Everything but the Born term vanishes now, because the negative-norm contributions cancel all positive-norm contributions. The same mechanism is responsible for making all the vacuum expectation values (20) equal to zero.

*Note added in proof*. After this article was submitted for publication, I learned that the same model had been discovered by V. Glaser, in *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958), p. 130. I would like to thank Dr. V. Glaser for correspondence on this subject.

**ACKNOWLEDGMENTS**

The author would like to thank Professor R. H. Pratt and the Department of Physics of the University of Pittsburgh for the kind hospitality extended to him during the academic year 1969–1970. Discussions with Professor B. Schroer and Dr. R. Seiler were very enlightening.

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PHYSICAL REVIEW D VOLUME 1, NUMBER 8 15 APRIL 1970

# Relativistic Balmer Formula Including Recoil Effects

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(Received 10 October 1969)

It is shown that an approximate summation of the "crossed-ladder" Feynman diagrams for the scattering of two charged particles leads to the formula $s=m_1^2+m_2^2+2m_1m_2[1+Z^2\alpha^2/(n-\epsilon_j)^2]^{-1/2}$ for the squared mass of bound states. This formula neglects radiative corrections. It includes recoil effects properly, and reduces in the limit of one infinite mass to the corresponding spectrum of a relativistic particle in a static Coulomb potential. In the particular case of positronium, its expansion in powers of $\alpha$ coincides up to order $\alpha^4$ with the singlet energy levels. In an appendix we investigate some properties (gauge invariance, static limit) of this series of graphs.

## I. INTRODUCTION

The derivation of a reliable and systematic procedure to find the bound-state energies in relativistic theories remains a difficult task.[1] There does not seem to be any better definition than to look for poles in Green's functions. For long-range interactions, such as those encountered in quantum electrodynamics, a fortunate circumstance permits an interesting approximation to the scattering amplitude near the forward direction. It provides, therefore, in this case a procedure to calculate energy levels. Through short and elementary calculations, one obtains a compact formula for the bound states which incorporates the proper relativistic recoil effects for arbitrary mass ratios. Furthermore, this formula reduces, in the limit of one large mass, to the known spectra for the Klein-Gordon or Dirac equation in a Coulomb potential.

It remains a task for the future to develop a systematic perturbation theory in order to incorporate radiative corrections as well as some magnetic effects due to spin.

The idea of the present investigation relies on the remark that the eikonal or classical approximation gives a very accurate description of Coulomb nonrelativistic scattering at small angles. The amplitude computed in this way has poles in the energy variable which coincide with the known energy levels for the hydrogenlike atoms. This is also true for the one-body relativistic equations in the same potential up to centrifugal barrier shifts (short-range effects) to be discussed below. It is then tempting to investigate the generalization of this behavior in the relativistic two-body case. Recently it has been noted that the eikonal asymptotic behavior is recovered as an approximate summation of the "crossed-ladder" Feynman diagrams. This set of diagrams, which does not include radiative corrections of the self-interaction type, presents, however, a number of attractive features in contrast, say, with the ladder diagrams (see Fig. 1): (i) Gauge

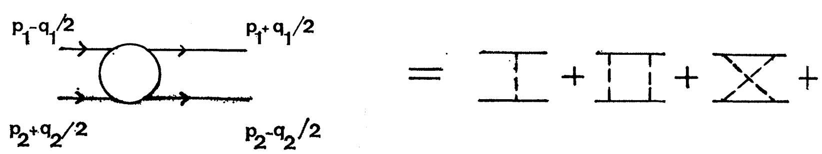


Fig. 1. Energy-momentum conservation implies $q_1=q_2$.

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[1] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957).

invariance is maintained at all stages. (ii) Proper behavior is obtained when one of the charged particles is very heavy. In this limit one recovers the perturbation expansion pertaining to the one-body relativistic amplitude in an external field.[2] (iii) The eikonal limit is obtained for large energies as already mentioned.[3] An appendix is devoted to a discussion of points (i) and (ii). The price to be paid is, however, that we are unable to produce a "useful" equation which generates the corresponding four-point Green's function. We only know a formal equation in terms of functional derivatives or we can write a Feynman path integral with a two-time Lagrangian in the kernel, reminiscent of the classical formulation of relativistic electrodynamics without self-interactions. The basic difficulty is that in the $n$th order of perturbation theory we have $n!$ terms, which, in the general case, do not recombine into a manageable expression. We are thus prevented from writing a recursive equation of the Bethe-Salpeter type with a simple kernel.

Nevertheless, from the example of one-body dynamics it is not totally unrealistic to expect that the eikonal approximation might lead at all energies but for very small angles to an amplitude whose poles are close to the correct answer. Using our method we have to face the problem of infrared divergences. These divergences dominate the behavior at small angles, once an infinite phase is extracted from the amplitude in analogy with the nonrelativistic case. Since infrared divergences are correctly treated in this eikonal approximation, this is another way of understanding why one can expect to obtain a reasonable result using the present method.

