Operator Formulation of the Droplet Model*

BENJAMIN W. LEE
Institute for Theoretical Physics, State University of New York, Stony Brook, New York, 11790
(Received 15 December 1969)

We study in detail the implications of the operator formulation of the droplet model. The picture of high-energy scattering that emerges from this model attributes the interaction between two colliding particles at high energies to an instantaneous, multiple exchange between two extended charge distributions. Thus the study of charge correlation functions becomes the most important problem in the droplet model. We find that in order for the elastic cross section to have a finite limit at infinite energy, the charge must be a conserved one. In quantum electrodynamics the charge in question is the electric charge. In hadronic physics, we conjecture, it is the baryonic charge. Various arguments for and implications of this hypothesis are presented. We study formal properties of the charge correlation functions that follow from microcausality, T, C, P invariances, and charge conservation. Perturbation expansion of the correlation functions is studied, and their cluster properties are deduced. A cluster expansion of the high-energy T matrix is developed, and the exponentiation of the interaction potential in this scheme is noted. The operator droplet model is put to the test of reproducing the high-energy limit of elastic scattering in quantum electrodynamics found by Cheng and Wu in perturbation theory. We find that the droplet model reproduces exactly the results of Cheng and Wu as to the impact factor. In fact, the "impact picture" of Cheng and Wu is completely equivalent to the droplet model in the operator version. An appraisal is made of the possible limitation of the model.

I. INTRODUCTION

SOME time ago, Chou and Yang[1] proposed an operator version of the droplet model which had been developed by Yang and co-workers.[2] The proposal was made at that time specifically to study diffractive dissociation phenomena. As we shall discuss, however, the operator formulation is necessary even for elastic scattering in many instances, and, as Chou and Yang have already pointed out,[1] the implications of the q-number version of the model are not always identical to those of the classical version even for elastic scattering.

In the present paper, we shall study in detail the implications of the operator formulation of the droplet model. The picture of high-energy scattering that emerges from this study attributes the interaction between two particles at high energies to an instantaneous, multiple exchange of quanta between two extended charge distributions. Thus the study of a charge distribution function and a hierarchy of charge correlation functions within a particle becomes the most important problem in the droplet model. We shall study various properties of the relevant charge correlation functions, such as their symmetry properties, reality properties, and cluster properties.

We will find that in order for the high-energy limit of the elastic scattering to be finite, the charge we referred to above should be a conserved one (i.e., the time component of a conserved current). In quantum electrodynamics (QED), the charge in question is, of course, the electric charge; for hadronic interactions, we shall tentatively identify this charge to be the one associated with the baryon number conservation. We shall present various arguments for this suggestion and study some of its implications, but a detailed analysis of the existing data from this point of view is deferred to future publications.

The operator droplet model, whose precise formulation we shall present in the text, will be put to the test of reproducing the high-energy limits of QED in perturbation theory found recently by Cheng and Wu.[3,4] We shall find, notwithstanding the earlier assertion by Cheng and Wu[5] to the contrary, that the model gives precisely the high-energy limits of elastic scattering in QED found by these authors up to, say, the sixth order in perturbation theory. The impact factor of Cheng and Wu is essentially the Fourier transform of certain charge correlation functions. It is perhaps worth emphasizing that the agreement is not only in spirit, but in details as well. In fact the "impact picture" of high-energy elastic scattering of Cheng and Wu[4] is completely equivalent to the operator droplet model insofar as QED in perturbation theory is concerned. This gives us confidence in the soundness of the operator formulation and in the ability of the model to handle various relativistic quantum effects.

The plan of the paper is as follows.

In Sec. II, the physical motivation for the droplet model is briefly reviewed, and the need for the operator formulation is explained. The density operator in QED is identified to be that of the electric charge. We present here the speculation that the corresponding density operator in hadronic physics is that of the baryonic charge.

In Sec. III, the precise formulation of the operator droplet model is presented. Charge correlation functions

* Research supported in part by AEC Contract No. AT(30-1) 3668B.
[1] T. T. Chou and C. N. Yang, Phys. Rev. **175**, 1832 (1968).
[2] T. T. Chou and C. N. Yang, Phys. Rev. **170**, 1591 (1968), and references cited therein.
[3] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969); **182**, 1868 (1969); **182**, 1873 (1969); **182**, 1899 (1969).
[4] H. Cheng and T. T. Wu, Phys. Rev. D **1**, 459 (1970).
[5] See the first-cited article of Ref. 3.

are defined and the high-energy limit of the T matrix is written out in terms of them.

In Sec. IV, we study formal properties of correlation functions which can be deduced from such general principles as charge commutation relation at equal times, T, C, and P invariances, and charge conservation. Perturbation treatment of the correlation function is outlined here, and cluster properties of the correlation functions are deduced. A cluster expansion of the T matrix at high energies, analogous to that in statistical mechanics,[6] is developed; in this connection we observe the familiar exponentiation of the interaction potential at high energies. In this section we also explain why the density should be a conserved one in order for the elastic scattering cross section to have a finite limit.

In Sec. IV, we compare the model with the high-energy limits of elastic scattering in QED studied by Cheng and Wu in perturbation theory. The impact factor of the photon is studied in detail, and the equivalence of the "impact picture" and the droplet model is established.

In the concluding section, we shall appraise a possible limitation of the droplet model and say a few words about the implication of the model on high-energy collisions in general.

II. DROPLET MODEL

To understand the motivation for the droplet model of Chou and Yang,[2] let us consider the passage of light through a medium containing scatterers in the eikonal picture (i.e., in the approximation of assuming the optical path to be a straight line). The amplitude at the point z along the optical path which is characterized by the impact parameter $\mathbf{b}$ satisfies the equation

$$\frac{\partial}{\partial z}\Psi(\mathbf{b},z)=-\rho(\mathbf{b},z)\Psi(\mathbf{b},z)\,, \tag{2.1}$$

where $\rho(\mathbf{b},z)$ is the local density of matter responsible for absorption and dispersion (ρ is in general complex). The transmission coefficient $S(\mathbf{b})$ is given by

$$S(\mathbf{b})=\frac{\Psi(\mathbf{b},\,\infty)}{\Psi(\mathbf{b},\,-\infty)}=\exp\left[-\int_{-\infty}^{\infty}dz\,\rho(\mathbf{b},z)\right] \tag{2.2}$$

and the T matrix for the scattering by

$$T(\mathbf{q})=-i\int d^2b\,e^{-i\mathbf{q}\cdot\mathbf{b}}[S(\mathbf{b})-1]. \tag{2.3}$$

In analogy to this, Chou and Yang are led to postulate that the high-energy limit of the elastic collision of two hadrons A and B is given by

$$S(\mathbf{b})=\exp\left[\int d^2x\int d^2y\,\sigma_A(\mathbf{x})\sigma_B(\mathbf{y})V(\mathbf{b}+\mathbf{x}-\mathbf{y})\right], \tag{2.4}$$

where $\sigma(\mathbf{x})$ is the two-dimensional density of "opaqueness"[2]:

$$\sigma(\mathbf{x})=\int dz\,\rho(\mathbf{x},z)\,, \tag{2.5}$$

$\rho(x,y,z)$ being a spherically symmetric distribution function; V is a function of two-dimensional length which is put in to represent an instantaneous, long-range interaction among constituents of hadrons A and B. The reason that a two-dimensional distribution is relevant in the above formula is that, at high energies, longitudinal distances suffer Lorentz contractions, so the target appears as a disk to the projectile (and vice versa).

