get the following expression for the two-body amplitude:

$$-[i/(2\pi)^4]\delta^{(4)}(q_1-q_2)$$
$$\times(p_1+\tfrac{1}{2}q_1,\,p_2-\tfrac{1}{2}q_2|T|p_1-\tfrac{1}{2}q_1,\,p_2+\tfrac{1}{2}q_2)$$
$$\underset{m_2\text{ large}}{\simeq}-2im_2\int\frac{d^4x_2}{(2\pi)^4}\delta(x_2\cdot n_2)e^{iq_2x_2}(p_1+\tfrac{1}{2}q_1|T$$
$$\times\left[-ie_2\int_{-\infty}^{+\infty}dt\,D^{\mu0}(x_2-x_1-n_2t)\right]|p_1-\tfrac{1}{2}q_1).$$

Let us observe that

$$-ie_2\int_{-\infty}^{+\infty}dt\,D^{\mu0}(x_2-x_1-n_2t)=\frac{e_2}{4\pi}\frac{g^{\mu0}}{|\mathbf{x}_2-\mathbf{x}_1|}.$$

Using the time independence of this result and performing the translation $\mathbf{x}_1\to\mathbf{x}_1+\mathbf{x}_2$, we can factor out $\delta(q_1{}^0)e^{i\mathbf{q}_1\cdot\mathbf{x}_2}$. The integral over x_2 gives finally $\delta(q_1{}^0)\times\delta^{(3)}(\mathbf{q}_1-\mathbf{q}_2)=\delta^{(4)}(q_1-q_2)$, since $q_2{}^0=0$. Therefore the two-body amplitude, in the limit m_2 large, is proportional to the scattering amplitude of the first particle in the static field, $A^0(x_1)=(e_2/4\pi)(1/|x_1|)$, produced by the second particle in its rest frame. If we are interested in finding the bound states, the problem reduces to solving the one-body Klein-Gordon equation

$$\left[\left(p_1{}^0-\frac{e_1e_2}{4\pi}\frac{1}{|x_1|}\right)^2-(p_1)^2-m_1{}^2\right]\psi=0.$$

For spin-$\frac{1}{2}$ particles the same argument can be repeated. In this case, it leads to the Dirac amplitude of the light particle in the Coulomb field produced by the heavy one. But another effect can be included here, if one regards the magnetic moment of the heavy particle as a nonvanishing static property. Then, in addition to the Coulomb field, the heavy particle produces a magnetic field responsible for the hyperfine splitting of the Dirac levels. We then have to add an extra term in the potential equal to $(e_2/2m_2)\sigma_2{}^{\mu\nu}F_{\mu\nu}{}^2$, where we can even replace $e_2/2m_2$ by the actual magnetic moment $\mu_2=(e_2/2m_2)(1+x)$. In the same spirit as above, we have to neglect the noncommutativity of the spin matrices and set $\sigma_2{}^{0\nu}\sim0$ as if the operators were replaced by their mean values. Following the previous recipe, we find that particle one moves in a static field with four-potential given by

$$A_0(\mathbf{x}_1)=\frac{e_2}{4\pi|\mathbf{x}_1|},\quad \mathbf{A}(\mathbf{x}_1)=\mu_2\,\mathrm{curl}_{\mathbf{x}_1}\left(\frac{\boldsymbol{\sigma}_2}{4\pi|\mathbf{x}_1|}\right).$$

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Parameter-Free Regularization of One-Loop Unitary Dual Diagram*

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We propose a parameter-free regularization of the one-loop planar unitary dual diagram, very similar to the renormalization in quantum field theory, compatible with duality, Regge behavior, unitarity, and crossing symmetry.

I. INTRODUCTION

RECENTLY, there have been several attempts[1–3] to put the Veneziano model on the same footing as quantum electrodynamics, by considering the N-point functions as tree diagrams of the theory. This leads to a finite result when factorization problems are neglected.[1] However, when factorization is taken into account, the one-loop diagram exhibits an exponential divergence due to the large degeneracy of levels.[2,3]

It would then seem that a renormalization procedure would need an infinite number of subtractions, and thus an infinite number of parameters. However, we show that in the case of the one-loop diagram, the divergent part of the new trajectory, $\alpha_t+g^2\Sigma(\alpha_t)$, is independent of t and can be removed by a single subtraction. This can be extended to the amplitude itself and one finds that the subtraction of a crossing-symmetric, Regge-behaved, dual amplitude, having only single and double poles in the external variables s and t, is enough to make it finite. We then find that the renormalized amplitude still Reggeizes at $s\to-\infty$. We have extended this to the case of the one-loop diagram with N external legs.