In the absence of a good equation, we cannot at present derive a systematic procedure to compute radiative corrections. In general it is hard to include short-range effects. This is why we have to devote special attention to the $\alpha^2/r^2$ terms induced by the minimal coupling of photons, which are responsible for the breaking of the $O(4)$ degeneracy of the spectrum.

In order to introduce our procedure, let us first present briefly a derivation for the nonrelativistic potential problem.

## II. NONRELATIVISTIC HYDROGEN ATOM

The Coulomb scattering amplitude will be studied as the limit of the Yukawa amplitude corresponding to the potential

$$V(r)=-Z\alpha e^{-\mu r}/r\,,\quad \alpha=e^2/4\pi\quad(\hbar=c=1)\qquad(1)$$

when $\mu$ goes to zero. It will, of course, be necessary in this limit to factorize an infinite phase, which turns out to be

$$\exp\left[+\frac{2imZ\alpha}{p}\ln\left(2p\frac{e^{-c}}{\mu}\right)\right],$$

$$c=0.577 \text{ (Euler's constant)}\qquad(2)$$

where $p$ is the particle momentum.

We can proceed in two different ways. We can first study the $\mu=0$ limit of the forward amplitude. This will allow us to justify the use of an approximation for small-angle scattering, which turns out to be the eikonal approximation.

The amplitude $T_\mu(E,\cos\theta)$ is, for fixed $\theta$, an analytic function of $E$ in a cut plane. The right-hand cut extends from 0 to $+\infty$; on the negative real axis there is a pole at $-(\mu^2/4m)(1-\cos\theta)$ and a cut which starts at $-(\mu^2/m)(1-\cos\theta)$ $[m\equiv(m_em_p)/(m_e+,m_p)]$. We look for an approximate $T$ which has, in the limit $\mu=0$, the correct poles of the Coulomb problem. In order to avoid in the approximation the effect of the left-hand singularities, we notice that they are removed to $-\infty$ in the forward direction ($\cos\theta=1$).

Let us then study the behavior as $\mu\to0$ of $T_\mu(E,\cos\theta=1)$. The $n$th-order Born approximation for $T(E,1)$ is given by

$$T^{(n)}\left(E=\frac{p^2}{2m},1\right)=-\frac{1}{\mu^2}\left(\frac{Z\alpha}{2\pi^2}\right)^n(2m)^{n-1}$$

$$\times\int d^3q_1\cdots d^3q_n\,\delta^{(3)}(\mathbf{q}_1+\cdots+\mathbf{q}_n)$$

$$\times\left(\prod_{i=1}^n\frac{1}{q_i^2+1}\right)\frac{1}{2\mathbf{p}\mathbf{q}_1+\mu q_1^2-i\epsilon}$$

$$\cdots\frac{1}{2\mathbf{p}(\mathbf{q}_1+\cdots+\mathbf{q}_{n-1})+\mu(\mathbf{q}_1+\cdots+\mathbf{q}_{n-1})^2-i\epsilon}.\qquad(3)$$

$T^{(n)}(E)$ diverges like $\mu^{-2}$, and the coefficient of $\mu^{-2}$ is obtained by setting $\mu=0$ in the integrand. This has the effect of replacing the propagator $[(\mathbf{p}+\mathbf{q})^2-p^2-i\epsilon]^{-1}$ by $(2\mathbf{p}\cdot\mathbf{q}-i\epsilon)^{-1}$. When this replacement is made, it is very easy to calculate the scattering amplitude to all orders, not only in the forward direction, but also for small-angle scattering. The most elementary way, if not the shortest, is to solve the Lippmann-Schwinger integral equation with this linearized propagator. The corresponding solution is simply the eikonal approximation:

$$T_{\mathbf{p}-\mathbf{q}/2\to\mathbf{p}+\mathbf{q}/2}=\frac{ip}{m}\int\frac{d^2b}{(2\pi)^3}$$

$$\times e^{i\mathbf{q}\cdot\mathbf{b}}\left\{\exp\left[-\frac{im}{p}\int_{-\infty}^{+\infty}V((b^2+z^2)^{1/2})dz\right]-1\right\}.\qquad(4)$$

[2] One of us (C.I.) learned this at SLAC from S. Brodsky and L. Brown (private communication), who attributed this remark to D. R. Yennie.

[3] For an elementary treatment, see, for instance, H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969).