Equation (2.4) is essentially classical in that $\sigma_A(\mathbf{x})$ is a c-number distribution. In order to discuss diffractive dissociations, Chou and Yang[1] proposed a q-number version of Eq. (2.4). In the operator version of the droplet model $\sigma_A(\mathbf{x})$ is replaced by an operator. In quantum electrodynamics, it is natural to identify $\rho(\mathbf{x},z)$ in Eq. (2.5) with the charge density operator:

$$\begin{aligned}\sigma(\mathbf{x})&=\int_{-\infty}^{\infty}dz\,j_0(\mathbf{x},\,z;\,t=0)\,,\\ j_\mu(x)&=\bar{\Psi}(x)\gamma_\mu\Psi(x)\,,\end{aligned} \tag{2.6}$$

where $\Psi(x)$ is the electron field operator (in the Heisenberg picture). An operator transcription of Eq. (2.4) is in any case necessary in order to discuss, for example, Delbrück scattering: The charge distribution of a photon is identically zero,

$$\sigma_\gamma(\mathbf{x})=0\,,$$

so the classical expression (2.4) cannot describe the high-energy limit of Delbrück scattering correctly. In quantum electrodynamics, $V(\mathbf{b})$ in Eq. (2.3) comes from Coulomb potential, so we assume

$$\begin{aligned}V(\mathbf{b})&=e^2\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dz\,D_F{}^{00}(\mathbf{b},\,z,\,t=0)\\ &=e^2i\int\frac{d^2q}{(2\pi)^2}\frac{e^{i\mathbf{q}\cdot\mathbf{b}}}{\mathbf{q}^2+\mu^2},\end{aligned} \tag{2.7}$$

where $D_F{}^{\mu\nu}(x)$ is the usual Feynman photon propagator and μ^2 is a fictitious photon mass.

The operator version of the droplet model will be developed fully in Sec. III. Let it suffice to say for the moment that we will be led in this model to describe high-energy scattering in terms of instantaneous, two-dimensional correlation functions of charge densities of participating particles. Thus, the configuration space description of particles in terms of charge densities acquires a paramount importance. While our consideration will be oriented primarily toward QED in this paper, this aspect of the model may be generalized

[6] See, for example, J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, New York, 1940).

readily to hadronic strong interactions. The analog of the electric charge would be the baryonic charge; both charges are conserved, so that many of the results we shall obtain in QED (such as constancy of high-energy cross sections) which follow from the conserved vector interaction should also hold in strong interactions if the density of "opaqueness" is identified with the baryonic charge of the hadron. Furthermore, the selection rule observed in hadronic diffraction scattering, i.e., no exchange of internal quantum numbers, follows immediately with this identification. We do not know exactly what mediates the force between baryonic charges, but presumably it is of very short range [$<(1\text{ BeV})^{-1}$], as the absence of any particle with the vacuum quantum number in the energy range below 2 BeV may suggest. In any case, it is likely that the observed diffraction peaks are not the reflection of the "potential" V in Eq. (2.4), but rather the reflection of an extended charge distribution inside the hadron. For our considerations, what matters primarily is the question "how is the baryonic charge distributed in a hadron?" and not the question "what carries the charge within a hadron?" The second question and various models for answering it (partons, quarks, etc.) are of interest to us only insofar as they help us to answer the first question.

III. OPERATOR FORMULATION

Let us first consider scattering of a particle (electron, positron, or photon) in an external field. The operator droplet model gives the high-energy limit of this amplitude $T(p,\mathbf{q})$ as

$$iT(p,\mathbf{q})(2\pi)^3\delta(p-p')\delta^2(\mathbf{q}-\mathbf{q}')$$
$$=\lim_{p\to\infty}\langle(\mathbf{p}'-\tfrac{1}{2}\mathbf{q})\lambda'|\int d^2b\, e^{-i\mathbf{q}'\cdot\mathbf{b}}$$
$$\times[S(\mathbf{b})-1]|(\mathbf{p}+\tfrac{1}{2}\mathbf{q})\lambda\rangle, \quad (3.1)$$

where λ and λ' are helicity labels of the initial and final particles. $S(\mathbf{b})$ is obtained from Eqs. (2.4), (2.6), and (2.7) and the external-field approximation, $\sigma_B(\mathbf{x})=Z\delta^2(\mathbf{x})$:

$$S(\mathbf{b})=\exp\left[i\lambda\int d^2\mathbf{x}\,\sigma(\mathbf{x})F(\mathbf{b}+\mathbf{x})\right], \quad (3.2)$$

where

$$\sigma(\mathbf{x})=\int_{-\infty}^{\infty}j_0(\mathbf{x},z;t=0)dz, \quad (3.3)$$

$$F(\mathbf{b})=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{b}}\frac{1}{\mathbf{q}^2+\mu^2}, \quad (3.4)$$

$$\lambda=-Ze^2.$$

Expanding the exponential (3.2), we obtain

$$i(2\pi)^3\delta(p-p')\delta^2(\mathbf{q}-\mathbf{q}')T(p,\mathbf{q})$$
$$=\sum_{n=1}\frac{(i\lambda)^n}{n!}\int d^2b\, e^{-i\mathbf{q}'\cdot\mathbf{b}}\left[\prod_{i=1}^{n}\int d^2x_iF(\mathbf{b}+\mathbf{x}_i)\right]$$
$$\times\lim_{p\to\infty}\langle(\mathbf{p}'-\tfrac{1}{2}\mathbf{q})\lambda'|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|(\mathbf{p}+\tfrac{1}{2}\mathbf{q})\lambda\rangle. \quad (3.5)$$

An important property of the matrix element appearing in Eq. (3.4) is its behavior under translation (we shall omit helicity labels when they are not essential):

$$\langle\mathbf{p}'-\tfrac{1}{2}\mathbf{q}|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|\mathbf{p}+\tfrac{1}{2}\mathbf{q}\rangle$$
$$=e^{i\mathbf{q}\cdot\mathbf{z}}\langle\mathbf{p}'-\tfrac{1}{2}\mathbf{q}|\sigma(\mathbf{x}_1+\mathbf{z})\sigma(\mathbf{x}_2+\mathbf{z})\cdots\sigma(\mathbf{x}_n+\mathbf{z})|\mathbf{p}+\tfrac{1}{2}\mathbf{q}\rangle. \quad (3.6)$$

We now define the quantity $\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle$ by the Fourier transform:

$$(2\pi)\delta(p-p')\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{\lambda'\lambda}\equiv\lim_{p\to\infty}\int\frac{d^2q}{(2\pi)^2}$$
$$\times\langle(\mathbf{p}'-\tfrac{1}{2}\mathbf{q})\lambda'|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|(\mathbf{p}+\tfrac{1}{2}\mathbf{q})\lambda\rangle. \quad (3.7)$$