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[1] K. Kikkawa, B. Sakita, and M. Virasoro, Phys. Rev. **184**, 1701 (1969); K. Kikkawa, S. Klein, B. Sakita, and M. Virasoro, University of Wisconsin Report No. 248, 1969 (unpublished).
[2] D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).
[3] K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969).

Our regularization technique is very different from any procedure using finite cutoffs, which essentially amounts to the subtraction of an entire function of s and t.

Because we subtract a sum of poles and double poles, our process is rather similar to the renormalization of second-order diagrams in field theory. So, one may hope that the higher-order diagrams can be made finite by the counter terms thus defined.

II. FORMAL REGGEIZATION OF SECOND-ORDER FOUR-POINT FUNCTION AT $s\to-\infty$

We start with the second-order four-point function as computed in Refs. 1 and 2:

$$F(p_1,p_2,p_3,p_4)=-ig^4\int d^4k\int_0^1 dx_1dx_2dx_3dx_4 \times x_1^{-\alpha(k^2)-1}\cdots x_4^{-\alpha(k_3^2)-1}\prod_{i=1}^4(1-x_i)^{-c}q_0^{-4}(\sqrt{w}) \times\exp\left(-\sum_n\sum_{i,j=1}^4\frac{C_{ij}{}^np_ip_j}{n(1-w^n)}\right),\quad(1)$$

where

$$q_0(Z)=\prod_{n=1}^\infty(1-Z^{2n}),$$

$$w=x_1x_2x_3x_4,$$

$$\begin{aligned}C_{ij}&=x_1\cdots x_i &&\text{if } j=4\\ &=(x_1\cdots x_i)(x_{j+1}\cdots x_4) &&\text{if } i\le j\le 3\\ &=x_{j+1}\cdots x_i &&\text{if } i>j,\end{aligned}$$

$$\alpha(k^2)=\alpha_0-\tfrac{1}{2}k^2.$$

When all the relations found by Fubini and Veneziano are taken into account (Ref. 3), one finds the same expression except for an extra $(1-w)$ factor. This alteration has no consequences for the rest of this paper.

Expression (1) diverges near $w\to1$ as $\exp[\frac{2}{3}\pi^2/(1-w)]$. We introduce the temporary cutoff $\theta(4-\epsilon-x_1-x_2-x_3-x_4)$. This cutoff, as already noticed, does not change the imaginary part and preserves Reggeization at $s\to-\infty$.

After Wick rotation and integration over d^4k, one finds

$$F(\alpha_s,\alpha_t)=4\pi^2g^4\int_0^1\prod_{i=1}^4 dx_ix_i^{-\alpha_0-1}(1-x_i)^{-c}\frac{q_0^{-4}}{\ln^2w}e^{\alpha_0h}T_1^{-\alpha_s} \times T_2^{-\alpha_t}\theta(4-\epsilon-x_1-x_2-x_3-x_4),\quad(2)$$

where

$$h(x_i)=\frac{1}{2}\left(\ln w-\sum_i\frac{\ln^2x_i}{\ln w}\right)+\sum_n\sum_i\frac{-8w^n+x_i{}^n+w^n/x_i{}^n}{n(1-w^n)},$$

$$T_1(x_1,x_2,x_3,x_4)=\exp\frac{\ln x_1\ln x_3}{\ln w}-\sum_{n=1}^\infty\frac{(1-x_1{}^n)(1-x_3{}^n)(x_2{}^n+x_4{}^n)}{n(1-w^n)},\quad(3)$$

$$T_2(x_1,x_2,x_3,x_4)=T_1(x_2,x_1,x_4,x_3),\quad(4)$$

$$\alpha_s=\alpha_0-\tfrac{1}{2}(p_1+p_2)^2.$$

e^{α_0h}, T_1, and T_2 are well-behaved functions when $w\to1$. When $s\to-\infty$, $F(\alpha_s,\alpha_t)$ Reggeizes and one finds, neglecting terms of order $(-\alpha_s)^{\alpha_t-1}\ln s$,

$$F(\alpha_s,\alpha_t)=g^4(-\alpha_s)^{\alpha_t}[\ln(-\alpha_s)\Gamma(-\alpha_t)\Sigma(\alpha_t)+\Gamma(-\alpha_t)\beta(t)-\Gamma'(-\alpha_t)\Sigma(\alpha_t)]+O(\alpha_s{}^{\alpha_t-1}\ln s).\quad(5)$$

