The predictions of expression (2.7) for all possible cases of interest are displayed in Table I along with the earlier calculations of Refs. 9 and 10 for comparison. As we noted earlier, these values are still consistent with the present experimental upper limit. The photon energy distributions for various cases are ploted in Figs. 1 and 2.

PHYSICAL REVIEW D VOLUME 1, NUMBER 1 1 JANUARY 1970

Radiative Corrections to $K_{\mu 3}$ Decays*

EDWARD S. GINSBERG†
Department of Physics, Technion—Israel Institute of Technology, Haifa, Israel
(Received 11 August 1969)

We have calculated the radiative corrections to the Dalitz plot for $K_{\mu 3}{}^{\pm}$ and $K_{\mu 3}{}^0$ decays, assuming a phenomenological weak K-π vertex and using perturbation theory. The answer depends logarithmically on a cutoff. We have evaluated all terms which contribute to order α analytically, without any approximations concerning the smallness of the muon mass or the "real inner bremsstrahlung." Thus, the dependence on the parameter ξ (the ratio of the form factor f_-/f_+) is exact. The radiative corrections to the $K_{\mu 3}{}^0$ Dalitz plot, muon spectrum, and lifetime average around 2% over most of their respective ranges and are not especially sensitive to the cutoff. The radiative corrections to $K_{\mu 3}{}^{\pm}$ decays is a fraction of a percent over most of the Dalitz plot and is sensitive to the cutoff. The radiative correction to the $\Delta I=\frac{1}{2}$ rule prediction for the ratio of the charged and neutral decay rates is approximately 2%. The final-state Coulomb correction accounts for most of this numerical result, the rest being model-dependent noise.

I. INTRODUCTION

In two previous papers,[1,2] we have derived expressions for the radiative corrections to the Dalitz plots in $K_{l3}{}^{\pm}$ and $K_{l3}{}^0$ decays using a phenomenological model for the weak interaction and perturbation theory. The processes considered are examples of strangeness-changing leptonic weak decays which can be analyzed experimentally in great detail. The numerous theoretical predictions for the form factors involved in these decays can, in principle, be tested by sufficiently fine measurements. Such measurements, of course, require an estimate of the radiative corrections for their interpretation. The numerical estimates which we have given were limited to the electron modes, where the approximation $m_e \to 0$ is valid. Unfortunately, in this limit the dependence on one of the form factors, f_-, is neglected, since these terms are proportional to $m_e{}^2$. In this paper we remove this restriction and present numerical estimates of the radiative corrections applicable to the muon modes. We have performed all the necessary integrations by analytical means, thus avoiding some lengthy numerical computations. Therefore, within the limitations of our model, the dependence on ξ, the ratio of the form factors f_-/f_+, is evaluated exactly.

Briefly, let us recall the assumptions underlying our previous calculations. First, we assume a phenomenological weak interaction for the hadrons using vector currents and characterized by the usual form factors f_+ and f_-. In momentum space, the Lagrangian takes the familiar form

$$\mathcal{L}\sim[(p_K+p_\pi)_\alpha f_+ +(p_K-p_\pi)_\alpha f_-]\bar{u}_\nu\gamma_\alpha\tfrac{1}{2}(1-i\gamma_5)v_l\,.$$

Our normalization is such that in the limit of unitary symmetry, the form of the weak K-π vertex is the same as the weak π-π vertex, the latter being given by the conserved vector current hypothesis.[3] Assuming also the octet hypothesis of Cabibbo,[4] we have $f_+ \to G_\beta{}^V \tan\theta$, $\xi=f_-/f_+ \to 0$, where $G_\beta{}^V$ is the weak-coupling constant determined from O^{14} decay and θ is the Cabibbo angle. Second, we calculate the radiative corrections to lowest order in α using perturbation theory and assuming minimal electromagnetic coupling. In particular, the gauge-invariant substitution $p_\alpha \to p_\alpha - eA_\alpha$ for the charged particles present gives rise to Feynman diagrams in which the weak and electromagnetic currents act at the same vertex.[5] Third, electromagnetic corrections to strong-interaction renormalization graphs are ignored; instead, we use phenomenological form factors and the physical masses of the particles involved. Finally, in calculating the radiative corrections, we shall neglect the momentum dependence of the form factors. If the form factors are expanded in the usual manner, $f_\pm(q^2)=f_\pm(0)(1+\lambda_\pm q^2/m_\pi)$, this amounts to neglecting terms of order $\alpha\lambda_\pm$, where $\lambda_\pm$ are small parameters characterizing the energy dependence of

