Radiative Correction to Eikonal Functions in Electrodynamics*

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We show that the second-order radiative correction changes the electron eikonal function into $[1+F(k^2)]\times[1+F'(k^2)]E(k^2)$, where $E(k^2)$ is the bare eikonal, k is the momentum transfer between the electrons (e and e'), and $F(k^2)$ is the second-order electric form factor. This suggests that in high-energy near-forward electrodynamic processes, the eikonals should be regarded as the basic carriers of the interaction.

I. INTRODUCTION

LATELY there has been considerable interest in the high-energy behavior of amplitudes in quantum electrodynamics.[1–4] It has been shown that if an arbitrary number of photons are exchanged across the t channel, the invariant amplitudes near the forward direction for elastic processes $a+b\rightarrow a+b$ can all be distinctly characterized by two quantities: (i) the impact factors F_a and F_b,[1] and (ii) the electron-electron and the electron-positron eikonals.[2–5] The presence of the impact factors is a reflection that quantum fluctuation gives structures to particles, while the eikonals indicate that the t-channel-exchanged photons behave like classical potentials.

The importance of this result lends itself not only to electrodynamic processes, but also perhaps to other areas, since we may wish to abstract certain observations and apply them to high-energy processes involving hadrons. For this reason, the subject of electrodynamics is taken up once again.

Now, if the true situation can be qualitatively well described by the sets of Feynman diagrams discussed in Ref. 2, a physical picture seems to emerge.[6] It suggests that in a highly energetic near-forward process, as far as the leading order goes, we may regard the "electrons" and "positrons" as the more elementary constituents. The other energetic particles, i.e., the photons, interact via dissociation virtually into electron-positron pairs, which, incidentally, gives rise to impact factors. Then the electrons and the positrons interact amongst themselves in pairs via eikonal exchange. In other words, instead of individual particles, some collective entities, the eikonals, now serve as the basic agents that carry the interaction.

In order to make this picture more concrete, one should assign definite roles to, e.g., the effects of radiative corrections. To be specific, the eikonal function between electrons has the form

$$E(k^2)=\int d^2x_\perp\, e^{-ik\cdot x_\perp}\times\left\{\exp\left[-iee'\int\frac{d^2q_\perp}{(2\pi)^2}e^{iq_\perp\cdot x_\perp}\frac{1}{q_\perp^2+\lambda^2-i\epsilon}\right]-1\right\},$$

where k is the momentum that is exchanged. If this object is the basic interaction carrier, then radiative corrections should modify it to become

$$\text{(a)}\qquad E(k^2)\rightarrow[1+F(k^2)]E(k^2),$$

i.e., $F(k^2)$ gives structure to the eikonals. On the other hand, if the individual exchanged photons are still the basic agents, then we may expect a result

$$\text{(b)}\quad E(k^2)\rightarrow[1+g(k^2)]\int d^2x_\perp\, e^{-ik\cdot x_\perp}\times\left\{\exp\left[-iee'\int\frac{d^2q_\perp}{(2\pi)^2}I(q_\perp^2)I'(q_\perp^2)e^{iq_\perp\cdot x_\perp}\times\frac{1}{q_\perp^2+\lambda^2-i\epsilon}\right]-1\right\},$$

i.e., each photon vertex is expected to acquire some structure $I(q_\perp^2)$ and $I'(q_\perp^2)$. We have also introduced an additional factor $g(k^2)$ to account for the probable over-all modification.

The purpose of this work is to perform such an investigation. As a beginning, the simplest situation is studied here. We shall find out what the effects of second-order radiative correction are on an energetic electron line. As it turns out, after elaborate cancellation

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and combination, the eikonal is shown to modify to the form (a). In fact, we shall be concerned with the spin-nonflip part only, and $F(k^2)$ is none other than the electric form factor.

The plan of this paper is as follows: We shall find that in order to study the high-energy behavior of the electron-electron scattering amplitude with arbitrary number of photons exchanged in the t channel, it is convenient to split the amplitude into two halves. Therefore in Sec. II we shall first investigate the behavior of the n-photon absorption (production) amplitudes by an energetic electron. In Sec. III, we shall pair these amplitudes together and show that the electron-electron eikonal now takes on the form (a). In Sec. IV, we shall discuss our results.

II. PARTIAL AMPLITUDES

In order to calculate scattering amplitudes involving "electrons" at high energy, it is convenient to look at the partial diagrams, in which virtual photons are emitted or absorbed from an electron line. We shall subdivide this section into parts according to the number of photons coming out. We shall call a photon line which begins from and ends on the same electron line a radiative-correction photon line. We include effects of one radiative-correction photon line only.

We choose the coordinate system so that the electron is moving approximately in the z direction with very high momentum. Let p_1 and p_3 be the initial and the final energy-momentum vectors, respectively; then

$$p_1=p-\tfrac{1}{2}k\,,\quad p_3=p+\tfrac{1}{2}k\,,$$
$$p^\mu=((|p|^2+\tfrac{1}{4}k^2+m^2)^{1/2},0,0,|p|)\,,$$

and

$$k^\mu=(0,k_1,k_2,0)\,.$$

Obviously, k is the momentum transfer. We shall look for the leading behavior of the following amplitudes when $p\to\infty$, while the other quantities are held finite and fixed. Furthermore, we shall confine our attention to the spin-nonflip amplitudes.

A. One-Photon Absorption

One-photon absorption is represented by Fig. 1. It gives an amplitude[7]

$$M_1=\bar{u}(p_3)e\Lambda_{\text{ren}}{}^{\alpha_1}u(p_1)\,, \tag{1}$$

where $\Lambda_{\text{ren}}{}^{\alpha}$ is the renormalized part of the vertex, and

$$\Lambda^{\alpha_1}=e^2\int\frac{d^4q}{(2\pi)^4i}\gamma_\mu\frac{1}{m+\gamma\cdot(p-q+\frac{1}{2}k)-i\epsilon}\gamma^{\alpha_1}\times\frac{1}{m+\gamma\cdot(p-q-\frac{1}{2}k)-i\epsilon}\gamma^\mu\frac{1}{q^2+\lambda^2-i\epsilon}\,,$$

in which we have used m and λ^2 to indicate the masses of the electron and the photon, respectively.

For the spin-nonflip amplitude, the dominant term comes from the electric part, which yields

$$M_1=e(p^\alpha/m)F(k^2)\,,$$

with

$$F(k^2)=\frac{e^2}{16\pi^2}\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\bigg(\frac{-4m^2[1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2]-2k^2(1-x_2)(1-x_3)}{m^2(x_2+x_3)^2+k^2x_2x_3+\lambda^2x_1}+\frac{4m^2[1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2]}{m^2(x_2+x_3)^2+\lambda^2x_1}-2\ln\frac{m^2(x_2+x_3)^2+k^2x_2x_3+\lambda^2x_1}{m^2(x_2+x_3)^2+\lambda^2x_1}\bigg)\,. \tag{2}$$