This is the term in g^4 of the expansion in powers of g^2 of

$$g^2\beta_{\text{new}}(t)\Gamma(-\alpha_{\text{new}})(-\alpha_s)^{\alpha_{\text{new}}(t)},$$

where

$$\alpha_{\text{new}}(t)=\alpha_t+g^2\Sigma(\alpha_t)+O(g^4)$$

is the new trajectory,

$$\beta_{\text{new}}(t)=1+g^2\beta(t)+O(g^4)$$

is the new residue,

$$\Sigma(\alpha_t)=4\pi^2\int_0^1\frac{dx_2dx_4}{\ln^2x_2x_4}(x_2x_4)^{-\alpha_0-1}(1-x_2)^{-c}(1-x_4)^{-c} \times e^{\alpha_0h_1}q_0^{-4}(x_2x_4)^{1/2}e^{\alpha_tH(x_2,x_4)} \times\theta(2-\epsilon-x_2-x_4),\quad(6)$$

$$h_1=\frac{\ln x_2\ln x_4}{\ln x_2x_4}+2\sum_n\frac{-2x_2{}^nx_4{}^n+x_2{}^n+x_4{}^n}{n(1-x_2{}^nx_4{}^n)},$$

and

$$H(x_2,x_4)=2\sum_{n=1}^\infty\frac{2x_2{}^nx_4{}^n-x_2{}^n-x_4{}^n}{n(1-x_2{}^nx_4{}^n)}-\frac{\ln x_2\ln x_4}{\ln x_2x_4}+\ln\left(\sum_{n=1}^\infty\frac{n(x_2{}^n+x_4{}^n)}{1-x_2{}^nx_4{}^n}-\frac{1}{\ln x_2x_4}\right).\quad(7)$$

$\beta(t)$ has a similar expression.

$\Sigma(\alpha_t)$ is also exponentially divergent near $x_2x_4\simeq1$, and we study the properties of this integral in Sec. III.

III. REGULARIZATION OF $\Sigma(\alpha_t)$

$e^{H(x_2,x_4)}$ is given in terms of T_1 and T_2 by

$$e^{H(x_2,x_4)}=-\frac{\partial^2}{\partial x_1\partial x_3}T_1\Big|_{x_1=x_3=1}\left(\frac{\partial^2}{\partial x_1\partial x_3}T_2\Big|_{x_1=x_3=1}\right)^{-1}.\quad(8)$$

Using the theory of elliptic functions, it is shown in the

Appendix that T_1 can be written

$$T_1=\frac{\sin(\pi\ \ln x_2/\ln w)\ \sin(\pi\ \ln x_4/\ln w)}{\sin(\pi\ \ln x_1x_2/\ln w)\ \sin(\pi\ \ln x_1x_4/\ln w)}\times\prod_{n=1}^{\infty}\frac{[1-2q^{2n}\cos(2\pi\ \ln x_2/\ln w)+q^{4n}][1-2q^{2n}\cos(2\pi\ \ln x_4/\ln w)+q^{4n}]}{[1-2q^{2n}\cos(2\pi\ \ln x_1x_2/\ln w)+q^{4n}][1-2q^{2n}\cos(2\pi\ \ln x_1x_4/\ln w)+q^{4n}]},\quad(9)$$

where $q=e^{2\pi^2/L nw}$. Notice that the infinite product is equal to 1 up to a null asymptotic expansion in $1-w$. More precisely, it is equal to $1+O(e^{-2\pi^2/(1-w)})$ near $w\simeq1$. By computing $e^{H(x_2,x_4)}$, one finds that the sines drop out and that $e^{H(x_2,x_4)}=1+O(e^{-4\pi^2/(1-x_2x_4)})$ near $x_2x_4\simeq1$.