* Supported by the National Science Foundation.
† Permanent address: Department of Physics, University of Massachusetts at Boston, Boston, Mass. 02116.
1 E. S. Ginsberg, Phys. Rev. **162**, 1570 (1967).
2 E. S. Ginsberg, Phys. Rev. **171**, 1675 (1968); **174**, 2169(E) (1968).
3 R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
4 N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).
5 E. S. Ginsberg, Phys. Rev. **142**, 1035 (1966).

the form factors.[6] The model here envisioned is characterized by two phenomenological constants, namely, f_+ and ξ, and in addition, by a cutoff Λ. The results depend only logarithmically on Λ, as do all other radiative corrections to weak processes, with one exception. The presence of the cutoff is due to the nonrenormalizability of the weak Lagrangian given above. We regard the estimates of the radiative corrections as useful, provided the numerical result is not particularly sensitive to reasonable variations in a cutoff around the value of a nucleon mass.

In Sec. II we present the results of our calculation of the radiative corrections to the charged kaon decays $K_{\mu3}^{\pm}$. In Sec. III we give similar results for the neutral kaon decays $K_{\mu3}^0$. In the final section we make some brief comments on these results and on the accuracy of the numerical estimates. Expressions for many of the integrals involved are contained in the Appendices.

II. $K_{\mu3}^{\pm}$ RADIATIVE CORRECTION

The calculation of the radiative corrections to the $K_{\mu3}^{\pm}$ Dalitz plot is based on the expressions given in Ref. 1. With some minor exceptions, we will use the notation of that paper. We shall not attempt to indicate, by means of cumbersome superscripts and subscripts, the charge state to which each variable refers, it being understood that all symbols occurring in this section refer to the charge states appropriate to $K_{\mu3}^{\pm}$ decays. Thus, m_K, I_0, etc., in this section refer to the mass of the positive kaon, an integral expression given in Ref. 1, etc., while the same symbols used in the next section will refer to the mass of the neutral kaon, an integral expression given in Ref. 2, etc. We shall write down all expressions in the center-of-mass system of the decaying particles, i.e., the kaon rest frame.

The zero-order pion-muon energy correlation (the Dalitz plot) is given by

$$\Gamma_0(E_\mu,E_\pi)=(2\pi)^{-3}|f_+|^2\{[2m_KE_\mu-m_\mu^2\,\mathrm{Re}(1-\xi)]E_\nu-(m_K^2-\tfrac{1}{4}m_\mu^2|1-\xi|^2)(W_\pi-E_\pi)\}\,. \quad (1)$$

This expression is linear in the pion energy and quadratic in the muon energy.

The radiative corrections to lowest order in α originate from two sources, the virtual corrections and the inner bremsstrahlung. The latter can be conveniently split into two parts, the first is infrared-divergent and the second is the so-called real inner bremsstrahlung. Thus,

$$\Gamma_{\mathrm{RC}}(E_\mu,E_\pi)=\Gamma_v(E_\mu,E_\pi)+\Gamma_{\mathrm{IR}}(E_\mu,E_\pi)+\Gamma_{\mathrm{RIB}}(E_\mu,E_\pi)\,. \quad (2)$$

The virtual corrections[7] to the $K_{\mu3}^{\pm}$ Dalitz plot are given by

$$\begin{aligned}\Gamma_v(E_\mu,E_\pi)&=(2\pi)^{-3}|f_+|^2\{[(m_K^2-m_\mu^2)(W_\mu-E_\mu)-H_\mu^2E_\nu]\,\mathrm{Re}A\\&\quad+\tfrac{1}{2}m_\mu^2(W_\mu-E_\mu)\,\mathrm{Re}[(A^*+B)(1+\xi)]\\&\quad+\tfrac{1}{4}m_\mu^2(W_\pi-E_\pi)|1+\xi|^2\,\mathrm{Re}B\}\,,\end{aligned} \quad (3)$$

where

$$A=(\alpha/\pi)[\tfrac{3}{2}\ln(\Lambda/m_\mu)-1+t_1-\tfrac{1}{2}m_\mu^2(1+\xi)t_2/H_\mu^2]\,, \quad (4)$$

$$B=(\alpha/\pi)[-\tfrac{3}{2}\ln(\Lambda/m_\mu)-(7/4)+t_1-2t_2/(1+\xi)] \quad (5)$$

and

$$\begin{aligned}t_1=\frac{E_\mu}{p_\mu}\Bigg\{&\mathrm{Li}_2\left(\frac{2p_\mu}{m_K-E_\mu+p_\mu}\right)-\mathrm{Li}_2\left(\frac{2m_Kp_\mu}{m_K(E_\mu+p_\mu)-m_\mu^2}\right)\\&-\ln\left(\frac{m_K-E_\mu+p_\mu}{m_K-E_\mu-p_\mu}\right)\ln\left(\frac{E_\mu+p_\mu}{m_K}\right)\\&+\left[1-\ln\left(\frac{E_\mu+p_\mu}{m_\mu}\right)\right]\ln\left(\frac{E_\mu+p_\mu}{m_\mu}\right)\Bigg\}\,,\end{aligned} \quad (6)$$

$$t_2=\left(\frac{m_K-E_\mu}{p_\mu}\right)\ln\left(\frac{E_\mu+p_\mu}{m_\mu}\right)-\ln\left(\frac{m_K}{m_\mu}\right), \quad (7)$$

and $\mathrm{Li}_2(z)$ is the dilogrithm function. In these and subsequent expressions we have omitted the infrared-divergent terms, which cancel out exactly when the various contributions to the radiative corrections are combined.