B. Two-Photon Absorption

In order to preserve gauge invariance, we have to include the diagrams of Figs. 2(a)–2(d). The first diagram corresponds to the amplitude

$$M_{2a}=e^4 3!\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\int\frac{d^4q}{(2\pi)^4i}\frac{N^{\alpha_1\alpha_2}}{D_{2a}{}^4}\,, \tag{3}$$

$$\begin{aligned}N^{\alpha_1\alpha_2}=\;&\bar{u}(p_3)\gamma^\mu[m-\gamma\cdot(px_4+\tfrac{1}{2}k(1+x_1-x_2+x_3)-rx_3)]\gamma^{\alpha_1}\\&\times[m-\gamma\cdot(px_4+\tfrac{1}{2}k(x_1-x_2+x_3-1)-r(x_3-1))]\gamma^{\alpha_2}[m-\gamma\cdot(px_4-\tfrac{1}{2}k(1-x_1+x_2-x_3)-rx_3)]\gamma_\mu u(p_1)\\&+\tfrac{1}{4}q^2\bar{u}(p_3)\{\gamma^\mu\gamma^\nu\gamma^{\alpha_1}\gamma_\nu\gamma^{\alpha_2}[m-\gamma\cdot(px_4-\tfrac{1}{2}k(1-x_1+x_2-x_3)-rx_3)]\gamma_\mu\\&+\gamma^\mu[m-\gamma\cdot(px_4+\tfrac{1}{2}k(1+x_1-x_2+x_3)-rx_3)]\gamma^{\alpha_1}\gamma^\nu\gamma^{\alpha_2}\gamma_\nu\gamma_\mu\\&+\gamma^\mu\gamma^\nu\gamma^{\alpha_1}[m-\gamma\cdot(px_4+\tfrac{1}{2}k(x_1-x_2+x_3-1)-r(x_3-1))]\gamma^{\alpha_2}\gamma_\nu\gamma_\mu\}u(p_1)\,,\end{aligned}$$

$$D_{2a}=q^2+b^2-i\epsilon\,,$$

$$b^2=2p\cdot rx_3x_4+c^2\,,$$

$$c^2=\lambda^2x_4+m^2(x_1+x_2+x_3)^2+k^2x_2(x_1+x_3)-k\cdot rx_3(2x_2+x_4)+r^2x_3(1-x_3)\,.$$

[7] We use the metric $g^{\mu\nu}=(-1,1,1,1)$, $\gamma^k=-\gamma^{k+}$, $\gamma^0=\gamma^{0+}$, $\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$. The Dirac equation is $(m+\gamma\cdot p)\,u(p)=0$.

We have shifted the q integration to arrive at this form. Our problem now is to extract the leading terms as $p\to\infty$. Given the denominator function D_{2a}, it is seen that the major contribution comes from that region where $x_3\cong 0$ and/or $x_4\cong 0$.[8] On the other hand, it is easy to see (for example, by infinite-momentum technique[2]) that

$$N^{\alpha_1\alpha_2}\cong\bar{u}(p_3)\gamma^\mu[m-\gamma\cdot(px_4+\tfrac{1}{2}k(1+x_1-x_2+x_3)-rx_3)]$$
$$\times\gamma^{\alpha_1}(-\gamma\cdot px_4)\gamma^{\alpha_2}[m-\gamma\cdot(px_4-\tfrac{1}{2}k(1-x_1+x_2-x_3)-rx_3)]\gamma_\mu u(p_1)+\tfrac{1}{4}q^2\bar{u}(p_3)\gamma^\mu\gamma^\nu\gamma^{\alpha_1}(-\gamma\cdot px_4)\gamma^{\alpha_2}\gamma_\nu\gamma_\mu u(p_1).$$

The essential point to observe now is that $N^{\alpha_1\alpha_2}$ is proportional to x_4. For this reason, we can safely discard all factors with x_3, since a factor x_3x_4 will give rise to terms of order $1/2p\cdot r$ (up to powers of logarithm) lower than the other. Therefore,

$$N_{p\to\infty}{}^{\alpha_1\alpha_2}\cong 4mp^{\alpha_1}p^{\alpha_2}x_4[1-4x_4+x_4^2-(k^2/m^2)(1-x_1)(1-x_2)]+(2q^2/m)p^{\alpha_1}p^{\alpha_2}x_4.$$

The amplitude can be written in two parts:

$$M_{2a}=M_{2a}{}^{(1)}+M_{2a}{}^{(2)}, \tag{4}$$

corresponding to the first and the second term, respectively, of $N^{\alpha_1\alpha_2}$. After the q integration, we find

$$\begin{aligned}M_{2a}{}^{(1)}&=\frac{e^4}{16\pi^2}4mp^{\alpha_1}p^{\alpha_2}\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\frac{1}{(b^2-i\epsilon)^2}x_4\left[1-4x_4+x_4^2-\frac{k^2}{m^2}(1-x_1)(1-x_2)\right]\\
&\cong\frac{e^4}{16\pi^2}4mp^{\alpha_1}p^{\alpha_2}\int_0^1 dx_1dx_2dx_4\,\delta(1-x_1-x_2-x_4)x_4[1-4x_4+x_4^2-(k^2/m^2)(1-x_1)(1-x_2)]\\
&\qquad\times\int_0^1 dx_3\,[2p\cdot rx_3x_4+m^2(x_1+x_2)^2+k^2x_1x_2+\lambda^2x_4-i\epsilon]^{-2}\\
&\cong\frac{e^4}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r-i\epsilon}\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\\
&\qquad\times\frac{-4m^2(1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2)-2k^2(1-x_2)(1-x_3)}{m^2(x_2+x_3)^2+k^2x_2x_3+\lambda^2x_1}.\end{aligned}\tag{5}$$

Also,

$$M_{2a}{}^{(2)}=\frac{e^4}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}}{m}\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\frac{x_4}{b^2-i\epsilon}. \tag{6}$$

In order to extract the first two leading terms in p of this integral, it is convenient to introduce the following additional integral:

$$\begin{aligned}\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\frac{1}{sx_3+a^2}&=\frac{1}{s}\left\{\frac{s^2-a^4}{2s^2}\ln\left(1+\frac{s}{a^2}\right)+\frac{a^2}{2s}+\frac{a^2}{s}\left[\frac{a^2+s}{s}\ln\left(1+\frac{s}{a^2}\right)-1\right]-\frac{3}{4}\right\}\\
&\cong(1/2s)\ln s-(1/2s)\ln a^2-3/4s,\end{aligned}\tag{7}$$

where a is some arbitrary nonzero number with the dimension of mass. Using also the identity

$$\frac{1}{\alpha}-\frac{1}{\beta}=-\int_0^1 dz\frac{\alpha-\beta}{[(\alpha-\beta)z+\beta]^2},$$

we write ($s=2p\cdot r-i\epsilon$)

$$\begin{aligned}M_{2a}{}^{(2)}&\cong\frac{e^4}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}}{m}\left[\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)x_4\left(\frac{1}{sx_3x_4+c^2}-\frac{1}{sx_3x_4+a^2x_4}\right)+\frac{1}{2s}\ln\frac{s}{a^2}-\frac{3}{4s}\right]\\
&\cong\frac{e^4}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}}{m}\left\{-\int_0^1 dz\int_0^1 dx_1dx_2dx_3dx_4\,\delta(1-x_1-x_2-x_3-x_4)\frac{x_4(c^2-a^2x_4)}{[(c^2-a^2x_4)z+sx_3x_4+a^2x_4]^2}+\frac{1}{2s}\ln\frac{s}{a^2}-\frac{3}{4s}\right\}\\
&\cong\frac{e^4}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}}{m}\left\{-\int_0^1 dz\int_0^1 dx_1dx_2dx_4\,\delta(1-x_1-x_2-x_4)\frac{c'^2-a^2x_4}{(c'^2-a^2x_4)z+a^2x_4}\frac{1}{s}+\frac{1}{2s}\ln\frac{s}{a^2}-\frac{3}{4s}\right\},\end{aligned}$$

