2n}g^{2n}(2n+1)^2(2n)!$$

$$=\langle 1|:q^2:|1\rangle\sum_{n=0}^{\infty}\Delta^{2n}g^{2n}[(2n+2)!-(2n+1)!]$$

$$=\langle 1|:q^2:|1\rangle\int_0^\infty e^{-z}\frac{z^2-z}{1-\Delta^2g^2z^2}dz. \tag{A6}$$

Since

$$\langle n|:q^{2l}:|n\rangle=\frac{(2l)!n!}{l!l!(n-l)!}\Delta^l \tag{A7}$$

$$\langle 1|U^2|1\rangle=\Delta\int_0^\infty e^{-z}\frac{1}{1-\Delta^2g^2z^2}\times(z+2\Delta gz^2+2z^2-2z)dz, \tag{A8}$$

letting $x=g\Delta$, and setting

$$h_n(x,z)=\frac{e^{-z/x}}{(1+z)x^{n+1}},$$

we obtain

$$\langle 1|U^2|1\rangle=\Delta\int_0^\infty\{-h_1(x,z)+2h_2(x,z)+2xh_3(x,z)\}\frac{1}{1-z}dz. \tag{A9}$$

The LZ prescription for treating the singularities is, as shown in the text,

$$\int_0^\infty\rightarrow P\int_0^\infty-2\pi b\int\delta(z-1). \tag{A10}$$

We conclude, therefore, that

$$\langle 1|U^2|1\rangle=\Delta(A_{11}-bB_{11})$$

$$=\Delta\left[[-f_1(x)+2f_2(x)+2xf_3(x)]-2\pi b\times\left(-\frac{e^{-1/x}}{2x^2}+\frac{2e^{-1/x}}{x^3}\right)\right], \tag{A11}$$

where

$$f_n(x)\equiv\frac{1}{x^{n+1}}P\int_0^\infty\frac{e^{-z/x}z^n}{1-z^2}dz. \tag{A12}$$

Systematic evaluation is made somewhat simpler by the use of Feynman graphs for various terms. In Fig. 3 we display those which contribute to $\langle 1|U^2|1\rangle$ *and* $\langle 2|U^2|2\rangle$.

Applying the same techniques, we find the following results:

$$A_{22}(x)=-3f_1+4f_2+(3x^2-2x)f_3+(2x-6x^2)f_4+2x^2f_5,$$

$$B_{22}(x)=(\pi e^{-1/x}/x^3)(-4+4/x),$$

$$A_{33}(x)=-5f_1+6f_2+(-66x+16x^2)f_3+(24x-12x^2)f_4+(-16x^2+60x^3)f_5+[(16/3)x^2-40x^3]f_6+(16/3)x^3f_7,$$

$$B_{33}(x)=(\pi e^{-1/x}/x^2)(11-12/x+32/x^2+32/3x^3).$$

APPENDIX B

In this appendix we shall prove that the matrix elements of

$$U\equiv\sum_{l=0}^{\infty}g^l:q^{2l+1}:$$

equal those of

$$V\equiv\frac{1}{2g^{1/2}}\int_0^\infty e^{-(gt^2/4\omega)-t}(e^{g^{1/2}tq}-e^{-g^{1/2}tq})dt.$$

The proof is by induction. We first write

$$U_{nm}=\sum_{l=0}^{\infty}g^l\langle n|:q^{2l+1}:|m\rangle, \tag{B1}$$

$$\langle n|:q^2:U|m\rangle=\sum_{l=0}^{\infty}g^l\langle n|:q^2::q^{2l+1}:|m\rangle. \tag{B2}$$

Now

$$:q^2:=(a^2-2a^\dagger a+a^{\dagger 2})/(-2\omega),$$

so

$$\langle n|:q^2:=-(1/2\omega)\{[(n+1)(n+2)]^{1/2}\times\langle n+2|-2n\langle n|+[n(n-1)]^{1/2}\langle n-2|\}. \tag{B3}$$

By Wick's theorem, however,

$$:q^2::q^{2l+1}:=\,:q^{2l+3}:+2(2l+1):q^{2l+1}:(1/2\omega)+(2l+1)(2l):q^{2l-1}:(1/2\omega)^2. \quad \text{(B4)}$$

It is not difficult to see that

$$\sum_{l=0}^{\infty}\langle n|:q^{2l+3}:|m\rangle g^l=\frac{1}{g}(U_{nm}-\langle n|q|m\rangle), \quad \text{(B5)}$$

$$\frac{1}{\omega}\sum_{l=0}^{\infty}(2l+1)\langle n|:q^{2l+1}:|m\rangle g^l=\frac{1}{\omega}\left(2g\frac{d}{dg}U_{nm}+U_{nm}\right), \quad \text{(B6)}$$

and

$$(1/2\omega)^2\sum_{l=0}^{\infty}(2l)(2l+1)\langle n|:q^{2l-1}:|m\rangle g^l=\frac{1}{2\omega^2}\left(2g^3\frac{d^2}{dg^2}U_{nm}+7g^2\frac{d}{dg}U_{nm}+3gU_{nm}\right). \quad \text{(B7)}$$