For $V(r)=-Z\alpha e^{-\mu r}/r$, this gives

$$T_{\mathbf{p}-\mathbf{q}/2\to\mathbf{p}+\mathbf{q}/2}=\frac{ip}{m}\int\frac{d^2b}{(2\pi)^3}\times e^{i\mathbf{q}\cdot\mathbf{b}}\left\{\exp\left[\frac{2imZ\alpha}{p}K_0(\mu b)\right]-1\right\},\quad(5)$$

where

$$K_0(z)\equiv\int_0^\infty e^{-z\cosh\varphi}d\varphi.$$

In the forward direction the result behaves, as expected, like $\mu^{-2}$:

$$T\left(\frac{p^2}{2m},1\right)=\frac{1}{\mu^2}\frac{ip}{m}\int_0^\infty\frac{b\,db}{(2\pi)^2}\times\left\{\exp\left[\frac{2imZ\alpha}{p}K_0(b)\right]-1\right\}.\quad(6)$$

Using the asymptotic expansion of $K_0(b)$ in the vicinity of $b=0$,

$$K_0(b)=-J_0(ib)\ln(\tfrac{1}{2}b)+\sum_{m=0}^{\infty}\psi(m+1)(\tfrac{1}{2}b)^m\frac{1}{(m!)^2},\quad(7)$$

one finds that $T(p^2/2m,\,1)$ is a meromorphic function of $p$ with poles at

$$imZ\alpha/p_n=+n\quad(n\geq1)\quad(8\text{a})$$

or

$$E_n=p_n{}^2/2m=-mZ^2\alpha^2/2n^2,\quad(8\text{b})$$

which are the correct bound-state energies of the Coulomb problem.

However, the amplitude given by Eq. (5) has a nonuniform behavior as $\mu$ and $q^2$ go to zero. If we first let $\mu$ tend to zero, we expect to get an almost correct expression for the Coulomb amplitude. For fixed $q^2$ the behavior in $\mu$ is different and can be entirely absorbed into the phase factor given by (2). The proper Coulomb amplitude is then simply

$$T_c=\frac{ip}{m}\int\frac{d^2b}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{b}}(pb)^{-2im\alpha Z/p},\quad(9\text{a})$$

which, after integration, reads

$$T_c=-\frac{Z\alpha}{2\pi^2}\frac{1}{q^2}\left(\frac{4p^2}{q^2}\right)^{-iZ\alpha m/p}\frac{\Gamma(1-iZ\alpha m/p)}{\Gamma(1+iZ\alpha m/p)}.\quad(9\text{b})$$

Expression (9) would be the exact Coulomb amplitudes if $E$ were $p^2/2m$, instead of $(p^2+\frac{1}{4}q^2)/2m$. If we keep in mind this replacement [namely, $p$ is taken as $(2mE)^{1/2}$], (9a) is then a very simple expansion of $T_c$ in terms of impact parameters $b$. The poles are again given by (8).

## III. KLEIN-GORDON AND DIRAC EQUATIONS FOR COULOMB POTENTIAL

Some new features appear for a relativistic particle in a Coulomb potential. For a spinless particle in a potential $V(r)$, the Klein-Gordon equation reads

$$(p_0{}^2-m^2)\psi=(p^2+2p_0V-V^2)\psi.$$

The term $V^2=Z^2\alpha^2/r^2$ induces simply a shift on the eigenvalues of the angular momentum operator $L^2$. If we forget this term for a moment, the eikonal approximation will now give

$$E_n=m(1+Z^2\alpha^2/n^2)^{-1/2}.$$

But in a given partial wave $l$, we know that $n=n'+l+1$, $n'\geq0$; to take into account the effect of $V^2$, we replace $l$ by $\lambda$, the solution of $\lambda(\lambda+1)=l(l+1)-Z^2\alpha^2$. It gives then the correct Klein-Gordon spectrum

$$E_{n,l}=m[1+Z^2\alpha^2/(n-\epsilon_l)^2]^{-1/2},$$

where

$$\epsilon_l=l+\tfrac{1}{2}-[(l+\tfrac{1}{2})^2-Z^2\alpha^2]^{1/2}.\quad(10)$$

For a Dirac particle, the same analysis can be repeated; there are both $V^2$ and magnetic terms, whose combined effect is to replace in (10) $l$ by $j$ and $\epsilon_l$ by $\epsilon_j$. This is most easily performed by squaring the Dirac equation in the form ($p$ four dimensional)

$$[(p-eA)^2-m^2+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}]\psi=0,$$

and using a diagonal representation for $\sigma^{\mu\nu}$. One then recognizes that the extra terms $\frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}$ are equal to $i\boldsymbol{\sigma}\cdot\mathbf{E}$, where the electric field $\mathbf{E}$ behaves again like $1/r^2$. Hence the previous treatment of such terms applies after the diagonalization of a $2\times2$ matrix.

## IV. QUANTUM ELECTRODYNAMICS

We first consider two scalar charged particles of masses $m_1$ and $m_2$, exchanging a vector boson of mass $\mu\to0$.