[8] There are, of course, many equivalent ways to extract the high-energy behavior of amplitudes. We follow the method of P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) **22**, 263 (1969); **22**, 299 (1963); J. C. Polkinghorne, J. Math. Phys. **4**, 503 (1963).

where to come to the last step, we have once again made use of the fact that the high-s behavior comes from the region where $x_3 \cong 0$, resulting in

$$c^2 \to c'^2 = m^2(x_1+x_2)^2 + k^2 x_1 x_2 + \lambda^2 x_4$$

and

$$M_{2a}{}^{(2)} \cong \frac{e^4}{16\pi^2} \frac{2p^{\alpha_1}p^{\alpha_2}}{m} \frac{1}{2p\cdot r - i\epsilon} \left\{ \ln(2p\cdot r - i\epsilon) - \tfrac{3}{2} - \int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) 2\ln\left[\frac{m^2(x_2+x_3)^2 + k^2 x_2 x_3 + \lambda^2 x_1}{x_1}\right]\right\}. \quad (8)$$

Our contention is that parts of the vertex correction and the self-energy effect of the internal electron line cancel the term which is proportional to $[1/(2p\cdot r - i\epsilon)] \ln(2p\cdot r - i\epsilon)$. Let us then turn first to the self-energy figure 2(b), which gives

$$M_{2b} = e^2 \bar{u}(p_3)\gamma^{\alpha_1}[m+\gamma\cdot(p+r-\tfrac{1}{2}k) - i\epsilon]^{-1}[-\Sigma_{\rm ren}(p+r-\tfrac{1}{2}k)][m+\gamma\cdot(p+r-\tfrac{1}{2}k)-i\epsilon]^{-1}\gamma^{\alpha_2}u(p_1), \quad (9)$$

where $\Sigma_{\rm ren}$ is the renormalized part of the self-energy; i.e.,

$$-\Sigma_{\rm ren}(p+r-\tfrac{1}{2}k = p') = (m+\gamma\cdot p')^2 \left\{ -\frac{1}{8\pi^2}\int_0^1 dz \int_0^1 dx_1 dx_2\, \delta(1-x_1-x_2) x_1 x_2 \times \frac{m(1+x_2) + (\gamma\cdot p' - m)(1-x_2)[1-2m^2x_2(1+x_2)z/(m^2x_2{}^2+\lambda^2 x_1)]}{m^2 x_2{}^2 + (p'^2+m^2)x_2(1-x_2)z + \lambda^2 x_1}\right\}. \quad (10)$$

It is easy to extract the dominant contribution, which comes from the portion proportional to $(\gamma\cdot p' - m)$,

$$M_{2b} \cong \frac{e^4}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r - i\epsilon}\left[\ln\frac{2p\cdot r - i\epsilon}{m^2} - 2 - 2\int_0^1 dx\,(1-x)\ln\left(x^2 + \frac{\lambda^2}{m^2}(1-x)\right) - \int_0^1 dx_1 dx_2\,\delta(1-x_1-x_2)\frac{4m^2x_2(1-x_2{}^2)}{m^2x_2{}^2+\lambda^2x_1}\right]. \quad (11)$$

We now come to the vertex correction. Note that one of the electron lines is off mass shell. Figure 2(c) gives an amplitude

$$M_{2c} \cong \bar{u}(p_3)\gamma^{\alpha_1}[m+\gamma\cdot(p+r-\tfrac{1}{2}k)-i\epsilon]^{-1} \times \Lambda_{\rm ren}{}^{\alpha_2}u(p_1), \quad (12)$$

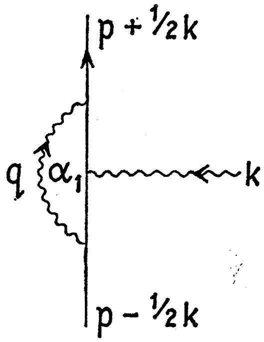

FIG. 1. One-photon absorption with radiative correction.

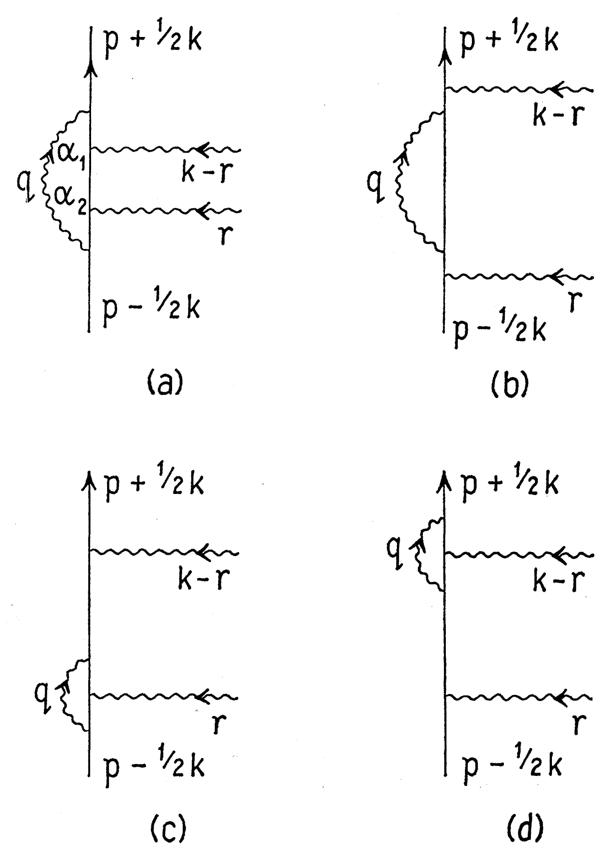

FIG. 2. Two-photon absorption with radiative correction.

where $\Lambda_{\rm ren}{}^{\alpha_2}$ is the renormalized vertex,

$$\Lambda_{\rm ren}{}^{\alpha_2}=2!\int dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\int\frac{d^4q}{(2\pi)^4i}\times\left(\frac{N^{\alpha_2}}{D_{2c}{}^3}-\frac{\gamma^{\alpha_2}\{q^2-4m^2[1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2]\}}{[q^2+m^2(x_2+x_3)^2+\lambda^2x_1-i\epsilon]^3}\right). \quad (13)$$

In this expression, the second term is due to subtraction, and

$$D_{2c}=q^2+m^2(x_2+x_3)^2+\lambda^2x_1+2(p-\tfrac{1}{2}k)\cdot rx_1x_2+r^2x_2(1-x_2)-i\epsilon\,,$$
$$N^{\alpha_2}=\gamma_\mu\{m-\gamma\cdot[px_1-\tfrac{1}{2}kx_1+r(1-x_2)-q]\}\gamma^{\alpha_2}\times[m-\gamma\cdot(px_1-\tfrac{1}{2}kx_1-rx_2-q)]\gamma_\mu\,. \quad (14)$$

It is straightforward to show that the dominant contribution is

$$u(p_3)\gamma^{\alpha_1}[m-\gamma\cdot(p+r-\tfrac{1}{2}k)]N^{\alpha_2}u(p_1)\cong\bar{u}(p_3)\gamma^{\alpha_1}[m-\gamma\cdot(p+r-\tfrac{1}{2}k)]\gamma^{\alpha_2}q^2u(p_1)\cong2(p^{\alpha_1}p^{\alpha_2}/m)q^2\,.$$

The standard technique gives

$$M_{2c}\cong\frac{e^4}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r-i\epsilon}\int dx_1dx_2dx_3\times\delta(1-x_1-x_2-x_3)\left(-2\ln\frac{(2p\cdot r-i\epsilon)x_1x_2}{m^2(x_2+x_3)^2+\lambda^2x_1}+\frac{4m^2[1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2]}{m^2(x_2+x_3)^2+\lambda^2x_1}\right). \quad (15)$$

We point out that the first term comes from

$$\frac{N^{\alpha_2}}{D_{2c}{}^3}-\frac{\gamma^{\alpha_2}q^2}{[q^2+m^2(x_2+x_3)^2+\lambda^2x_1-i\epsilon]^3}$$

in Eq. (13). It was pointed out that the part which contributes to N^{α_2} is $N^{\alpha_2}\cong\gamma^{\alpha_2}q^2$. As we shall see, this is in conformity with a general rule which states that for any graphs (except self-energy correction) such that the radiative-correction photon line has at least one end tacked onto an off-mass-shell electron line, then the only part of the numerator function which can contribute to the first two leading terms in p of the amplitude must be proportional to q^2. Furthermore, these two powers of q must come from the electron lines which touch the two ends of the radiative-correction photon line.