Equation (3.7) can be inverted: By virtue of Eq. (3.6), we may write

$$(2\pi)\delta(p-p')\langle\sigma(\mathbf{x}_1+\mathbf{z})\cdots\sigma(\mathbf{x}_n+\mathbf{z})\rangle$$
$$=\lim_{p\to\infty}\int\frac{d^2q}{(2\pi)^2}e^{-i\mathbf{q}\cdot\mathbf{z}}\langle\mathbf{p}'-\tfrac{1}{2}\mathbf{q}|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|\mathbf{p}+\tfrac{1}{2}\mathbf{q}\rangle.$$

Therefore,

$$\lim_{p\to\infty}\langle(\mathbf{p}'-\tfrac{1}{2}\mathbf{q})\lambda'|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|(\mathbf{p}+\tfrac{1}{2}\mathbf{q})\lambda\rangle$$
$$=(2\pi)\delta(p-p')\int d^2z\, e^{i\mathbf{q}\cdot\mathbf{z}}$$
$$\times\langle\sigma(\mathbf{x}_1+\mathbf{z})\cdots\sigma(\mathbf{x}_n+\mathbf{z})\rangle_{\lambda'\lambda}. \quad (3.8)$$

Substituting Eq. (3.8) into Eq. (3.5), we obtain, after some manipulation, the desired formula:

$$iT(p,\mathbf{q})=\int d^2b\, e^{-i\mathbf{q}\cdot\mathbf{b}}\sum_{n=1}^{\infty}\frac{(i\lambda)^n}{n!}\left(\prod_{i=1}^{n}\int d^2\mathbf{x}_i\,F(\mathbf{b}+\mathbf{x}_i)\right)$$
$$\times\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{\lambda'\lambda}. \quad (3.9)$$

In order for the formula (3.9) to be meaningful, the right-hand side of Eq. (3.7) must have a finite limit. $T(p,\mathbf{q})$ in Eq. (3.9) will then be independent of p, the incident energy, and the elastic cross section

$$\lim_{p\to\infty}\frac{d\sigma}{dt}\sim|T(p,\mathbf{q})|^2$$

will have a finite limit.

Equation (3.9) can be generalized to the case of elastic scattering $A(p+\frac{1}{2}\mathbf{q},\lambda)+B(-p-\frac{1}{2}\mathbf{q},\mu)\to A(p-\frac{1}{2}\mathbf{q},\lambda')+B(-p+\frac{1}{2}\mathbf{q},\mu')$:

$$i\lim_{p\to\infty}T^{AB}(p,\mathbf{q})=2\int d^2b\,e^{-i\mathbf{q}\cdot\mathbf{b}}\sum_n\frac{(i\lambda)^n}{n!}$$
$$\times\left(\prod_{i=1}^n\int d^2x_i\int d^2y_i\,F(\mathbf{b}+\mathbf{x}_i-\mathbf{y}_i)\right)$$
$$\times\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{\lambda'\lambda}{}^A$$
$$\times\langle\sigma(\mathbf{y}_1)\cdots\sigma(\mathbf{y}_n)\rangle_{-\mu_1'-\mu}{}^B\times(-)^{2s_B},\quad(3.10)$$

where s_B is the spin of the particle B, and

$$2\pi\delta(p-p')\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{\lambda'\lambda}{}^A$$
$$=\lim_{p\to\infty}\int\langle A(\mathbf{p}'-\tfrac{1}{2}\mathbf{q},\lambda')|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)$$
$$\times|A(p+\tfrac{1}{2}\mathbf{q},\lambda)\rangle.\quad(3.11)$$

In deriving Eq. (3.10) we have made use of the fact that the initial and final states of particle B are given by[7]

$$(-)^{s_B-\mu_B}e^{-i\pi J_2}|B(\mathbf{p}\pm\tfrac{1}{2}\mathbf{q},\mu_B)\rangle$$
$$=(-)^{s_B-\mu_B}PY|B(\mathbf{p}\pm\tfrac{1}{2}\mathbf{q},\mu_B)\rangle$$
$$=\eta_B(-)^{s_B-\mu_B}P|B(\mathbf{p}\pm\tfrac{1}{2}\mathbf{q},-\mu_B)\rangle$$
$$=\eta_B(-)^{s_B}Pe^{-i\pi J_3}|B(\mathbf{p}\pm\tfrac{1}{2}\mathbf{q},-\mu_B)\rangle,\quad|\eta_B|=1$$

where $Y=Pe^{-i\pi J_2}$, P being the parity operator, and the fact that

$$Pe^{-i\pi J_3}\sigma(\mathbf{x})e^{+i\pi J_3}P^{-1}=\sigma(\mathbf{x}).$$

Here again, the right-hand side of Eq. (3.11) must be finite in order for the elastic cross section to reach a finite limit as $p\to\infty$. As we shall see in detail later, the limit of Eq. (3.11) exists if $\sigma(\mathbf{x})$ is the two-dimensional projection of a conserved charge density (i.e., the time component of a *conserved* vector current). In the following we shall assume $\sigma(\mathbf{x})$ to be of the form

$$\sigma(\mathbf{x})=\int_{-\infty}^{\infty}dz\,j_0(\mathbf{x},z;t=0)$$

with $\partial^\mu j_\mu(x)=0$.

IV. CORRELATION FUNCTIONS

In the droplet model, the high-energy limit of elastic scattering is expressed in terms of correlation functions of the form of Eq. (3.11). In this section we will study some formal properties of these correlation functions.

[7] M. Jacob and G. C. Wick, Am. J. Phys. **7**, 404 (1959).

A. Symmetry

The correlation function $\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{\lambda\lambda'}$ is a symmetric function of its arguments $\mathbf{x}_1,\ldots,\mathbf{x}_n$. (4.1) This is so because $\sigma(\mathbf{x})$ and $\sigma(\mathbf{y})$ commute:

$$[\sigma(\mathbf{x}),\sigma(\mathbf{y})]=0.\quad(4.2)$$

Equation (4.2), in turn, follows from the commutativity of charge densities at equal times.

B. Reality

For the purpose of this discussion it is convenient to define single-particle states which are eigenstates of combined TP operation. Recalling that, according to Jacob-Wick convention,[7]

$$TP|\mathbf{p},\lambda\rangle=\eta(-)^{s-\lambda}|\mathbf{p},-\lambda\rangle,\quad(4.3)$$

where η is a phase factor and s is the spin of the particle, we define

$$\begin{aligned}|\mathbf{p};|\lambda|+\rangle&\equiv(1/\sqrt{2})[|\mathbf{p},\lambda\rangle+(-)^{s-\lambda}|\mathbf{p},-\lambda\rangle],\\|\mathbf{p};|\lambda|-\rangle&\equiv(i/\sqrt{2})[|\mathbf{p},\lambda\rangle-(-)^{s-\lambda}|\mathbf{p},-\lambda\rangle].\end{aligned}\quad(4.4)$$

Both are eigenstates of TP with the eigenvalue η (note that T is antiunitary). Denoting $\lambda\pm$ collectively by c, we define

$$2\pi\delta(p-p')\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{c'c}$$
$$\equiv\lim_{p\to\infty}\int d^2\mathbf{q}\langle\mathbf{p}+\tfrac{1}{2}\mathbf{q},c'|\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)|\mathbf{p}-\tfrac{1}{2}\mathbf{q},c\rangle.$$