Thus, the divergent part of $\Sigma(\alpha_t)$ is independent of t when $\epsilon\to0$. Moreover, a single subtraction is enough to make $\Sigma(\alpha_t)$ finite.

$\Sigma(\alpha_t)$ appears as the renormalization of the leading trajectory. Since the scalar external particles belong to it and are stable, we demand that their mass be the physical mass. Thus, we subtract the divergent quantity $\Sigma(0)$.

Since

$$q_0^{-4}(x_2x_4)^{1/2}[e^{\alpha_t H(x_2,x_4)}-1]=O\left(\exp\left(-\frac{10}{3}\frac{\pi^2}{1-x_2x_4}\right)\right)\quad\text{as}\quad x_2x_4\to1,$$

$\Sigma(\alpha_t)-\Sigma(0)$ is now given by a convergent integral for t below threshold. Above threshold, it can be shown by using standard techniques that $\Sigma(\alpha_t)-\Sigma(0)$ has branch points for every two-particle threshold and pseudo-threshold.[4]

All this regularization supposes that there is only one stable particle in the theory, or equivalently, that the intercept α_0 of the leading trajectory is not too negative. If there were more than one stable particle, we would have to add new parameters in the theory from the beginning, for instance by adding satellites, in order to fit the renormalized values of the masses of stable particles with their physical values.

IV. RENORMALIZATION OF AMPLITUDE $F(\alpha_s,\alpha_t)$

Following the guidelines of the usual Feynman graphs renormalization, we look for a function $\tilde{F}(\alpha_s;\alpha_t)$ obeying the following conditions:

(1) $F-\tilde{F}$ is finite for the cutoff ϵ going to zero.

(2) $\tilde{F}$ is crossing symmetric and in each channel can be written as a sum of single and double poles, the residues at the poles being polynomials in the other variable.

(3) $F-\tilde{F}$ Reggeizes at $s\to-\infty$.

(4) $\tilde{\Sigma}(\alpha_t)=\Sigma(0)$ so that $F-\tilde{F}$ behaves like $g^4(-\alpha_s)^{\alpha_t}\times\ln(-\alpha_s)\ \Gamma(-\alpha_t)[\Sigma(\alpha_t)-\Sigma(0)]$ when $s\to-\infty$.

Notice that in expression (9) the poles and the double poles in the s channel arise from the behavior of the sines when $x_2=1$ or $x_4=1$, whereas the infinite product gives rise to the cut when $w\to0$.

We are thus led to define a function $\tilde{F}(\alpha_s,\alpha_t)$ by Eq. (2) except that T_1 and T_2 are replaced by

$$\tilde{T}_1=\frac{\sin(\pi\ \ln x_2/\ln w)\ \sin(\pi\ \ln x_4/\ln w)}{\sin(\pi\ \ln x_1x_2/\ln w)\ \sin(\pi\ \ln x_1x_4/\ln w)},\quad(10)$$

$$\tilde{T}_2=\tilde{T}_1(x_2,x_1,x_4,x_3).$$

$F-\tilde{F}$ is evidently finite when $\epsilon\to0$, because near $w=1$

$$\tilde{T}_1-T_1=O(e^{4\pi^2/\ln w}).$$

$\tilde{F}$ has poles and double poles located like those of F. $\tilde{T}_1$ and $\tilde{T}_2$ have the following property:

$$\tilde{T}_2=1-\tilde{T}_1.$$

From the set of variables x_i, we change to the set x_1, x_3, z_2, and z_4, definied by[5]

$$z_2=\frac{\sin(\frac12\pi\ \ln x_2/\ln w)\ \sin(\frac12\pi\ \ln x_1x_2x_3/\ln w)}{\sin(\frac12\pi\ \ln x_2x_1/\ln w)\ \sin(\frac12\pi\ \ln x_2x_3/\ln w)},$$
$$z_4=\frac{\sin(\frac12\pi\ \ln x_4/\ln w)\ \sin(\frac12\pi\ \ln x_3x_4x_1/\ln w)}{\sin(\frac12\pi\ \ln x_1x_4/\ln w)\ \sin(\frac12\pi\ \ln x_3x_4/\ln w)},\quad(11)$$