Similarly, we can show that

$$M_{2d}=M_{2c}\,. \quad (16)$$

When we add these four amplitudes up [Eqs. (4), (5), (8), (11), (15), and (16)], we see that the $\ln(2p\cdot r-i\epsilon)/(2pr-i\epsilon)$ term cancels out, as we promised. Besides, a remarkable thing happens, because

$$M_{2a}+M_{2c}=M_{2a}+M_{2d}=e^2\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r-i\epsilon}\times\left[\frac{e^2}{16\pi^2}\left(\tfrac{3}{2}-2\int_0^1dx_1dx_2dx_3\times\delta(1-x_1-x_2-x_3)\ln x_2\right)+F(k^2)\right].$$

It is easy to show that

$$-\tfrac{3}{2}-2\int_0^1dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\ln x_2=0\,.$$

Let us also define

$$e^2\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r-i\epsilon}G=M_{2b}+M_{2c}=M_{2b}+M_{2d}\,,$$

or

$$G=\frac{e^2}{16\pi^2}\left[1-2\int_0^1dx\,(1-2x)\ln\left(x^2+\frac{\lambda^2}{m^2}(1-x)\right)-\int_0^1dx\frac{4m^2x^2(1-\frac{1}{2}x)}{m^2x^2+\lambda^2(1-x)}\right] \quad (17)$$

after some simplification. Then,

$$M_2=M_{2a}+M_{2b}+M_{2c}+M_{2d}=e^2\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r-i\epsilon}[F(k^2)+G]. \quad (18)$$

C. Three-Photon and Multiphoton Absorption

We are now in a position to attack the general case. However, it is perhaps more pedagogical to work out the case with three external photon lines and then discuss the general situation, since otherwise we shall have to write down large numbers of superscripts and subscripts, which is quite confusing.

There are eight diagrams we have to contend with when three photons are absorbed [Figs. 3(a)–3(h)], out of which four have been done already [3(e)–3(h)]. The most complicated one is Fig. 3(a), which gives

$$M_{3a}=e^54!\int_0^1dxdx_1dx_2dz_1dz_2\times\delta(1-x-x_1-x_2-z_1-z_2)\int\frac{d^4q}{(2\pi)^4i}\frac{N^{\alpha_1\alpha_2\alpha_3}}{D_{3a}{}^5}\,, \quad (19)$$

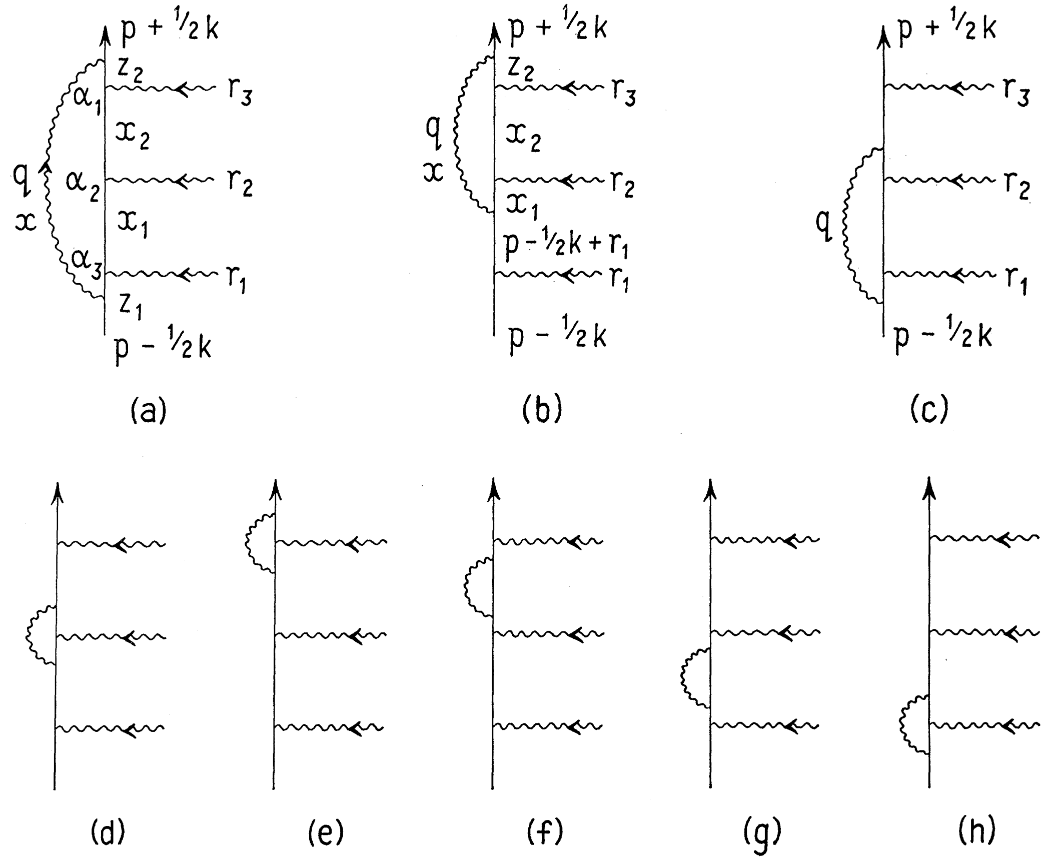

FIG. 3. Three-photon absorption with radiative correction.

where

$$D_{3a}=q^2+m^2(x_1+x_2+z_1+z_2)^2 +\lambda^2 x+2x(p-\tfrac{1}{2}k)\cdot[x_2(r_1+r_2)+x_1r_1] -[x_2(r_1+r_2)+x_1r_1]^2-2z_2k\cdot[x_2(r_1+r_2)+x_1r_1] +k^2z_2(x_1+x_2+z_1)+x_2(r_1+r_2)^2+x_1r_1^2-i\epsilon,$$

and by the infinite-momentum technique

$$N^{\alpha_1\alpha_2\alpha_3}\cong\bar{u}(p_3)\gamma_\mu[m-\gamma\cdot(px+\tfrac{1}{2}k(1+z_1-z_2+x_1+x_2) -x_1r-x_2(r_1+r_2))]\gamma^{\alpha_1}(-\gamma\cdot px)\gamma^{\alpha_2}(-\gamma\cdot px)\gamma^{\alpha_3} \times[m-\gamma(px-\tfrac{1}{2}k(1-z_1+z_2-x_1-x_2) -x_1r_1-x_2(r_1+r_2))]\gamma_\mu u(p_1)+\tfrac{1}{4}q^2\bar{u}(p_3) \times\gamma_\mu\gamma_\nu\gamma^{\alpha_1}(-\gamma\cdot px)\gamma^{\alpha_2}(-\gamma\cdot px)\gamma^{\alpha_3}\gamma_\nu\gamma_\mu u(p_1). \quad (20)$$