If we write the contribution for an $n$th-order ladder diagram in the forward direction, we find again that in the limit $\mu\to0$, this diagram diverges like $1/\mu^2$. The coefficient is obtained by replacing the propagators $1/[(p+q)^2-m_i{}^2+i\epsilon]$ by $1/(2p\cdot q+i\epsilon)$. However, in the same order, there are other diagrams which behave like $1/\mu^2$; in particular, this is true of the crossed ladders, obtained by crossing the photon lines in $n!$ different ways.

These diagrams are generated by the following expression (see Fig. 1):

$$-\frac{i}{(2\pi)^4}\delta^{(4)}(q_1-q_2)(p_1+\tfrac{1}{2}q_1,\,p_2-\tfrac{1}{2}q_2|T|p_1-\tfrac{1}{2}q_1,\,p_2+\tfrac{1}{2}q_2)=K(p_1+\tfrac{1}{2}q_1|T(A_1)|p_1-\tfrac{1}{2}q_1)\times(p_2-\tfrac{1}{2}q_2|T(A_2)|p_2+\tfrac{1}{2}q_2)\Big|_{A_1=A_2=0}.\quad(11)$$

The symbol $K$ stands for

$$K=\exp\left[\int\int d^4x_1 d^4x_2 \frac{\delta}{\delta A_1{}^{\mu}(x_1)} \times D^{\mu\nu}(x_1-x_2)\frac{\delta}{\delta A_2{}^{\nu}(x_2)}\right]. \quad (12)$$

The photon propagator in an arbitrary gauge is

$$D^{\mu\nu}(x)=-\frac{i}{(2\pi)^4}\int\frac{d^4q}{q^2+i\epsilon} \times e^{iqx}[g^{\mu\nu}+q^{\mu}n^{\nu}(q)+n^{\mu}(q)q^{\nu}+q^{\mu}q^{\nu}f(q)]. \quad (13)$$

Finally, the one-particle amplitude reads

$$(p'|T(A)|p) =(p'|T\exp\left[i\int_0^\infty \mathcal{V}(X-2P\tau,P)d\tau\right]\mathcal{V}(X,P)|p), \quad (14)$$

where $T$ is the time-ordering symbol. The conjugate operators $P$ and $X$ satisfy the commutation relations $[X_\mu,P_\nu]=ig_{\mu\nu}$ and $\mathcal{V}(X,P)=-e[A(X)P+PA(X)]+e^2 \times A^2(X)$. Note that in Eq. (11) the states refer to particles on their mass shells, i.e.,

$$(p_i+\tfrac{1}{2}q_i)^2=(p_i-\tfrac{1}{2}q_i)^2=m_i^2 \quad (i=1,2).$$

The same formalism applies to spin-$\frac{1}{2}$ particles, provided one adds to $\mathcal{V}(X,P)$ the magnetic term $\frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(X)$ and that the external states are meant to include Dirac projectors on positive energies. It can be checked that these rules for spinor electrodynamics are equivalent to the usual Feynman rules. They allow one to treat on a similar footing spin-0 and spin-$\frac{1}{2}$ particles.

The amplitude (11) is, in fact, gauge invariant—i.e., it is independent of the arbitrary functions $n^\mu(q)$ and $f(q)$ which enter into the definition of the photon propagator (13). The proof given in the Appendix is modeled on an argument presented by Feynman in Ref. 4. Hence, we can set $n^\nu(q)$ and $f(q)$ equal to zero in (13).

Unfortunately we do not know how to proceed with the exact evaluation of (11). But in analogy with the previous two cases, we expect a reliable result for small scattering angles from the linearization of the Green's functions. This simply amounts to replacing the operator $P$ by the $c$ number $\frac{1}{2}(p+p')$ in the matrix element (14).[3]

The result is very similar to the nonrelativistic one; there are only kinematical modifications. It gives for the amplitude $T$ in the center-of-mass frame, $p_1=(p_1{}^0,\mathbf{p})$ and $p_2=(p_2{}^0,-\mathbf{p})$,

$$T=\frac{i|\mathbf{p}|(p_1{}^0+p_2{}^0)}{m_1m_2}\int\frac{d^2b}{(2\pi)^3} \times e^{i\mathbf{q}\cdot\mathbf{b}}\left\{\exp\left[\frac{2i(p_1\cdot p_2)Z\alpha K_0(\mu b)}{|\mathbf{p}|(p_1{}^0+p_2{}^0)}\right]-1\right\}. \quad (15)$$