The property

$$TPj_0(x)(TP)^{-1}=+j_0(-x)$$

implies

$$[\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle_{c'c}]^*=\langle\sigma(-\mathbf{x}_1)\cdots\sigma(-\mathbf{x}_n)\rangle_{c'c}.\quad(4.5)$$

C. Charge Conservation

The quantity

$$\int d^2x\,\sigma(\mathbf{x})=\int d^3x\,j_0(\mathbf{x};t=0)\equiv Q$$

is the charge operator. Therefore,

$$\int d^2x\langle\sigma(\mathbf{x})\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle^A$$
$$=Q_A\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle^A,\quad(4.6)$$

where Q_A is the charge of the particle A. In particular, for neutral particles we have

$$\int d^2x\,\langle\sigma(\mathbf{x})\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle^A=0\quad(A\text{ neutral}).\quad(4.7)$$

The remaining properties of correlation functions which we shall discuss are intuitively plausible, but I do not know of any general proofs. They have, however, been verified in perturbation theory and we shall indicate arguments for them based on perturbation theory. First, a few words about perturbative construction of the correlation function are in order. In perturbation theory we may write[8] (in the following, T means the time-ordered product, disregarding disconnected vacuum-to-vacuum diagrams)

$$2\pi\delta(p-p')\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle^A$$
$$=\lim_{p\to\infty}\int\frac{d^2q}{(2\pi)^2}\langle A(\mathbf{p}'+\tfrac{1}{2}\mathbf{q})^{\text{in}}|$$
$$\times T\bigg(\sigma_0(\mathbf{x}_1)\cdots\sigma_0(\mathbf{x}_n)\exp\Big[-i\int_{-\infty}^{\infty}dt\,H_I(t)\Big]\bigg)$$
$$\times|A(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^{\text{in}}\rangle,\quad(4.8)$$

where, in QED,

$$\sigma_0(\mathbf{x})=\int_{-\infty}^{\infty}dz\,\bar{\Psi}_{\text{in}}(\mathbf{x},z;0)\gamma_0\Psi_{\text{in}}(\mathbf{x},z;0)\qquad(4.9)$$

and $H_I(t)$ is the interaction Hamiltonian in which field operators are replaced by their "in"-fields. Since $\sigma_0(\mathbf{x}_i)$'s refer to time $t=0$, Eq. (4.8) may be written more conveniently as

$$\lim_{p\to\infty}\int d^2q\langle A(\mathbf{p}+\tfrac{1}{2}\mathbf{q})^{\text{in}}|$$
$$\times\Big[T\exp\Big(-i\int_0^{\infty}dt\,H_I(t)\Big)\Big]\sigma_0(\mathbf{x}_1)\cdots\sigma_0(\mathbf{x}_n)$$
$$\times\Big[T\exp\Big(-i\int_{-\infty}^{0}dt'H_I(t')\Big)\Big]|A(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^{\text{in}}\rangle.\quad(4.10)$$

The perturbation expansion is obtained, in the usual manner, if we expand the time-ordered exponentials in Eq. (4.10) and made use of the Dyson-Wick contraction theorem. The integrations over space-time can be done most easily in the following manner: First do integrations over the third space coordinates, z's (longitudinal to p), thereby obtaining δ functions expressing conservation of longitudinal momenta; then perform integrations over time $\int_0^\infty dt$ and $\int_{-\infty}^0 dt$ implied in Eq. (4.10). There results a product of energy denominators of the form of $(E_n-E_i)^{-1}$, where E_n is the energy of an intermediate state and

$$E_i=[\mathbf{p}^2+(\tfrac{1}{2}\mathbf{q})^2+m_A{}^2]^{1/2}$$
$$\simeq p+(1/2p)[(\tfrac{1}{2}\mathbf{q})^2+m_A{}^2]\quad\text{as }p\to\infty\,,$$

m_A being the mass of the particle A; now take the limit $p\to\infty$. Under this limiting process only a subset of terms survives and has finite limits. If $\sigma(\mathbf{x})$ referred to a scalar density, no term would survive; if $\sigma(\mathbf{x})$ referred to a component of a tensor or a nonconserved vector the limit would not exist.[9] The terms that are nonvanishing in Eq. (4.10) can be represented in terms of the "impact diagrams" of Cheng and Wu.[4] Impact diagrams are time-ordered diagrams in which time runs from left to right and in which there are no vertices at which particles are created out of (or, annihilated into) vacuum. The vertices at $\mathbf{x}_1,\ldots,\mathbf{x}_n$ must be on electron or positron lines and cannot create or annihilate an electron pair. The terms which do not correspond to any of impact diagrams have energy denominators which are in general of order of p and make vanishing contributions to Eq. (4.10).

D. Separation of Positive and Negative Charges

The electron field operator $\Psi(\mathbf{x},t=0)$, $\mathbf{x}=(x,y,z)$, in the Heisenberg picture can be decomposed as

$$\Psi(\mathbf{x},0)=\sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\Big(\frac{m}{E}\Big)^{1/2}$$
$$\times[b(\mathbf{p},s)u(\mathbf{p},s)e^{i\mathbf{p}\cdot\mathbf{x}}+d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{-i\mathbf{p}\cdot\mathbf{x}}],$$

where $b(\mathbf{p},s)$ and $d(\mathbf{p},s)$ are annihilation operators, $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ are the usual spinors, and $E=(p^2+m^2)^{1/2}$. We shall write

$$\Psi^{(+)}(\mathbf{x},0)=\sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\Big(\frac{m}{E}\Big)^{1/2}b(\mathbf{p},s)u(\mathbf{p},s)e^{i\mathbf{p}\cdot\mathbf{x}},$$

$$\Psi^{(-)}(\mathbf{x},0)=\sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\Big(\frac{m}{E}\Big)^{1/2}d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

We now assert that

$$\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle$$
$$=\sum_{i_1=\pm}\sum_{i_2=\pm}\cdots\sum_{i_n=\pm}\langle\sigma^{i_1}(\mathbf{x}_1)\sigma^{i_2}(\mathbf{x}_2)\cdots\sigma^{i_n}(\mathbf{x}_n)\rangle,\quad(4.11)$$

where

$$\sigma^{\pm}(\mathbf{x})=\int_{-\infty}^{\infty}dz\,\bar{\Psi}^{(\pm)}(\mathbf{x},z;0)\gamma_0\Psi^{(\pm)}(\mathbf{x},z;0)\,,$$

i.e., σ^+ (σ^-) is the charge operator for electrons (posi-

[8] See, for example, J. D. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Chap. 17, pp. 174–184.

[9] The statement here is related to that of S. J. Chang and S. Ma [Phys. Rev. Letters 22, 1334 (1969)] that the eikonal phase vanishes for a scalar exchange, has a finite limit for a vector exchange, and diverges for an exchange of $J\geq2$. In this connection, it is well to recall that the photon impact factor of Cheng and Wu (Ref. 3) is finite, precisely because the gauge invariance assures the cancellation of divergent parts. The case of an exactly or approximately conserved axial-vector current has not been studied.

trons). In perturbation theory, Eq. (4.11) is true because terms in $\sigma(\mathbf{x})$ other than $\sigma^{\pm}$, such as terms containing bd or $b^\dagger d^\dagger$, make vanishing contributions to Eq. (4.10).