The second term comes from an average over q. In fact, the factor q^2 is extracted from the lines with parameters z_1 and z_2. We can show that q^2 terms due to other terms are negligible. Suppose we take from lines with parameters x_1 and x_2 each a factor $\gamma\cdot q$; then we shall have in the numerator function a term

$$\sim\tfrac{1}{4}q^2\gamma^{\alpha_1}\gamma_\nu\gamma^{\alpha_2}\gamma^\nu\gamma^{\alpha_3}\sim q^2\gamma^{\alpha_1}\gamma^{\alpha_2}\gamma^{\alpha_3}. \quad (21)$$

Now, by the infinite-momentum technique, the large component of the vector γ^α is

$$\gamma_+=\gamma^3+\gamma^0\sim p,$$

the small component is

$$\gamma_-\sim\gamma^3-\gamma^0\sim 1/p,$$

and the components that are of order unity are

$$\gamma_\perp=\gamma_1 \quad\text{or}\quad \gamma_2.$$

Since $\gamma_+\gamma_+=0$ and $\{\gamma_+,\gamma_\perp\}=0$, we see that the leading term in Eq. (21) is two powers of p smaller than the second term of (20). By the same kind of argument, we can show that all the other q^2 and $(q^2)^2$ terms are small.

It is convenient to introduce the substitution

$$x_1=wy_1,\quad x_2=w(1-y_1),\quad dx_1dx_2=wdw\,dy_1; \quad (22)$$

then

$$D_{3a}\cong q^2+m^2(z_1+z_2)^2+\lambda^2x+2pwx\cdot(r_2y_1+r_1)+k^2z_1z_2-i\epsilon$$

and

$$N^{\alpha_1\alpha_2\alpha_3}\cong(4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}/m)x^2[2m^2(1-4x+x^2) -\tfrac{1}{2}k^2(1-z_1+z_2+x)(1+z_1-z_2+x)] +(4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}/m)x^2q^2.$$

The amplitude can be written as

$$M_{3a}=M_{3a}{}^{(1)}+M_{3a}{}^{(2)}\,, \tag{23}$$

where, after the q integration,

$$M_{3a}{}^{(1)}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}2\int_0^1 dz_1dz_2dwdx\,\delta(1-z_1-z_2-w-x)$$
$$\times wx^2\int_0^1 dy_1\frac{2m^2(1-4x+x^2)-\frac{1}{2}k^2(1-z_1+z_2+x)(1+z_1-z_2+x)}{[2pxw\cdot(r_2y_1+r_1)+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x-i\epsilon]^3}$$
$$\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\int_0^1 dz_1dz_2dw\,\delta(1-z_1-z_2-w-x)$$
$$\times x[2m^2(1-4x-x^2)-\tfrac{1}{2}k^2(1-z_1+z_2+x)(1+z_1-z_2+x)][-(2p\cdot r_2-i\epsilon)^{-1}]$$
$$\times\{[2pwx\cdot(r_2+r_1)+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x]^{-2}-[2pwx\cdot r_1+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x]^{-2}\}\,.$$

The range of integration where $w\sim 0$ gives us the leading term, which is

$$M_{3a}{}^{(1)}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_2-i\epsilon}\left(\frac{1}{2p\cdot r_1-i\epsilon}-\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\right)\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)$$
$$\times\frac{-4m^2(1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2)-2k^2(1-x_2)(1-x_3)}{k^2x_2x_3+m^2(x_2+x_3)^2+\lambda^2x_1}\,. \tag{24}$$

Similarly, after the q integration,

$$M_{3a}{}^{(2)}=\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}2\int_0^1 dz_1dz_2dx_1dx_2dx\,\delta(1-z_1-z_2-x_1-x_2-x)$$
$$\times x^2[2px\cdot((r_1+r_2)x_2+r_1x_1)+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x-i\epsilon]^{-2}$$
$$=\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}2\int_0^1 dz_1dz_2dxdw\,\delta(1-z_1-z_2-x-w)wx^2\int_0^1 dy_1$$
$$\times[2pxw\cdot(r_2y_1+r_1)+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x-i\epsilon]^{-2}$$
$$=\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}2\int_0^1 dz_1dz_2dxdw\,\delta(1-z_1-z_2-x-w)x[-(2p\cdot r_2-i\epsilon)^{-1}]$$
$$\times[(2p\cdot(r_1+r_2)xw+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x-i\epsilon)^{-1}-(2p\cdot r_1xw+k^2z_1z_2+m^2(z_1+z_2)^2+\lambda^2x-i\epsilon)^{-1}]\,.$$

As before, to facilitate the extraction of the first two leading terms, it is helpful to introduce the extra integral of Eq. (7). One then finds, following the same procedure that led to $M_{2a}{}^{(2)}$,

$$M_{3a}{}^{(2)}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\Big\{(2p\cdot r_2-i\epsilon)^{-1}\{(2p\cdot r_1-i\epsilon)^{-1}2\ln(2p\cdot r_1-i\epsilon)$$
$$-[2p\cdot(r_1+r_2)-i\epsilon]^{-1}2\ln[2p\cdot(r_1+r_2)-i\epsilon]\}+(2p\cdot r_2-i\epsilon)^{-1}[(2p\cdot r_1-i\epsilon)^{-1}-(2p\cdot(r_1+r_2)-i\epsilon)^{-1}]$$
$$\times\left[-\tfrac{3}{2}-\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)2\ln\left(\frac{k^2x_2x_3+m^2(x_2+x_3)^2+\lambda^2x_1}{x_1}\right)\right]\Big\}\,. \tag{25}$$

We see the alikeness of Eqs. (24) and (25) to Eqs. (5) and (8).

Now we take up Fig. 3(b), which gives

$$M_{3b}\cong e^2[2p^{\alpha_3}/(2p\cdot r_1-i\epsilon)]M^{\alpha_1\alpha_2}\,, \tag{26}$$

where

$$M^{\alpha_1\alpha_2}=3!\int_0^1 dx_1dx_2dxdz\,\delta(1-x_1-x_2-x-z)\int\frac{d^4q}{(2\pi)^4i}\frac{N^{\alpha_1\alpha_2}}{D_{3b}{}^4} \tag{27}$$

and

$$D_{3b}=q^2+2px\cdot[(r_1+r_2)x_2+r_1x_1]+d^2. \tag{28}$$

Here, d^2 is some function independent of q and p. Its exact structure is immaterial, as we shall see. Also,

$$N^{\alpha_1\alpha_2}\cong(2p^{\alpha_1}p^{\alpha_2}/m)q^2x. \tag{29}$$

There are, of course, many other terms in $N^{\alpha_1\alpha_2}$. However, we shall argue later on that they are indeed negligible. After the q integration we have

$$\begin{aligned}M^{\alpha_1\alpha_2}&\cong\frac{1}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\int_0^1 dx_1dx_2dxdz\\&\quad\times\delta(1-x_1-x_2-x-z)\\&\quad\times 2x/\{2px\cdot[(r_1+r_2)x_2+r_1x_1]+d^2\}\\&\cong\frac{1}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\int_0^1 dxdzdw\,\delta(1-x-z-w)xw\\&\quad\times\int_0^1 dy_1\frac{2}{2pxw\cdot(r_2y_1+r_1)+d^2},\end{aligned} \tag{30}$$

after introducing the same change of variables given in Eq. (22). The y_1 integration renders

$$M^{\alpha_1\alpha_2}\cong\frac{1}{16\pi^2}\frac{2p^{\alpha_1}p^{\alpha_2}}{m}\frac{1}{2p\cdot r_2-i\epsilon}\ln\frac{2p\cdot(r_1+r_2)-i\epsilon}{2p\cdot r_1-i\epsilon},$$

or

$$M_{3b}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\frac{1}{2p\cdot r_2-i\epsilon}\times\ln\frac{2p\cdot(r_1+r_2)-i\epsilon}{2p\cdot r_1-i\epsilon}. \tag{31}$$