After the factorization of an "infinite" phase, we find again poles for

$$\frac{iZ\alpha(p_1\cdot p_2)}{|\mathbf{p}|(p_1{}^0+p_2{}^0)}=n. \quad (16)$$

Defining, as usual, $s=(p_1{}^0+p_2{}^0)^2$, we get

$$2|\mathbf{p}|\sqrt{s}=[s-(m_1-m_2)^2]^{1/2}[s-(m_1+m_2)^2]^{1/2},$$
$$2(p_1\cdot p_2)=s-m_1{}^2-m_2{}^2.$$

Finally, Eq. (16) gives

$$s_n=m_1{}^2+m_2{}^2+2m_1m_2(1+Z^2\alpha^2/n^2)^{-1/2}, \quad n\geq 1. \quad (17)$$

Let us examine the limit of (17) when $m_2\to+\infty$. Defining $E_n=(s_n)^{1/2}-m_2$, we have

$$E_n=\lim_{m_2\to\infty}(s_n-m_2{}^2)/2m_2=m_1(1+Z^2\alpha^2/n^2)^{-1/2}.$$

This is just the result of the Klein-Gordon or Dirac equations if, as discussed above, we replace $n$ by $n-\epsilon_j$. If we make the same ansatz in (17), we finally get the simple and compact formula

$$s_{n,j}=m_1{}^2+m_2{}^2+2m_1m_2[1+Z^2\alpha^2/(n-\epsilon_j)^2]^{-1/2}. \quad (18)$$

The same calculation can be done for two spin-$\frac{1}{2}$ particles. The result is again given by (17). This is not surprising, since the spin-flip amplitude vanishes in the forward direction, and the approximation assumes essentially that at each step there is almost exclusively forward scattering. Therefore, spin effects are lost in this approximation.

Let us now expand formula (18) in powers of $\alpha$ up to terms in $\alpha^4$. It gives for the binding energy

$$E_{n,j}=s^{1/2}-m_1-m_2 =-\frac{Z^2\alpha^2}{2n^2}\frac{m_1m_2}{m_1+m_2}+\frac{Z^4\alpha^4}{8n^4}\frac{m_1m_2}{m_1+m_2}\left(3-\frac{m_1m_2}{(m_1+m_2)^2}\right) -\frac{Z^4\alpha^4}{n^3}\frac{m_1m_2}{m_1+m_2}\frac{1}{2j+1}+\cdots. \quad (19)$$

The first term is the nonrelativistic binding energy. The second term is a displacement of all the levels with same principal quantum number. It contains a relativistic correction to the relative mass of the two particles which is multiplied by the dimensionless quantity $3-m_1m_2/(m_1+m_2)^2$. The third term is the fine-structure splitting. In the particular case of positronium, $m_1=m_2$

[4] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

$=m$, $Z=1$, Eq. (19) reads

$$E_{n,j}=-\frac{m\alpha^2}{4n^2}+\frac{11}{64}\frac{m\alpha^4}{n^4}-\frac{m\alpha^4}{2n^3(2j+1)}+\cdots, \quad (20)$$

which turns out to be the correct expression in this order of the bound-state energies of the singlet states $s=0$, $j=l$.[5]

It must be admitted that the replacement $n\to n-\epsilon_j$ in (18) may seem a bit artificial. However, we can partially understand this substitution by the following remark: If we use the eikonal approximation for one of the charged particles only in (11), we generate the known amplitude for the other particle in an external field (essentially the Coulomb field of the first particle as seen in a frame where its momentum is $p$). The proof is similar to the one given in the Appendix. There we show that, in the case where one of the masses goes to infinity, the crossed-ladder series generates the proper amplitude for a relativistic particle in a static field. In any case, both approaches imply an angular shift for the poles as given by (10). Of course, one would hope to find a better justification for this procedure.

A formula similar to (18) is obtained in the hypothetical case of scalar photons interacting with "charged" particles with a coupling constant equal to $2m_ig_i$. Bound states occur for "charges" of equal sign and for total energies given by

$$s_n=m_1^2+m_2^2+2m_1m_2[1-(g_1g_2/4\pi)^2(1/n^2)]^{1/2}.$$

Some comments on the infrared divergence are in order. Within the set of diagrams that were considered here, the "infrared catastrophe" manifests itself by an infinite phase factor. It is interesting to observe that this factor becomes real below threshold and indeed infinite at the location of the poles. This phenomenon already occurs in the nonrelativistic case. But we know in this case that the remaining coventional amplitude nevertheless has poles at these points. On the other hand, had we included more terms in the amplitude—in particular, terms where photons would be emitted and absorbed by the same charged particle—we would have found a new type of infrared divergence in the modulus of the amplitude. To correct for this effect, we would have had to include the probability for real photon emission (inelastic effects) in order to get a finite result. Hence it is likely that a more correct treatment of the problem requires the investigation of the four-point Green's function (which does not suffer from these defects) rather than of the scattering amplitude.