The expression (4.10) may be written as

$$\lim_{p\to\infty}\int d^2q\sum_{\alpha}\sum_{\alpha'}\langle(\mathbf{p}'+\tfrac{1}{2}\mathbf{q})^{\text{in}}|\times T\exp\left(-i\int_0^\infty dt\,H_I(t)\right)|\alpha^{\text{in}}\rangle\times\langle\alpha^{\text{in}}|\sigma_0(\mathbf{x}_1)\cdots\sigma_0(\mathbf{x}_n)|\alpha'^{\text{in}}\rangle\langle\alpha'^{\text{in}}|\times T\exp\left(-i\int_{-\infty}^0 dt\,H_I(t)\right)|(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^{\text{in}}\rangle,$$

where the summations are over a complete set of "in" states, α and α'. The matrix element

$$\langle\alpha^{\text{in}}|\sigma_0(\mathbf{x}_1)\cdots\sigma_0(\mathbf{x}_n)|\alpha'^{\text{in}}\rangle$$

may again be interpreted as

$$\langle\alpha^{\text{in}}|\sigma_0(\mathbf{x}_1)\cdots\sigma_0(\mathbf{x}_n)|\alpha'^{\text{in}}\rangle=\sum_{\alpha_1}\sum_{\alpha_2}\cdots\sum_{\alpha_{n-1}}\langle\alpha^{\text{in}}|\sigma_0(\mathbf{x}_1)|\alpha_1^{\text{in}}\rangle\times\langle\alpha_1^{\text{in}}|\sigma_0(\mathbf{x}_2)|\alpha_2^{\text{in}}\rangle\cdots\langle\alpha_{n-1}^{\text{in}}|\sigma_0(\mathbf{x}_n)|\alpha'^{\text{in}}\rangle. \quad (4.12)$$

We are interested in the case in which a number k of $\sigma(x)$'s act on the same particle which carries a positive fraction $\beta<1$ of the incoming longitudinal momentum p. In such a case the term on the right-hand side of Eq. (4.12) will have a factor

$$\begin{aligned}\lim_{p\to\infty}&\int d^3p_1\int d^3p_2\cdots\int d^3p_{k-1}\\&\times\langle e^{\mp}(\beta'\mathbf{p}+\mathbf{r}')|\sigma_0^{\pm}(\mathbf{x}_1)|e^{\mp}(\mathbf{p}_1)\rangle\\&\times\langle e^{\mp}(\mathbf{p}_1)|\sigma_0^{\pm}(\mathbf{x}_2)|e^{\mp}(\mathbf{p}_2)\rangle\\&\times\cdots\langle e^{\mp}(\mathbf{p}_{k-1})|\sigma_0^{\pm}(x_k)|e^{\mp}(\beta\mathbf{p}+\mathbf{r})\rangle\\&=(\pm)^k2\pi\delta(\beta\mathbf{p}-\beta'\mathbf{p})e^{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{x}_1}\delta^2(\mathbf{x}_1-\mathbf{x}_2)\delta^2(\mathbf{x}_2-\mathbf{x}_3)\times\cdots\delta^2(\mathbf{x}_{k-1}-\mathbf{x}_k)\\&=(\pm)^{k-1}\lim_{p\to\infty}\langle e^{\mp}(\beta'\mathbf{p}+\mathbf{r}')|\sigma_0^{\pm}(\mathbf{x}_1)|e^{\mp}(\beta\mathbf{p}+\mathbf{r})\rangle\times\delta^2(\mathbf{x}_1-\mathbf{x}_2)\cdots\delta^2(\mathbf{x}_{k-1}-\mathbf{x}_k).\end{aligned} \quad (4.13)$$

E. Cluster Property

From the above observation we infer

$$\begin{aligned}\langle\sigma^+(\mathbf{x}_1)\cdots\sigma^+(\mathbf{x}_n)\sigma^-(\mathbf{y}_1)\cdots\sigma^-(\mathbf{y}_m)\rangle&=\sum_{(N)}\sum_{(M)}\Big[\prod_{l=1}^{N}{}'\prod_{j>1}\delta^2(\mathbf{x}_1^{(l)}-\mathbf{x}_j^{(l)})\Big]\\&\times\Big[\prod_{k=1}^{M}{}'(-)^{n_k-1}\prod_{j>1}\delta^2(\mathbf{y}_1^{(k)}-\mathbf{y}_j^{(k)})\Big]\\&\times\tau(\mathbf{x}_1^{(1)},\mathbf{x}_1^{(2)}\cdots\mathbf{x}_1^{(N)};\mathbf{y}_1^{(1)}\cdots\mathbf{y}_1^{(M)}),\end{aligned} \quad (4.14)$$

where the summation $\sum_{(N)}$ is over all possible partitions of m ordered points $\mathbf{x}_1\cdots\mathbf{x}_n$ into N nonempty sets S^1, $S^2,\ldots,S^N$; $1\leq N\leq n$; $\mathbf{x}_i^{(l)}$, $i=1,\ldots,n_l$ are the points in the set S^l, $l=1,\ldots,N$; $\sum_{l=1}^N n_l=n$. The product $\prod_l{}'$ is over all sets among $S^1, S^2,\ldots,S^N$ which have more than one element. Similarly for $\sum_M$, $\mathbf{y}_i^{(k)}$, and $\prod_k{}'$. The irreducible correlation function

$$\tau(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M)$$

corresponds to the impact diagram in which the vertices at $\mathbf{x}_1\cdots\mathbf{x}_N$ act on N distinct electron lines, and the vertices at $\mathbf{y}_1\cdots\mathbf{y}_M$ act on M distinct positron lines.

It is important to note that Eq. (4.14) is a definition and at the same time a theorem. It is a definition for the irreducible correlation function $\tau(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M)$; it is a theorem for other lower-order irreducible correlation functions which are coefficients of products of δ functions.

In terms of the irreducible correlation functions, Eq. (3.9) can be written as

$$\lim_{p\to\infty}iT(p,q)=\int d^2b\,e^{-\mathbf{q}\cdot\mathbf{b}}\sum_N\sum_M\frac{(-)^M}{N!M!}\times\Big[\prod_{i=1}^N d^2x_if_+(\mathbf{x}_i+\mathbf{b})\Big]\left[\prod_{j=1}^M\int d^2y_jf_-(\mathbf{y}_i+\mathbf{b})\right]\times\tau(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M), \quad (4.15)$$

where

$$f_{\pm}(\mathbf{x})=e^{\pm i\lambda F(\mathbf{x})}-1. \quad (4.16)$$

In QED, Eq. (4.16) is the familar exponentiation of the Coulomb potential.[10] The derivation of Eq. (4.15) from Eqs. (3.9) and (4.14) may appear tedious, but involves merely an exercise in combinatorics.