For the terms without the factor q^2 in the numerator function $N^{\alpha_1\alpha_2}$ [Eq. (29)], after the q integration, we shall have instead of Eq. (30) some quantity

$$\begin{aligned}&\sim\int_0^1 dxdzdw\,\delta(1-x-z-w)xw\int_0^1 dy_1\\&\qquad\times\frac{1}{[2pxw\cdot(r_2y+r_1)+d^2]^2}\\&\sim\int_0^1 dxdzdw\,\delta(1-x-z-w)\frac{-1}{2p\cdot r_2-i\epsilon}\\&\qquad\times\{[2pxw\cdot(r_2+r_1)+d^2]^{-1}-[2pxw\cdot r_1+d^2]^{-1}\}.\end{aligned}$$

Clearly, these terms are proportional to $(2p\cdot r_1-i\epsilon)^{-1}\times(2p\cdot r_2-i\epsilon)^{-1}$, etc. (again, up to powers of the logarithm), which are one power lower in p.

In a similar fashion, we obtain

$$M_{3c}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\frac{1}{2p\cdot r_2-i\epsilon}\times\ln\frac{2p\cdot(r_1+r_2)-i\epsilon}{2p\cdot r_1-i\epsilon}, \tag{32}$$

$$\begin{aligned}M_{3d}&\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\frac{1}{2p\cdot r_1-i\epsilon}\\&\quad\times\left(\frac{2p\cdot r_1}{2p\cdot r_2-i\epsilon}\ln(2p\cdot r_1-i\epsilon)\right.\\&\quad\left.-\frac{2p\cdot(r_1+r_2)}{2p\cdot r_2-i\epsilon}\ln[2p\cdot(r_1+r_2)-i\epsilon]+\bar{\Lambda}\right),\end{aligned} \tag{33}$$

$$M_{3e}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\times\{-\ln[2p\cdot(r_1+r_2)-i\epsilon]+\bar{\Lambda}\}, \tag{34}$$

$$M_{3f}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\times\{\ln[2p\cdot(r_1+r_2)-i\epsilon]+\bar{\Sigma}\}, \tag{35}$$

$$M_{3g}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\times\{\ln(2p\cdot r_1-i\epsilon)+\bar{\Sigma}\}, \tag{36}$$

$$M_{3h}\cong\frac{e^5}{16\pi^2}\frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\frac{1}{2p\cdot(r_1+r_2)-i\epsilon}\times\{-\ln(2p\cdot r_1-i\epsilon)+\bar{\Lambda}\}, \tag{37}$$

with

$$\begin{aligned}\bar{\Lambda}&=\int_0^1 dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\\&\quad\times\left(-2\ln\frac{x_1x_2}{m^2(x_2+x_3)^2+\lambda^2x_1}\right.\\&\quad\left.+\frac{4m^2[1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2]}{m^2(x_2+x_3)^2+\lambda^2x_1}\right)\end{aligned} \tag{38}$$

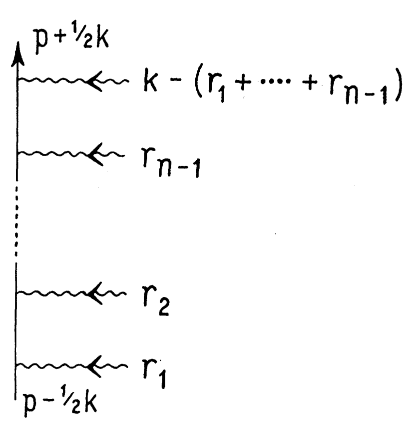

FIG. 4. Multiphoton absorption without radiative correction.

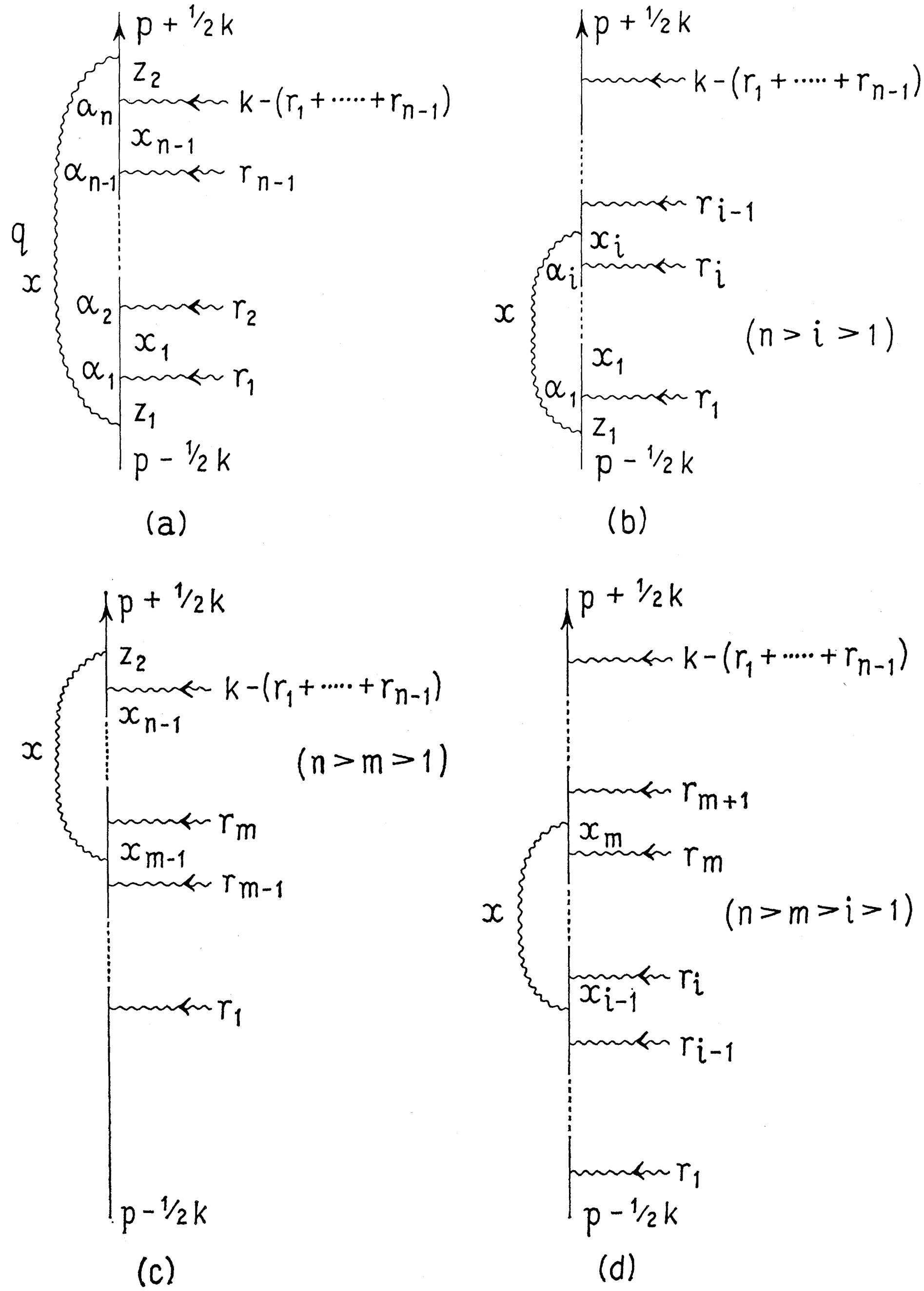

FIG. 5. Multiphoton absorption with radiative correction.