so that

$$0\le z_2\le1,\quad 0\le z_4\le1$$

and

$$\tilde{T}_1=z_4z_2,\quad \tilde{T}_2=1-z_2z_4.$$

By expanding in powers of z_2 and z_4, and integrating over z_2 and z_4, we find that $\tilde{F}$ is of the form

$$\tilde{F}(\alpha_s,\alpha_t)=g^4\sum_{n,p,q\ge0}\frac{1}{n+p-\alpha_s}\frac{1}{n+q-\alpha_s}\frac{\Gamma(\alpha_t+n+1)}{n!\Gamma(\alpha_t+1)}\times\int_0^1 dx_2dx_4\tilde{J}_{pq}(x_2,x_4).\quad(12)$$

This shows that $\tilde{F}$ has the very structure of a counter-

[4] For $t\to-\infty$ one can show that $\Sigma(\alpha_t)$ behaves like $(-t)^{1/6}$ up to logarithmic factors. Compare with

$$\int_{-\infty}^{+\infty}e^{bx}(e^{te^{-ax}}-1)dx=(1/a)\Gamma(-b/a)(-t)^{b/a}\quad\text{for } a>b>0.$$

[5] This change is somewhat similar to the one defined by Eq. (3.7) in Ref. 1.

term in the usual renormalization theory. As for F itself, the same change of variables casts it into the form

$$F(\alpha_s,a_t)=g^4 \sum_{n,p,q\geq 0} \frac{1}{n+p-\alpha_s}\frac{1}{n+q-\alpha_s} \times P_n(\alpha_s,p,q|\alpha_t), \quad (13)$$

where $P_n(\alpha_s,p,q|\alpha_t)$ is a polynomial of degree n in α_t, whose coefficients are functions of p, q, and α_s, and have unitarity cuts in the α_s plane.

One can check that $\tilde{F}$ Reggeizes for $s\to -\infty$,[6] and that the equality $\tilde{\Sigma}(\alpha_t)=\Sigma(0)$ is verified.

Notice that $\tilde{F}$ is not uniquely determined by conditions (1)–(4). One can always add to F a finite dual function, behaving like $(-\alpha_s)^{\alpha_t-1}$. Although we could not prove it, it seems very likely that it amounts to a change on the Born term. One encounters a similar problem in Lagrangian field theory for the definition of counterterms.

V. REGULARIZATION OF N-POINT FUNCTION

We integrate first the expression of $F_1^{(N)}(p_1,\ldots,p_N)$ over d^4k. We express then the p_ip_j in terms of the variables[7]

$$\alpha_{ij}=\alpha_0-\tfrac{1}{2}(p_i+\cdots+p_j)^2,\quad i<j.$$

Then

$$F_1^{(N)}(p_1\cdots p_N)=4\pi^2g^N\int_0^1\prod_{i=1}^N dx_i \prod_{i,j;\ i<j} T_{ij}^{-\alpha_{ij}} \times F_N(x_i)\theta(N-\epsilon-\sum x_i), \quad (14)$$

where $F_N(x_i)$ is independent of the α_{ij}.

We call T_{ij} the conjugate variable to the trajectory α_{ij}. One finds (see the Appendix) that

$$T_{ij}=\theta_1\left(\frac{\nu_{ij}}{\tau}\bigg|-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j+1}}{\tau}\bigg|-\frac{1}{\tau}\right)\Bigg/ \theta_1\left(\frac{\nu_{i,j+1}}{\tau}\bigg|-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j}}{\tau}\bigg|-\frac{1}{\tau}\right). \quad (15)$$

Keeping only the sines in the expansion of θ_1, we define $\tilde{F}_1^{(N)}(p_1\cdots p_N)$ by (14) with T_{ij} replaced by

$$\tilde{T}_{ij}=\frac{\sin(\pi\ln x_{i+1}\cdots x_j/\ln w)\sin(\pi\ln x_i\cdots x_{j+1}/\ln w)}{\sin(\pi\ln x_{i+1}\cdots x_{j+1}/\ln w)\sin(\pi\ln x_i\cdots x_j/\ln w)}. \quad (16)$$

$F_1^{(N)}-\tilde{F}_1^{(N)}$ is evidently finite for $\epsilon\to 0$. Let us look now at the factorization properties of $\tilde{F}_1^{(N)}$.