Putting all of this together, we find

$$[(n+1)(n+2)]^{1/2}U_{n+2,m}=-[n(n-1)]^{1/2}U_{n-2,m}+(2\omega g^{-1})q_{nm}-(2\omega g^{-1}+2+3g\omega^{-1}-2n)U_{nm}-(4g+7g^2\omega^{-1})\frac{d}{dg}U_{nm}-2g^3\omega^{-1}\frac{d^2}{dg^2}U_{nm}. \quad \text{(B8)}$$

Turning now to the matrix elements of V, we first observe that

$$\psi_n(q)\equiv\langle q|n\rangle=\frac{i^n}{N_n}H_n(\omega^{1/2}q)\exp(-\tfrac{1}{2}\omega g^2), \quad \text{(B9)}$$

with $N_n=[2^n n!(\pi/\omega)^{1/2}]^{1/2}$. Since

$$H_{n+2}(\omega^{1/2}q)=4\omega q^2H_n(\omega^{1/2}q)-4n(n-1)H_{n-2}(\omega^{1/2}q), \quad \text{(B10)}$$

we have

$$\frac{N_n}{N_{n+2}}V_{n+2,m}=-\frac{4\omega}{N_nN_m}i^{(m-n)}\int_{-\infty}^{\infty}e^{-\omega q^2}V(q)H_nH_mq^2dq-4n(n-1)(N_n/N_{n-2})V_{n-2,m}. \quad \text{(B11)}$$

To evaluate the integral, we note that we may write

$$V(q)=\frac{1}{2g}\int_0^{\infty}\exp[-(t^2/4\omega)-(t/g^{1/2})](e^{tq}-e^{-tq})dt=(1/2g)[h_+(q)-h_-(q)]. \quad \text{(B12)}$$

One then verifies that

$$qh_{\pm}=\mp1\pm\left(g^{-1/2}+g^{3/2}\omega^{-1}\frac{d}{dg}\right)h_{\pm}(q), \quad \text{(B13)}$$

from which it follows that

$$q^2h_{\pm}=\mp q-g^{-1/2}\pm[g^{-1}-(2\omega)^{-1}]h_{\pm}\pm(2g\omega^{-1}+\tfrac{3}{2}g^2\omega^{-2})\times\frac{d}{dg}h_{\pm}\pm g^3\omega^{-2}\frac{d^2}{dg^2}h_{\pm}. \quad \text{(B14)}$$

Using this, one shows that

$$\frac{i^{m-n}4\omega}{N_nN_m}\int_{-\infty}^{\infty}e^{-\omega q^2}V(q)H_nH_mdq=-4\omega g^{-1}q_{nm}+(4\omega g^{-1}+6+6g\omega^{-1})V_{nm}+(8g+14g^2\omega^{-1})\frac{d}{dg}V_{nm}+4g^3\omega^{-1}\frac{d^2}{dg^2}V_{nm} \quad \text{(B15)}$$

and that

$$[(n+1)(n+2)]^{1/2}V_{n+2,m}=-[n(n-1)]^{1/2}V_{n-2,m}+(2\omega g^{-1})q_{nm}-(2\omega g^{-1}+2+3g\omega^{-1}-2n)V_{nm}-(4g+7g^2\omega^{-1})\frac{d}{dg}V_{nm}-2g^3\omega^{-1}\frac{d^2}{dg^2}V_{nm}. \quad \text{(B16)}$$

This will be recognized as the recursion relation satisfied by U_{nm}.

It only remains, therefore, to show that $V_{10}=U_{10}$. (Notice that $U_{nm}=V_{nm}=0$ if $n=m+2i$.)

On the one hand, $U_{10}=\langle1|q|0\rangle=i/(2\omega)^{1/2}$. Now

$$V_{10}=\frac{i}{\sqrt{2}}(\omega/\pi)^{1/2}\int_{-\infty}^{\infty}e^{-\omega q^2}2q\omega^{1/2}dq\times\int_0^{\infty}\frac{1}{2g}e^{-(t/g^{1/2})-(t^2/4\omega)}(e^{tq}-e^{-tq})dt. \quad \text{(B17)}$$

In the above expression, we *may* change the order of integration and make a trivial change of variable to get

$$V_{10}=\frac{i}{g}(2\pi)^{-1/2}\int_0^{\infty}e^{-t/g^{1/2}}tdt\int_0^{\infty}e^{-\omega q^2}dq=\frac{i}{(2\omega)^{1/2}}. \quad \text{(B18)}$$

It follows, then, that $V_{nm}=U_{nm}$.