Let us also remark that all the formulas (for vector electrodynamics) discussed above allow a simple interpretation. They can be written

$$Z\alpha/v_{12}=-in_{\rm eff}. \quad (21)$$

Here $v_{12}$ is the relative velocity of particles 1 and 2 expressed in terms of their total energy on the mass shell, analytically continued to imaginary values. The quantity $n_{\rm eff}$ is an integer in the nonrelativistic case [or $O(4)$-symmetric case] and close to an integer $(n-\epsilon_j)$ for relativistic particles. This is clear for the nonrelativistic case [see (8a)] where $v_{12}=(2E/m)^{1/2}$. For a relativistic particle in a fixed potential, $v_{12}=[1-(m^2/E^2)]^{1/2}$, which, inserted in (21), yields formula (10). Finally, for two relativistic particles the relative velocity is given by

$$v_{12}=[\Delta(s,m_1^2,m_2^2)]^{1/2}/(s-m_1^2-m_2^2),$$

where

$$\Delta(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca.$$

With this value, expression (21) yields the result (18). The intuitive meaning of the remark is that the main contributions to the binding energy come from the Coulomb forces exerted by one particle in its rest frame on the second particle. This concept is invariant when interchanging the roles of the particles and can obviously be extended to all frames by expressing the relevant quantity, i.e., the velocity, in terms of the invariant total energy squared $s$.

Our conclusion is that the series of diagrams discussed in this paper deserves a more thorough study in view of its remarkable properties.

## ACKNOWLEDGMENTS

We have benefited from stimulating remarks from Dr. X. Artru, Dr. R. Stora, and Dr. I. Todorov.

## APPENDIX

In this appendix we give the proof of some statements made in the text.

### A. Ward Identity and Gauge Invariance

Since it may be less well known, we derive a "Ward identity" for the case of scalar charged particles. Let $G(A)$ be the Green's function in an external potential:

$$G(A)=1/\{[P-eA(X)]^2-m^2+i\epsilon\}.$$

The current operator $J_\mu(k,A)$ is defined through

$$J^\mu(k;A)=\frac{\delta}{\delta\tilde{A}_\mu(k)}G(A)=eG(A)\{P^\mu-eA^\mu(X),e^{ikX}\}G(A).$$

In this definition $\tilde{A}(k)$ stands for the Fourier transform of $A(x)$ and the symbol { , } means anticommutator. One has the "Ward identity"

$$k_\mu J^\mu(k;A)=e[e^{ik\cdot X},G(A)].$$

[5] Similar ideas have led M. Lévy and J. Sucher [Phys. Rev. **186**, 1656 (1969)] to propose for the positronium a different analytical expression which agrees up to order $\alpha^4$ with Eq. (20).

Indeed,

$$kJ(k;A)=eG(A)\{p\cdot k-eA(X)\cdot k,\, e^{ik\cdot X}\}G(A).$$

Now,

$$\{p\cdot k-eA(X)\cdot k,\, e^{ik\cdot X}\}=[G^{-1}(A),\, e^{ik\cdot X}],$$

from which the identity follows. With an obvious definition, we also find

$$k_{1\mu_1}J^{\mu_1,\mu_2,\cdots}(k_1,k_2,\ldots;A)=e[e^{ik_1\cdot X},\, J^{\mu_2\cdots}(k_2,\ldots;A)]. \quad \text{(A1)}$$

Let us apply these results to the mass-shell scattering amplitude to order $n$, which is proportional to

$$\begin{aligned}&\lim[(p_1-\tfrac{1}{2}q)^2-m_1^2][(p_1+\tfrac{1}{2}q)^2-m_1^2][(p_2+\tfrac{1}{2}q)^2-m_2^2]\\ &\times[(p_2-\tfrac{1}{2}q)^2-m_2^2]\int^{(n)}\frac{dk_1\cdots dk_n}{(k_1^2+i\epsilon)\cdots(k_n^2+i\epsilon)}\\ &\times\delta(\Sigma k-q)(p_1-\tfrac{1}{2}q|J_1^{\mu_1\mu_2\cdots\mu_n}(k_1,\ldots,k_n;0)|p_1+\tfrac{1}{2}q)\\ &\times\Lambda_{\mu_1\nu_1}(k_1)\Lambda_{\mu_2\nu_2}(k_2)\cdots\Lambda_{\mu_n\nu_n}(k_n)\\ &\qquad\times(p_2+\tfrac{1}{2}q|J_2^{\nu_1\cdots\nu_n}(k_1,\ldots,k_n;0)|p_2-\tfrac{1}{2}q),\end{aligned}$$

where $\Lambda_{\mu\nu}(k)$ stands for the gauge arbitrariness introduced in (13):

$$\Lambda_{\mu\nu}(k)=g_{\mu\nu}+k_\mu n_\nu(k)+n_\mu(k)k_\nu+k_\mu k_\nu f(k).$$