We have seen in the case $N=4$ that $\tilde{T}_2=1-\tilde{T}_1$; this can be generalized, and the $\tilde{T}_{ij}$ satisfy the general equation of duality:

$$T_{ij}=1-\prod_{p,q}T_{p,q}\prod_{r,s}T_{r,s},$$
$$1\leq p<i,\quad i\leq q<j,\quad i<r\leq j,\quad j<s\leq N-1. \quad (17)$$

This can be shown from the fact that

$$1-T_{ij}=\frac{\sin(\pi\ln x_i/\ln w)\sin(\pi\ln x_{j+1}/\ln w)}{\sin(\pi\ln x_{i+1}\cdots x_{j+1}/\ln w)\sin(\pi\ln x_i\cdots x_j/\ln w)}$$

and that

$$T_{p,q}T_{p,q+1}\cdots T_{p,q+r} =\frac{\sin(\pi\ln x_{p+1}\cdots x_q/\ln w)\sin(\pi\ln x_p\cdots x_{q+r+1}/\ln w)}{\sin(\pi\ln x_p\cdots x_q/\ln w)\sin(\pi\ln x_{p+1}\cdots x_{q+r+1}/\ln w)}.$$

The T_{ij} can be expressed in terms of the $N-3$ T_{ij} $(2\leq j\leq N-2)$ by the usual formulas. To see the poles and double poles in the $\alpha_{1,p}$ channel, we write, as previously, $T_{1,p}=z_1z_2$ and we choose two other independent variables ζ_1, ζ_2.

After development in powers of z_1^r, z_1^q and integration over z_1, z_2, ζ_1, ζ_2, we get

$$F_1^{(N)}=\sum_{\{m\}}\sum_{r,s=0}^{+\infty}\int_0^1\prod_{i=1;\ i\neq p}^{N=2} dT_{1,i}T_{1,i}^{-\alpha_{1i}-1}(1-T_{1,i})^{-c} \times\langle 0|V(p_{N-1})T_{1,N-2}^H\cdots V(p_{p+1})|\{m\}\rangle \times\frac{f_{r,s}^{(p)}(T_{1,i})}{(M+r-\alpha_{1p})(M+s-\alpha_{1s})} \times\langle\{m\}|V(p_{p+1})T_{1,p-1}^H\cdots V(p_2)|0\rangle. \quad (18)$$

The sum is taken over the occupation number basis, and M is defined by $H|\{m\}\rangle=M|\{m\}\rangle$.

The residue $f_{r,s}^{(p)}$ is independent of the external variables $\alpha_{i,j}$.

VI. CONCLUSION

We have shown that, by a procedure very similar to the renormalization in quantum field theory, the N-point one-loop amplitude can be made convergent. The renormalized amplitude satisfies perturbative unitarity, duality, and Regge behavior. The next step is to study the possibility of the renormalization of the nonplanar diagrams and of the N-loop diagrams. This problem is under investigation.

APPENDIX: COMPUTATION OF T_{ij}

We compute here the conjugate variable to the trajectory

$$\alpha_{i,j}=\alpha_0-\tfrac{1}{2}(p_i+\cdots+p_j)^2.$$

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[6] Whether the renormalized amplitude Reggeizes in any direction of the complex s plane except the positive real axis is still an open question, because of the singularities of θ functions one could encounter when distorting the integration contour over x_1 and x_3.

[7] Chan Hong-Mo and Tsou Sheung Tsun, Phys. Letters **28B**, 485 (1969).

The notations are those of Tannery and Molk (Ref. 8). The identities

$$p_ip_{i+j}=-\alpha_{i,i+j}+\alpha_{i,i+j-1}+\alpha_{i+1,i+j}-\alpha_{i+1,j-1}\quad(j>2),\qquad(19)$$

$$p_ip_{i+2}=-\alpha_{i,i+2}+\alpha_{i,i+1}+\alpha_{i+1,i+2}\,,\qquad(20)$$

$$p_ip_{i+1}=-\alpha_{i,i+1}-\alpha_0\qquad(21)$$

show that the integration over d^4k gives

$$\int d^4k\; x_1^{-\alpha(k^2)-1}x_N^{-\alpha(k+p_{N-1})^2-1}=\exp\left(-\sum_{i,j}\alpha_{i,j}\frac{\ln x_i\ln x_{j+1}}{\ln w}\right)f(x_i)\,,$$

where f depends only of x_i and α_0.