For two-body elastic scattering, Eq. (3.10) may be written as

$$\lim_{p\to\infty}iT^{AB}(p,\mathbf{q})=(-)^{2S_B}2\int d^2b\,e^{-i\mathbf{q}\cdot\mathbf{b}}\sum_N\sum_M\sum_L\sum_K\frac{(-)^{M+K}}{N!M!L!K!}\left(\prod_{i=1}^N\int d^2x_i\right)\left(\prod_{j=1}^M\int d^2y_j\right)\left(\prod_{k=1}^L\int d^2z_k\right)\left(\prod_{l=1}^K\int d^2\omega_l\right)\times\tau_{\lambda'\lambda}{}^A(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M)\tau_{-\mu',-\mu}{}^B(\mathbf{z}_1\cdots\mathbf{z}_L;\omega_1\cdots\omega_K)S(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M;\mathbf{z}_1\cdots\mathbf{z}_L;\omega_1\cdots\omega_K|\mathbf{b}). \quad (4.17)$$

[10] For recent discussions, see H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); S. J. Chang and S. Ma, *ibid.* **22**, 1334 (1969); H. D. I. Abarbanel and C. Itzykson, *ibid.* **23**, 53 (1969); H. Cheng and T. T. Wu, Ref. 4.

The factor $S(\{\mathbf{x}\};\{\mathbf{y}\};\{\mathbf{z}\};\{\boldsymbol{\omega}\}|\mathbf{b})$ is defined as follows:

$$S(\{\mathbf{x}\};\{\mathbf{y}\};\{\mathbf{z}\};\{\boldsymbol{\omega}\}|\mathbf{b}) \\ =\sum\prod f_+(\mathbf{x}_i-\mathbf{z}_k+\mathbf{b})\prod f_-(\mathbf{x}_i-\boldsymbol{\omega}_l+\mathbf{b}) \\ \times\prod f_-(\mathbf{y}_j-\mathbf{z}_k+\mathbf{b})\prod f_+(\mathbf{y}_j-\boldsymbol{\omega}_l+\mathbf{b}), \quad (4.18)$$

where the summation is over all possible products of the form on the right-hand side of Eq. (4.18) satisfying (a) and (b): (a) Any particular factor $f_+(\mathbf{x}_i-\mathbf{z}_k+\mathbf{b})$, $f_-(\mathbf{x}_i-\boldsymbol{\omega}_l+\mathbf{b})$, $f_-(\mathbf{y}_j-\mathbf{z}_k+\mathbf{b})$, or $f_+(\mathbf{y}_i-\boldsymbol{\omega}_l+\mathbf{b})$ appears *at most* once in the product. (b) Each $\mathbf{x}_i$, $\mathbf{y}_j$, $\mathbf{z}_k$, and $\boldsymbol{\omega}_l$ appears *at least* once in the arguments of factors f_+, f_- in the product. Each factor f_+ or f_- may be represented by a line connecting a point in the set

$$\alpha\equiv\{\mathbf{x}_1\cdots\mathbf{x}_N,\mathbf{y}_1\cdots\mathbf{y}_M\}$$

to a point in the set

$$\beta\equiv\{\mathbf{z}_1\cdots\mathbf{z}_L,\boldsymbol{\omega}_1\cdots\boldsymbol{\omega}_K\}.$$

The summation in Eq. (4.18) is over all diagrams in which every point $a\in\alpha$ is connected to at least one point $b\in\beta$, and every point $b\in\beta$ is connected to at least one point $a\in\alpha$, and in which no pair (a,b), $a\in\alpha$, $b\in\beta$ is connected by more than one line.

The Glauber theory of nuclear diffraction scattering[11] is obtained if we identify $F(x)$ in Eq. (4.16) with a suitable two-nucleon potential, $\sigma^+(\mathbf{x})$ with the two-dimensionally projected nucleon density, and if we set

$$\tau(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M)=0 \quad \text{for } M\geq 1$$

in Eqs. (4.15) and (4.17). We shall not elaborate on this subject further in this paper, since the results are readily available in the literature.

The properties of the correlation function

$$\langle\sigma(\mathbf{x}_1)\cdots\sigma(\mathbf{x}_n)\rangle$$

we have discussed imply the following.

(i) $\tau(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M)$ is symmetric in $\mathbf{x}_i$,

$$i=1,\ldots,N, \text{ and in } \mathbf{y}_j,\ j=1,\ldots,M. \quad (4.19)$$

(ii) $\tau(\{\mathbf{x}\},\{\mathbf{y}\})$ is a real function in the sense that

$$[\tau_{c'c}(\{\mathbf{x}\},\{\mathbf{y}\})]^*=\tau_{c'c}(\{-\mathbf{x}\},\{-\mathbf{y}\}), \quad (4.20)$$

where c and c' label combination of helicity states which are TP eigenstates.

(iii) $$\int d^2z[\tau^A(\mathbf{z},\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M) \\ +\tau^A(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{z}\mathbf{y}_1\cdots\mathbf{y}_M)] \\ =(Q_A-N+M)\tau^A(\mathbf{x}_1\cdots\mathbf{x}_N;\mathbf{y}_1\cdots\mathbf{y}_M), \quad (4.21)$$

where Q_A is the charge of the particle A. Equation (4.21) follows from Eq. (4.6).

[11] R. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Britten and L. G. Dunham (Interscience, New York, 1961).

V. IMPACT FACTORS—DELBRÜCK SCATTERING

In this section we wish to demonstrate that the considerations on the operator droplet model of the previous sections produce the high-energy limit of QED in perturbation theory found by Cheng and Wu. We have already seen that the perturbative construction of the correlation functions leads naturally to the concept of the impact diagrams introduced by Cheng and Wu, and that the cluster property of the correlation functions leads to the exponentiation of the Coulomb potential in high-energy scattering. We need further to show that the droplet model gives the high-energy limit for elastic scattering which agrees *exactly* with that found in perturbation theory. This will be done by establishing the connection between the correlation functions of the droplet model and the impact factors of Cheng and Wu.

Consider first the elastic scattering of an electron in an external field. In the approximation of retaining only the term with $N=1$, $M=0$ in Eq. (4.18), we obtain

$$\lim_{p\to\infty} iT^e(p,\mathbf{q})\cong\int d^2b\, e^{-i\mathbf{q}\cdot\mathbf{b}}\int d^2x\, f_+(\mathbf{x}+\mathbf{b})\tau_{\lambda'\lambda}{}^e(\mathbf{x};), \quad (5.1)$$

where

$$2\pi\delta(p-p')\tau_{\lambda'\lambda}{}^e(\mathbf{x};) \\ =\lim_{p\to\infty}\int d^2\mathbf{q}\,\langle e^-(\mathbf{p}'+\tfrac{1}{2}\mathbf{q},\lambda')| \\ \times\int dz\, j_0(\mathbf{x},z;0)|e^-(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\lambda)\rangle \\ =\int dz\, e^{i(p'-p)z}\lim_{p\to\infty}\int d^2q \\ \times e^{+i\mathbf{q}\cdot\mathbf{x}}\langle e^-(\mathbf{p}'+\tfrac{1}{2}\mathbf{q},\lambda')|j_0(0)|e^-(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\lambda)\rangle. \quad (5.2)$$