We have the products

$$\Lambda_{\mu_1\nu_1}(k_1)\cdots\Lambda_{\mu_n\nu_n}(k_n)=g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n}+\sum \text{ of terms},$$

which all involve at least one four-momentum $k$ acting either on $J_1$ or $J_2$. Making use of identity (A1), we reduce the matrix element of this divergence to the difference of two matrix elements of lower order between external states, one of which has a momentum shifted by the corresponding $k$. By applying the mass-shell conditions, such terms are seen to vanish. Therefore, it is proved that all the terms, except $g_{\mu\nu}$ in $\Lambda_{\mu\nu}(k)$, give no contribution. The proof extends easily to spin-$\frac{1}{2}$ particles.

### B. Infinite-Mass Limit

The formulas (11)–(14) reduce this problem to the study of the scattering of a heavy particle by an external field. In the scattering process, a heavy particle is hardly deflected. Therefore, it is not surprising, and indeed it will be proved, that the scattering amplitude tends, in this limit, to the eikonal expression. Another way to express it is that for a heavy particle in an external field we expect the wave function to pick a phase factor and its spin (if any) to precess in the magnetic field. This is what we show now.

Consider the matrix element $(p-\frac{1}{2}q|T(A)|p+\frac{1}{2}q)$, in which the infinite-mass limit is taken as follows: Let $p=\lambda n$, where $n$ is a fixed four-vector orthogonal to $q$, normalized to $n^2=1$; hence $m^2=\lambda^2+\frac{1}{4}q^2$. The infinite-mass limit is defined by letting $\lambda\to\infty$. A simple transformation leads to

$$\begin{aligned}&(\lambda n-\tfrac{1}{2}q|T(A)|\lambda n+\tfrac{1}{2}q)\\ &=(n-\tfrac{1}{2}q|e^{-i(\lambda-1)n\cdot X}T(A)e^{i(\lambda-1)n\cdot X}|n+\tfrac{1}{2}q)\\ &=(n-\tfrac{1}{2}q|T\exp\Big[i\int_0^\infty d\tau\, \mathcal{V}\{X-2\tau[P+(\lambda-1)n],\\ &\qquad P+(\lambda-1)n\}\Big]\mathcal{V}(X,P+(\lambda-1)n)|n+\tfrac{1}{2}q).\end{aligned}$$

It is now natural, when $\lambda$ is large, to neglect the operator $P$, as compared to $(\lambda-1)n$, and, consequently, to replace $P+(\lambda-1)n$ by the $c$ number $p$. For scalar particles, the time-ordering symbol becomes irrelevant and the amplitude reduces to

$$\begin{aligned}(p-\tfrac{1}{2}q|T(A)|p+\tfrac{1}{2}q)\underset{m\text{ large}}{\simeq}2im\int\frac{d^4x}{(2\pi)^4}\delta(X\cdot n)e^{iqx}\\ \times\left[\exp\left(\frac{i}{2m}\int_{-\infty}^{+\infty}dt\,\mathcal{V}(x+nt,p)\right)-1\right]. \quad \text{(A2)}\end{aligned}$$

Let us now examine the effect of inserting this approximation in the four-point amplitude given by (11). From Eq. (A2) we see that we have to investigate the behavior of $m_2^{-1}\mathcal{V}_2(X_2,p_2)$ when $m_2$ becomes very large. We choose the reference frame in which $p_2$ is close to $(m_2,0)$ and hence $q_2^0=0$. In this limit,

$$(1/m_2)\mathcal{V}_2(X_2,p_2)=-2e_2A_2^0(X_2)+(e_2^2/m_2)A_2^2(X_2),$$

and, for large $m_2$, we can neglect the second term. Equation (A2) can then be simplified:

$$\begin{aligned}(p_2-\tfrac{1}{2}q_2|T(A_2)|p_2-\tfrac{1}{2}q_2)\sim-2im_2\int\frac{d^4x_2}{(2\pi)^4}\delta(x_2\cdot n_2)e^{iq_2x}\\ \times\left\{\exp\left[-ie_2\int_{-\infty}^{+\infty}dt\,A_2^0(x_2+n_2t)\right]-1\right\}.\end{aligned}$$