and

$$\bar{\Sigma} = -\ln m^2 - 2 - 2\int_0^1 dx\,(1-x) \times \ln\left(x^2 + \frac{\lambda^2}{m^2}(1-x)\right) + \int_0^1 dx_1 dx_2\,\delta(1-x_1-x_2)\frac{4m^2x_2(1-x_2{}^2)}{m^2x_2{}^2+\lambda^2 x_1}. \quad (39)$$

Once again, all the p-dependent logarithmic terms cancel completely among themselves. The final result is

$$M_3 \cong e^3 \frac{4p^{\alpha_1}p^{\alpha_2}p^{\alpha_3}}{m} \frac{1}{2p\cdot r_1 - i\epsilon} \frac{1}{2p\cdot(r_1+r_2) - i\epsilon} \times [F(k^2)+2G]. \quad (40)$$

By now, it should be obvious what the trend is for the general case. Thus, to include second-order radiative correction to a graph with n absorbed photons (Fig. 4), we shall have to consider three classes of diagrams, besides vertex and self-energy insertion: (i) The radiative-correction photon line is inserted into the two on-mass-shell electron lines [Fig. 5(a)]; (ii) it is inserted into one on- and one off-mass-shell electron line [Figs. 5(b) and 5(c)]; and (iii) it is inserted into two off-mass-shell electron lines [Fig. 5(d)]. The previous examples reveal that, after introducing Feynman parameters and shifting the origin of q integration, the relevant terms in the numerator functions will be

$$N_{5a}\sim(2p^{\alpha_1})\cdots(2p^{\alpha_{n-1}})(p^{\alpha_n}/m)\times x^{n-1}\{-4m^2[1-(z_1+z_2)-\tfrac{1}{2}(z_1+z_2)^2]-2k^2(1-z_1)(1-z_2)+q^2\},$$

$$N_{5b}\sim(2p^{\alpha_1})\cdots(2p^{\alpha_{i-1}})(p^{\alpha_i}/m)x^{i-1}q^2,$$

$$N_{5c}\sim(2p^{\alpha_m})\cdots(2p^{\alpha_{n-1}})(p^{\alpha_n}/m)x^{n-m}q^2,$$

$$N_{5d}\sim(2p^{\alpha_i})\cdots(2p^{\alpha_{m-1}})(p^{\alpha_m}/m)x^{m-i}q^2,$$

while the denominator functions will be

$$D_{5a}\sim q^2+2px\cdot[r_1x_1+\cdots+(r_1+\cdots+r_{n-1})x_{n-1}]+m^2(z_1+z_2)^2+k^2z_1z_2+\lambda^2x-i\epsilon,$$

$$D_{5b}\sim q^2+2px\cdot[r_1x_1+\cdots+(r_1+\cdots+r_i)x_i]-i\epsilon,$$

$$D_{5c}\sim q^2+2px\cdot[(r_1+\cdots+r_{m-1})x_{m-1}+\cdots+(r_1+\cdots+r_{n-1})x_{n-1}]-i\epsilon,$$

$$D_{5d}\sim q^2+2px\cdot[(r_1+\cdots+r_{i-1})x_{i-1}+\cdots+(r_1+\cdots+r_m)x_m]-i\epsilon.$$

To be specific, we shall discuss Fig. 5(a) in greater detail. After the q integration, we make a change of variables

$$\begin{aligned}x_1&=wy_1,\\x_2&=w(1-y_1)y_2,\\&\cdots\\x_{n-2}&=w(1-y_1)\cdots(1-y_{n-3})y_{n-2},\\x_{n-1}&=w(1-y_1)\cdots(1-y_{n-1})(1-y_{n-2}),\end{aligned}\tag{41}$$

which has the properties that

$$x_1+\cdots+x_{n-1}=w,$$

$$dx_1\cdots dx_{n-1}=w^{n-2}(1-y_1)^{n-3}\cdots(1-y_{n-3})dwdy_1\cdots dy_{n-2},$$

and that the ranges of integration for the y's are from 0 to 1. It is then simple to carry out the y integration, starting from y_{n-2}, after which we see that both the Jacobian and the factor x^{n-1} from the numerator function are cancelled away. We are left with expressions very similar to those in the previous examples. Once again, introducing the integral expression in Eq. (7) and going through that same manipulation, we find

$$M_{5a}\cong\frac{\alpha^{n+2}}{16\pi^2}(2p^{\alpha_1})\cdots(2p^{\alpha_{n-1}})\frac{p^{\alpha_n}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\cdots\frac{1}{2p\cdot(r_1+\cdots+r_{n-1})-i\epsilon}\times\bigg[(p\text{-dependent logarithmic terms})-\tfrac{3}{2}+\int_0^1dx_1dx_2dx_3\,\delta(1-x_1-x_2-x_3)\times\bigg(\frac{-4m^2(1-(x_2+x_3)-\frac{1}{2}(x_2+x_3)^2)-2k^2(1-x_2)(1-x_3)}{m^2(x_2+x_3)^2+k^2x_2x_3+\lambda^2x_1}-2\ln\frac{m^2(x_2+x_3)^2+k^2x_2x_3+\lambda^2x_1}{x_1}\bigg)\bigg].$$

After a similar procedure, we also find that Figs. 5(b)–5(d) contribute to only p-dependent logarithmic terms. As before, all these cancel with those which come from mass and vertex renormalization effects. We are finally left with

$$M_n\cong e^n(2p^{\alpha_1})\cdots(2p^{\alpha_{n-1}})\frac{p^{\alpha_n}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\cdots\frac{1}{2p\cdot(r_1+\cdots+r_{n-1})-i\epsilon}[F(k^2)+(n-1)G].\tag{42}$$

The factor $F(k^2)$ is due to M_{5a} plus one of the vertex corrections. The rest of the vertex corrections $(n-1)$ pairs with the $(n-1)$ self-energy insertions to give the factor $(n-1)G$.