Therefore, we obtain

$$\tau_{\lambda'\lambda}{}^e(\mathbf{x};)=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{x}}\frac{1}{(2\pi)^3} \\ \times[F_1(q^2)-(i/m)F_2(q^2)\boldsymbol{\sigma}\cdot(\hat{\mathbf{p}}\times\mathbf{q})]_{\lambda'\lambda}, \quad (5.3)$$

which expresses the correlation function $\tau(\mathbf{x};)$ in terms of form factors F_1 and F_2. If we define the Fourier transform of $f_\pm$

$$f_\pm(\mathbf{x})=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{f}_\pm(\mathbf{q}),$$

we obtain

$$\lim_{p\to\infty} iT^e(p,\mathbf{q})=(2\pi)^{-3}\tilde{f}_+(\mathbf{q}) \\ \times[F_1(q^2)-(i/m)F_2(q^2)\boldsymbol{\sigma}\cdot(\hat{\mathbf{p}}\times\mathbf{q})]_{\lambda'\lambda}. \quad (5.4)$$

Next let us consider the elastic scattering of a photon in an external potential. To order α, only the irreducible correlation functions with $N\leq 1$, $M\leq 1$ are nonzero, so that Eq. (4.18) becomes

$$\lim_{p\to\infty} iT^{\gamma}(\mathbf{p},\mathbf{q})\sim\int d^2b\, e^{-i\mathbf{q}\cdot\mathbf{b}}$$
$$\times\int d^2x\int d^2y[-f_+(\mathbf{x}+\mathbf{b})f_-(\mathbf{y}+\mathbf{b})\tau(\mathbf{x};\mathbf{y})$$
$$+\delta^2(\mathbf{x}-\mathbf{y})\tau(\mathbf{x};)f_+(\mathbf{x}+\mathbf{b})$$
$$-\delta^2(\mathbf{x}-\mathbf{y})\tau(;\mathbf{y})f_-(\mathbf{y}+\mathbf{b})]_{\lambda'\lambda}. \quad (5.5)$$

Making use of the relations

$$\tau(\mathbf{x};)=-\tau(;\mathbf{x}),$$

as follows from the charge conjugation properties $C\sigma^+(\mathbf{x})C^{-1}=-\sigma^-(\mathbf{x})$ and $C|\gamma\rangle=-|\gamma\rangle$, and

$$f_+(\mathbf{x})+f_-(\mathbf{x})=-f_+(\mathbf{x})f_-(\mathbf{x}),$$

we rewrite Eq. (5.5) in the form

$$\lim_{p\to\infty} iT^{\gamma}(p,\mathbf{q})=-\int d^2b\, e^{-i\mathbf{q}\cdot\mathbf{b}}$$
$$\times\int d^2x\, f_+(\mathbf{x}+\mathbf{b})\int d^2y\, f_-(\mathbf{y}+\mathbf{b})I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y}), \quad (5.6)$$

where

$$I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=\tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})+\delta^2(\mathbf{x}-\mathbf{y})\tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{x};). \quad (5.7)$$

The configuration "impact factor" $I(\mathbf{x},\mathbf{y})$ satisfies the condition

$$\int d^2\mathbf{x}\, I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=0,$$
$$\int d^2y\, I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=0, \quad (5.8)$$

because, according to Eq. (4.24),

$$\int d^2x\, \tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=\tau_{\lambda'\lambda}{}^{\gamma}(;\mathbf{y})=-\tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{y};),$$
$$\int d^2y\, \tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=-\tau_{\lambda'\lambda}{}^{\gamma}(\mathbf{x};).$$

We note further that, to order α,

$$I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=\langle\sigma^+(\mathbf{x})\sigma^-(\mathbf{y})\rangle_{\lambda'\lambda}{}^{\gamma}+\langle\sigma^+(\mathbf{x})\sigma^+(\mathbf{y})\rangle_{\lambda'\lambda}{}^{\gamma}. \quad (5.9)$$

We now define the Fourier transform of $I(x,y)$:

$$\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})=\int d^2X\, e^{i\mathbf{q}\cdot\mathbf{X}}$$
$$\times\int d^2Y\, e^{i\mathbf{r}\cdot\mathbf{Y}}I_{\lambda'\lambda}{}^{\gamma}(\mathbf{X}+\tfrac{1}{2}\mathbf{Y},\mathbf{X}-\tfrac{1}{2}\mathbf{Y}), \quad (5.10)$$

or

$$I_{\lambda'\lambda}{}^{\gamma}(\mathbf{x},\mathbf{y})=\int\frac{d^2q}{(2\pi)^2}\int\frac{d^2r}{(2\pi)^2}$$
$$\times e^{-i(\mathbf{r}+\mathbf{q}/2)\cdot\mathbf{x}-i(-\mathbf{r}+\mathbf{q}/2)\cdot\mathbf{y}}\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r}).$$

We assert that $\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})$ is (perhaps to within a numerical factor) exactly the impact factor of Cheng and Wu. First, the scattering amplitude (5.6) is written in momentum space as

$$\lim_{p\to\infty} iT_{\lambda'\lambda}{}^{\gamma}(\mathbf{p},\mathbf{q})$$
$$=-\int\frac{d^2r}{(2\pi)^2}\tilde{f}_+(\mathbf{r}+\tfrac{1}{2}\mathbf{q})\tilde{f}_-(-\mathbf{r}+\tfrac{1}{2}\mathbf{q})\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r}), \quad (5.11)$$

which agrees with the result of Cheng and Wu as to the dependence of the high-energy amplitude on the impact factor. We note further that

$$\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\tfrac{1}{2}\mathbf{q})=\int d^2x\, e^{i\mathbf{q}\cdot\mathbf{x}}\int d^2y\, I_{\lambda'\lambda}{}^{\gamma}(x,y)$$
$$=0,$$

owing to the second of Eqs. (5.8). The first and the second of Eqs. (5.8) imply

$$\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\pm\tfrac{1}{2}\mathbf{q})=0, \quad (5.12)$$

which are consequences of charge conservation. To show in detail that $\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})$ we defined is identical with the quantity Cheng and Wu computed, we begin by substituting Eq. (5.9) into Eq. (5.10) and recalling Eq. (3.7):

$$(2\pi)\delta(p-p')\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})$$
$$=\lim_{p\to\infty}\int d^2X\, e^{i\mathbf{q}\cdot\mathbf{X}}\int d^2Y\, e^{i\mathbf{r}\cdot\mathbf{Y}}\int d^2q'$$
$$\times\langle\gamma(\mathbf{p}'-\tfrac{1}{2}\mathbf{q}',\lambda')|\sigma^+(\mathbf{X}+\tfrac{1}{2}\mathbf{Y})\sigma^-(\mathbf{X}-\tfrac{1}{2}\mathbf{Y})$$
$$+\sigma^+(\mathbf{X}+\tfrac{1}{2}\mathbf{Y})\sigma^+(\mathbf{X}-\tfrac{1}{2}\mathbf{Y})|\gamma(\mathbf{p}+\tfrac{1}{2}\mathbf{q}',\lambda)\rangle. \quad (5.13)$$