The effect of the operator $K$, given in Eq. (12), on this amplitude amounts to replacing $A_2^0(x_2)$ in the above formula by

$$\int d^4x_1\frac{\delta}{\delta A_1(x_1)}D^{\mu 0}(x_1-x_2).$$

Then $K(p_2-\frac{1}{2}q_2|T(A_2)|p_2-\frac{1}{2}q_2)|_{A_2=0}$ becomes a displacement operator acting on the field $A_1$. Therefore we get the following expression for the two-body amplitude:

$$-[i/(2\pi)^4]\delta^{(4)}(q_1-q_2)$$
$$\times(p_1+\tfrac{1}{2}q_1, p_2-\tfrac{1}{2}q_2|T|p_1-\tfrac{1}{2}q_1, p_2+\tfrac{1}{2}q_2)$$
$$\underset{m_2\text{ large}}{\simeq} -2im_2\int\frac{d^4x_2}{(2\pi)^4}\delta(x_2\cdot n_2)e^{iq_2x_2}(p_1+\tfrac{1}{2}q_1|T$$
$$\times\left[-ie_2\int_{-\infty}^{+\infty}dt\,D^{\mu 0}(x_2-x_1-n_2t)\right]|p_1-\tfrac{1}{2}q_1).$$

Let us observe that

$$-ie_2\int_{-\infty}^{+\infty}dt\,D^{\mu 0}(x_2-x_1-n_2t)=\frac{e_2}{4\pi}\frac{g^{\mu 0}}{|\mathbf{x}_2-\mathbf{x}_1|}.$$

Using the time independence of this result and performing the translation $\mathbf{x}_1\to\mathbf{x}_1+\mathbf{x}_2$, we can factor out $\delta(q_1^0)e^{i\mathbf{q}_1\cdot\mathbf{x}_2}$. The integral over $x_2$ gives finally $\delta(q_1^0)\times\delta^{(3)}(\mathbf{q}_1-\mathbf{q}_2)=\delta^{(4)}(q_1-q_2)$, since $q_2^0=0$. Therefore the two-body amplitude, in the limit $m_2$ large, is proportional to the scattering amplitude of the first particle in the static field, $A^0(x_1)=(e_2/4\pi)(1/|x_1|)$, produced by the second particle in its rest frame. If we are interested in finding the bound states, the problem reduces to solving the one-body Klein-Gordon equation

$$\left[\left(p_1^0-\frac{e_1e_2}{4\pi}\frac{1}{|x_1|}\right)^2-(p_1)^2-m_1^2\right]\psi=0.$$

For spin-$\frac{1}{2}$ particles the same argument can be repeated. In this case, it leads to the Dirac amplitude of the light particle in the Coulomb field produced by the heavy one. But another effect can be included here, if one regards the magnetic moment of the heavy particle as a nonvanishing static property. Then, in addition to the Coulomb field, the heavy particle produces a magnetic field responsible for the hyperfine splitting of the Dirac levels. We then have to add an extra term in the potential equal to $(e_2/2m_2)\sigma_2^{\mu\nu}F_{\mu\nu}^2$, where we can even replace $e_2/2m_2$ by the actual magnetic moment $\mu_2=(e_2/2m_2)(1+x)$. In the same spirit as above, we have to neglect the noncommutativity of the spin matrices and set $\sigma_2^{0\nu}\sim 0$ as if the operators were replaced by their mean values. Following the previous recipe, we find that particle one moves in a static field with four-potential given by

$$A_0(\mathbf{x}_1)=\frac{e_2}{4\pi|\mathbf{x}_1|},\quad \mathbf{A}(\mathbf{x}_1)=\mu_2\,\mathrm{curl}_{\mathbf{x}_1}\left(\frac{\boldsymbol{\sigma}_2}{4\pi|\mathbf{x}_1|}\right).$$

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PHYSICAL REVIEW D VOLUME 1, NUMBER 8 15 APRIL 1970

# Parameter-Free Regularization of One-Loop Unitary Dual Diagram*

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(Received 15 December 1969)

We propose a parameter-free regularization of the one-loop planar unitary dual diagram, very similar to the renormalization in quantum field theory, compatible with duality, Regge behavior, unitarity, and crossing symmetry.

## I. INTRODUCTION

RECENTLY, there have been several attempts[1–3] to put the Veneziano model on the same footing as quantum electrodynamics, by considering the $N$-point functions as tree diagrams of the theory. This leads to a finite result when factorization problems are neglected.[1] However, when factorization is taken into account, the one-loop diagram exhibits an exponential divergence due to the large degeneracy of levels.[2,3]

It would then seem that a renormalization procedure would need an infinite number of subtractions, and thus an infinite number of parameters. However, we show that in the case of the one-loop diagram, the divergent part of the new trajectory, $\alpha_t+g^2\Sigma(\alpha_t)$, is independent of $t$ and can be removed by a single subtraction. This can be extended to the amplitude itself and one finds that the subtraction of a crossing-symmetric, Regge-behaved, dual amplitude, having only single and double poles in the external variables $s$ and $t$, is enough to make it finite. We then find that the renormalized amplitude still Reggeizes at $s\to-\infty$. We have extended this to the case of the one-loop diagram with $N$ external legs.

* This research partially supported by the Air Force Office of Scientific Research under Contract No. AF-49(638)-1545.
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