We now compute

$$I=\exp\left[-\sum_{i,j}C_{ij}{}^n\frac{p^ip_j}{n(1-w^n)}\right].$$

We note first that for $i<j$, $C_{ji}C_{ij}=w$, so that

$$A_{ij}=\prod_n\exp\left(-\frac{C_{ij}{}^n+C_{ji}{}^n}{n(1-w^n)}\right)=(1-x_{i+1}\cdots x_j)\times\prod_{n=1}^{\infty}\left(1-\frac{w^n}{x_{i+1}\cdots x_j}\right)(1-w^nx_{i+1}\cdots x_j)\,.$$

From Vol. II, p. 250, xxix (1) in Ref. 8,

$$A_{ij}=-2i\pi Z_{ij}q_0{}^2\frac{\theta_1(\nu_{ij}\,|\,\tau)}{\theta_1'(0\,|\,\tau)}\,,\qquad(22)$$

where

$$q_0=\prod(1-q^{2n})\,,\quad q=\sqrt{w}\,,\quad Z_{ij}=(x_{i+1}\cdots x_j)^{1/2}\,,$$

$$\nu_{ij}=(1/2i\pi)\ln x_{i+1}\cdots x_j\,,\quad \tau=(1/2i\pi)\ln w\,,$$

$$\theta_1(\nu)=2q_0q^{1/4}\sin\nu\pi\prod_{n=1}^{\infty}(1-2q^{2n}\cos2\nu\pi+q^{4n})\,.\qquad(23)$$

[8] J. Tannery and J. Molk, *Elements de la Théorie des Fonctions Elliptiques* (Gauthier-Villars, Paris, 1893).

On the other hand,

$$A_{ii}=\exp\left(-\sum_n\frac{w^n}{n(1-w^n)}\right)=q_0\,,$$

so that

$$I=\prod_{i<j}\left[-2i\pi Z_{ij}\frac{\theta_1(\nu_{ij}\,|\,\tau)}{\theta_1'(0\,|\,\tau)}\right]^{p_ip_j},$$

where q_0 has dropped out owing to momentum conservation. Applying the identities (19)–(21), one finds that this can be rewritten

$$I=\prod_{i<j}\left[\frac{\theta_1(\nu_{ij}\,|\,\tau)\theta_1(\nu_{i-1,j+1}\,|\,\tau)}{\theta_1(\nu_{i,j+1}\,|\,\tau)\theta_1(\nu_{i-1,j}\,|\,\tau)}\,\frac{Z_{ij}Z_{i-1,j+1}}{Z_{i,j+1}Z_{i-1,j}}\right]^{-\alpha_{i,j}}g(x_i)\,,$$

where g depends only on the x_i and on α_0.

We apply now the transformation formula obtained by the theory of the elliptic functions:

$$\theta_1(\nu\,|\,\tau)=i\left(\frac{i}{\tau}\right)^{1/2}e^{-i\pi(\nu^2/\tau)}\theta_1\left(\frac{\nu}{\tau}\,\middle|\,-\frac{1}{\tau}\right).$$

One finds then that the conjugate factor to $\alpha_{i,j}$ is

$$\exp\left(-\frac{\ln x_i\ln x_{j+1}}{\ln w}\right)\left[\theta_1\left(\frac{\nu_{ij}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j+1}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\Big/\theta_1\left(\frac{\nu_{i,j+1}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\right].\qquad(24)$$

Collecting the term coming from the integration over d^4k, we find that the conjugate variable to the trajectory α_{ij} is

$$\theta_1\left(\frac{\nu_{ij}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j+1}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\Big/\theta_1\left(\frac{\nu_{i,j+1}}{\tau}\,\middle|\,-\frac{1}{\tau}\right)\theta_1\left(\frac{\nu_{i-1,j}}{\tau}\,\middle|\,-\frac{1}{\tau}\right).\qquad(25)$$

Using (23), we then prove (9).