Making use of the translational invariance of the matrix element, we obtain

$$(2\pi)\delta(p-p')\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})$$
$$=\lim_{p\to\infty}\int d^2z\, e^{i\mathbf{r}\cdot\mathbf{z}}\langle\gamma(\mathbf{p}'-\tfrac{1}{2}\mathbf{q},\lambda')|\sigma^+(\tfrac{1}{2}\mathbf{z})\sigma^-(-\tfrac{1}{2}\mathbf{z})$$
$$+\sigma^+(\tfrac{1}{2}\mathbf{z})\sigma^+(-\tfrac{1}{2}\mathbf{z})|\gamma(\mathbf{p}+\tfrac{1}{2}\mathbf{q},\lambda)\rangle. \quad (5.14)$$

If we now apply Eq. (4.10) and the discussion following it in Sec. IV to the right-hand side of Eq. (5.14), the result to order α is precisely that of Cheng and Wu computed in terms of impact diagrams:

$$\tilde{I}_{\lambda'\lambda}{}^{\gamma}(\mathbf{q},\mathbf{r})=\epsilon_i{}^*(\lambda')\tilde{I}_{ij}{}^{\gamma}(\mathbf{q},\mathbf{r})\epsilon_j(\lambda)\,,\quad \epsilon(\pm1)=(1/\sqrt{2})(\mp\hat{e}_x-i\hat{e}_y)\,, \tag{5.15}$$

$$\tilde{I}_{ij}{}^{\gamma}(\mathbf{q},\mathbf{r})\sim\alpha\int\frac{d^2p}{(2\pi)^2}\int_0^1 d\beta\{[\tfrac{1}{4}\delta_{ij}\beta^2\mathbf{q}^2+2\beta(1-\beta)(p_ip_j-\tfrac{1}{4}\beta^2q_iq_j)][(\mathbf{p}-\tfrac{1}{2}\beta\mathbf{q})^2+m^2]^{-1}[(\mathbf{p}+\tfrac{1}{2}\beta\mathbf{q})^2+m^2]^{-1}$$
$$-[\delta_{ij}Q^2+2\beta(1-\beta)(p_ip_j-Q_iQ_j)][(\mathbf{p}-\mathbf{Q})^2+m^2]^{-1}[(\mathbf{p}+\mathbf{Q})^2+m^2]^{-1}\}\,, \tag{5.16}$$

where m is the electron mass and $\mathbf{Q}=\frac{1}{2}\mathbf{r}+\frac{1}{4}(1-2\beta)\mathbf{q}$.

It is instructive to study the impact factor in the configuration space. After some calculation, the spin-nonflip impact factor $I^{\gamma}(\mathbf{x},\mathbf{y})$ turns out to be

$$I^{\gamma}(\mathbf{x},\mathbf{y})=\tau^{\gamma}(\mathbf{x};\mathbf{y})+\delta^2(\mathbf{x}-\mathbf{y})\tau^{\gamma}(\mathbf{x};),$$

$$\tau^{\gamma}(\mathbf{x};\mathbf{y})\sim\int_0^1 d\beta\delta^2[\beta\mathbf{x}+(1-\beta)\mathbf{y}]$$
$$\times\{(1-2\beta+2\beta^2)[\boldsymbol{\nabla}K_0(m|\mathbf{x}-\mathbf{y}|)]^2$$
$$-K_0(m|\mathbf{x}-\mathbf{y}|)\nabla^2K_0(m|\mathbf{x}-\mathbf{y}|)\}$$

$$\tau^{\gamma}(\mathbf{x};)=-\tau^{\gamma}(;\mathbf{x})$$
$$\sim\int_0^1 d\beta\left\{(1-2\beta+2\beta^2)\left[\boldsymbol{\nabla}K_0\left(\frac{m}{\beta}|\mathbf{x}|\right)\right]^2\right.$$
$$\left.-K_0\left(\frac{m}{\beta}|\mathbf{x}|\right)\nabla^2K_0\left(\frac{m}{\beta}|\mathbf{x}|\right)\right\}, \tag{5.17}$$

where $K_0(z)$ is the usual, modified Bessel function. The δ function in Eq. (5.16) expresses the center-of-mass effect that the electron and positron (virtual) must be found in opposite directions along a line passing through the center of the photon. Equations (5.16) and (5.17) tell us that *the radius of the photon* as seen by an external electromagnetic field *is* $1/2m_e$.

The foregoing example is instructive in that this provides us with a guide in understanding the mechanism of scattering of hadrons like pions which are neutral in baryonic charge. As we have stated in Sec. II, our view is that the hadronic diffraction scattering proceeds much the same way as the diffraction scattering in QED with the baryonic charge playing the role of the electric charge. In this picture, the high-energy interaction of pions is due to the polarizability of positive and negative baryonic charges. A phenomenological analysis along this line of thought is in progress.

VI. CONCLUSION

One may say that all we have done is to take the operator formulation of the droplet model seriously, and work out some consequences of the model. We have in particular found that the model gives the high-energy limit of elastic scattering which agrees exactly with that found in perturbation theory of QED.

But that would be losing the perspective. I hope that what we have done is to lay the foundation of a theory of high-energy scattering of hadrons formulated in configuration space in terms of correlations of charge densities. We do not know what is in store for this theory, but we must find out by looking at its predictions as to elastic scattering, diffractive dissociations, multipion emissions, etc., and testing the unitarity aspect of the theory. We are gratified to know that this theory passed one test—that of QED.

We must end this paper with a critical appraisal of the possible limitation of the droplet model. First, we recall that Cheng and Wu[12] found certain higher-order diagrams (eighth order and higher) to have the $s(\ln s)^n$, $n\geq 1$, behavior at high energies. The absorptive parts of these diagrams are typically those, or similar to those diagrams discussed in the multiperipheral model. The droplet model is incapable of describing the processes described by the multiperipheral model.[13] Since the $s(\ln s)^n$ behavior is not allowed by the Froissart-Martin bound, these diagrams must sum up in some sensible manner in order not to violate this bound. When they are summed, the effects of these diagrams may or may not dominate over the effects described adequately by the droplet model.

If these multiperipheral graphs in some generalized sense are more important at high energies, then the secondaries in high-energy collisions are produced through "pionization," and the droplet model will be inadequate to describe high-energy elastic scattering. On the other hand, if the sum of multiperipheral graphs actually has a slower rate of growth at high energies than those described by the droplet model (or the impact picture of Cheng and Wu), then the secondaries in high-energy collisions may best be looked at as fragmentations of the target and the projectile. That the latter possibility prevails is a basic premise of the droplet model.

[12] H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 1405 (1969).

[13] In the sense that such diagrams as are usually discussed in the multiperipheral model give rise to divergences in the momentum-space correlation functions (or impact factors), indicating that such Feynman diagrams have the asymptotic behavior $s(\ln s)^n$, $n\geq 1$. This is not to say that some suitable modification of the method discussed in this paper would not be able to overcome the difficulty associated with the presence of a logarithmic factor in the asymptotic behavior of the full amplitude. However, we have not considered such matters in detail in the present paper. I am indebted to Dr. T. T. Chou and Dr. R. Suitor for a discussion on this point.