Now, we shall show that G [Eq. (17)] actually vanishes. This is seen if we perform integration by parts of the term

$$\int_0^1dx\,(1-2x)\ln\left(x^2+\frac{\lambda^2}{m^2}(1-x)\right)=\int_0^1d(x-x^2)\ln\left(x^2+\frac{\lambda^2}{m^2}(1-x)\right)=-\int_0^1dx\frac{(x-x^2)(2x-\lambda^2/m^2)}{x^2+(\lambda^2/m^2)(1-x)}.$$

Consequently,

$$G=0.\tag{43}$$

III. FULL AMPLITUDES

For definiteness, we shall write down the high-energy spin-nonflip amplitude of "electron-electron" scattering $e(p_1)+e'(p_2)\to e(p_3)+e'(p_4)$, although what we say in the following can be applied directly to other cases involving electrons. The graphs we include in the present consideration are shown in Fig. 6. The number of photons that are exchanged in the t channel are arbitrary. For n photons, after fixing the labels on the photons which come out from one of the electrons, the

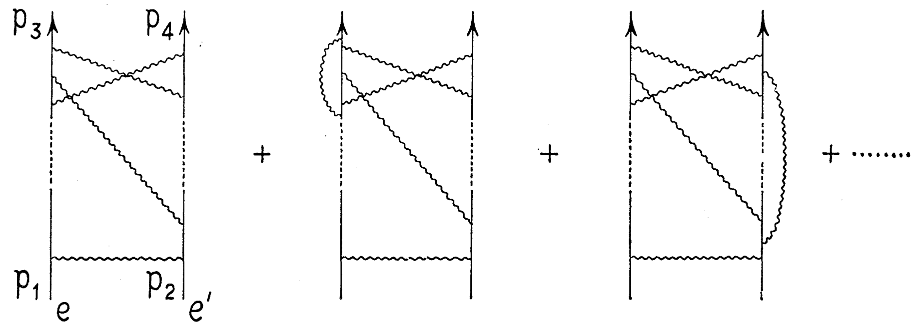

FIG. 6. Electron-electron scattering with radiative correction.

other ends are to be permuted in $n!$ different orders and tacked onto the other electron. The radiative photon lines on both sides are to be inserted in all manners as discussed before. It is well known that this procedure will preserve gauge invariance.

We shall choose the coordinate system in such a way that electrons 2 and 4 are moving almost in the $-z$ direction. Thus

$$p_2=p'+\tfrac{1}{2}k,\quad p_4=p'-\tfrac{1}{2}k,$$
$$p'^{\mu}=((|p'|^2+\tfrac{1}{4}k^2+m'^2)^{1/2},0,0,-|p'|).$$

We shall normalize the invariant amplitude in such a way that the Born amplitude is

$$T_{\text{Born}}=\bar{u}(p_4)\gamma^{\mu}u(p_2)\bar{u}(p_3)\gamma_{\mu}u(p_1)1/k^2 \cong (p\cdot p'/mm')1/k^2. \tag{44}$$

Then, the invariant amplitude is

$$T=i\int d^4x\,e^{-ik\cdot x}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\left(\int\frac{d^4r}{(2\pi)^4}e^{ir\cdot x}\frac{-i}{r^2+\lambda^2-i\epsilon}\right)_i\right]$$
$$\times\left(e^n[1+F(k^2)](2p)^{\alpha_1}\cdots(2p)^{\alpha_{n-1}}\frac{p^{\alpha_n}}{m}\frac{1}{2p\cdot r_1-i\epsilon}\cdots\frac{1}{2p\cdot(r_1+\cdots+r_{n-1})-i\epsilon}\right)$$
$$\times\left(e'^n[1+F'(k^2)](2p')_{\alpha_1}\cdots(2p')_{\alpha_{n-1}}\frac{p_{\alpha_n}'}{m'}\sum_{\text{perm}}^{n}\frac{1}{-2p'\cdot r_{\nu_1}-i\epsilon}\cdots\frac{1}{-2p'\cdot(r_{\nu_1}+\cdots+r_{\nu_{n-1}})-i\epsilon}\right), \tag{45}$$

where ν_i indices are to be permuted $n(n-1)$ times over $1,\ldots,n$. Now, we introduce the variables

$$r_+=r_3+r^0,\quad r_-=r_3-r^0;$$

then

$$p\cdot r=|p|r_-,\quad -p'\cdot r=|p'|r_+,$$

and

$$T=(2\pi)^2 i\int d^2x_\perp\,e^{-ik\cdot x_\perp}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\left(\int\frac{d^2r_\perp dr_+dr_-}{2(2\pi)^4}e^{ir_\perp\cdot x_\perp}\frac{-i}{r_+r_-+r_\perp^2-i\epsilon}\right)_i\right]$$
$$\times[1+F(k^2)][1+F'(k^2)](ee')^n\frac{(4p\cdot p')^{n-1}}{(4|p||p'|)^{n-1}}\frac{p\cdot p'}{mm'}\left(\frac{1}{r_1-i\epsilon}\right)_-\cdots\left(\frac{1}{r_1+\cdots+r_{n-1}-i\epsilon}\right)_-2\delta(r_1+\cdots+r_n)_-$$
$$\times\sum_{\text{perm}}^{n}\delta(r_1+\cdots+r_n)_+\left(\frac{1}{r_{\nu_1}-i\epsilon}\right)_+\cdots\left(\frac{1}{r_{\nu_1}+\cdots+r_{\nu_{n-1}}-i\epsilon}\right)_+.$$

Upon using the relations[2]

$$\sum_{\text{perm}}^{n}\delta(r_1+\cdots+r_n)_+\left(\frac{1}{r_{\nu_1}-i\epsilon}\right)_+\cdots\left(\frac{1}{r_{\nu_1}+\cdots+r_{\nu_{n-1}}-i\epsilon}\right)_+ =(2\pi i)^{n-1}\delta(r_1)_+\cdots\delta(r_n)_+$$

and

$$8|p||p'|=-4p\cdot p',$$

we find

$$T=i[1+F(k^2)][1+F'(k^2)]p\cdot p'/mm'$$
$$\times\int d^2x_\perp\,e^{-ik_\perp\cdot x_\perp}$$
$$\times\left[\exp\left(-iee'\int\frac{d^2r_\perp}{(2\pi)^2}e^{ir_\perp\cdot x_\perp}\frac{1}{r_\perp^2+\lambda^2-i\epsilon}\right)-1\right], \tag{46}$$

where $F(k^2)$ is given in Eq. (2) and is the second-order electric form factor.

IV. DISCUSSION

The second-order radiative-correction calculation once again suggests that in high-energy near-forward electrodynamical processes, the eikonals should be looked upon as the basic interaction carriers. It would be interesting to extend this investigation to include radiative effects to all orders. The major problem here is the complexity of the renormalization procedure, and we are looking into it.

One may also wonder what the effects of vacuum polarization are. Clearly, the photon self-energy effects will change the eikonals to something of the form (b) in Sec. I. However, it is unclear to us without detailed calculation what form it will assume after other diagrams, such as Fig. 7, are included.

It is amusing to notice that in Eq. (46), the amplitude is proportional to the square of the electric form factor. This resembles the conjecture of Wu and Yang,[9] except that they argued for large-angle scattering, whereas our deduction applies to small angles only.

Note added in manuscript. After the submission of this work, a paper by H. Cheng and T. T. Wu appeared [Phys. Rev. **184**, 1868 (1969)] in which second-order radiative correction to an electron which interacts with a Coulomb potential was considered. Results similar to those of our corresponding case were obtained. They also looked into the spin-flip amplitudes.

[9] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

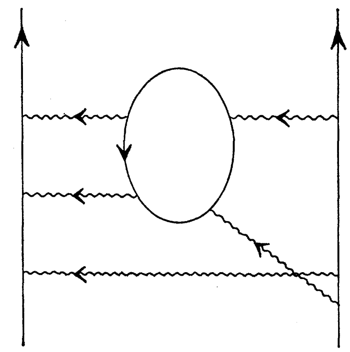

FIG. 7. An example due to vacuum polarization.

Higher-order radiative corrections to this problem were carried out independently by three groups: (1) H. Cheng and T. T. Wu, (2) S. J. Chang, and (3) Y.-P. Yao, using three different methods. To summarize the situation, the conclusion is that if vacuum polarization effects are neglected, and if Z diagrams are not included (Cheng and Wu, and Chang) and/or if soft-photon approximation is made to the t-channel exchange (Yao), then the results obtained in this paper and in the aforementioned work of Cheng and Wu remain valid to all orders.

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