The infrared contribution[8] to the radiative correction may be written

$$\Gamma_{\mathrm{IR}}(E_\mu,E_\pi)=(\alpha/\pi)\Gamma_0(E_\mu,E_\pi)I_0(E_\mu,E_\pi)\,, \quad (8)$$

where

$$\begin{aligned}I_0(E_\mu,E_\pi)=\frac{E_\mu}{p_\mu}\Bigg\{&\left[2\ln\left(\frac{2p_\mu}{m_\mu}\right)+\ln\left(\frac{m_\mu x_{\max}^2}{4(E_\mu+p_\mu)r_+}\right)\right]\\&\times\ln\left(\frac{E_\mu+p_\mu}{m_\mu}\right)+\mathrm{Li}_2\left(-\frac{\mathcal{A}^2}{4r_+}\right)-\mathrm{Li}_2\left(-\frac{4r_-}{\mathcal{A}^2}\right)\Bigg\}\\&-\ln\left(\frac{x_{\max}^2}{2m_\mu E_\nu(H_\pi^2-m_\mu^2)}\right),\end{aligned} \quad (9)$$

and

$$\begin{aligned}(E_\mu+p_\mu)r_\pm&=[E_\nu p_\mu^2(H_\pi^2-m_\mu^2)-\tfrac{1}{4}\mathcal{A}^2E_\mu]\\&\quad\pm\{[E_\nu p_\mu^2(H_\pi^2-m_\mu^2)-\tfrac{1}{4}\mathcal{A}^2E_\mu]^2-\tfrac{1}{16}m_\mu^2\mathcal{A}^4\}^{1/2}\,,\end{aligned} \quad (10)$$

$$\mathcal{A}^2=x_{\max}(4p_\mu p_\pi-x_{\max})\,. \quad (11)$$

The momentum transfer q^2 is here denoted by the more symmetrical notation

$$H_\pi^2\equiv q^2=m_K^2-2m_KE_\pi+m_\pi^2\,. \quad (12)$$

The real inner bremsstrahlung contribution to the $K_{\mu3}^{\pm}$ radiative correction has been expressed as a sum

[6] The experimental values of $\lambda_\pm$ are consistent with zero, i.e., constant form factors. See, for example, S. H. Aronson and K. Wendell Chen, Phys. Rev. **175**, 1708 (1968), for a summary of K_{e3} measurements, and C. Rubbia, in *Proceedings of the Topical Conference on Weak Interactions* (CERN, Geneva, 1969), p. 227, for results of the X_2 collaboration.

[7] Figures 1(b)–1(f) of Ref. 5.

[8] Figures 1(g)–1(i) of Ref. 5.

of integrals:

$$\Gamma_{\mathrm{RIB}}(E_\mu,E_\pi)=\frac{\alpha}{4\pi}\frac{|f_+|^2}{(2\pi)^3}\frac{1}{m_K}\times\int_0^{x_{\max}}dx\sum_{m,n}c_{m,n}I_{m,n}(p_\mu,p_K)\,,\quad(13)$$

where the $I_{n,m}$'s and $c_{m,n}$'s are given[9] in Ref. 1. The most general term in Eq. (13) is an integral of a rational function of polynomials in x and the square roots of polynomials in x times a logarithm of a similar rational function of x. We have been able to evaluate all these integrals analytically. We give the expressions for the nontrivial integrals in Appendix A. The real inner bremsstrahlung contribution can then be written

$$\Gamma_{\mathrm{RIB}}(E_\mu,E_\pi)=\frac{\alpha}{4\pi}\frac{|f_+|^2}{(2\pi)^3}\frac{1}{m_K}\sum_{i=0}^{7}U_i\,,\quad(14)$$

where the terms U_i are given in Appendix B.

We have evaluated the radiative corrections to the Dalitz plot [the sum of Eqs. (3), (8), and (14)] on a computer. This has been done for several real values of the parameter ξ and for cutoffs of one proton mass and two proton masses. In addition, the corrections to the Dalitz plot have been numerically integrated, yielding corrections to the muon (or pion) spectrum and to the decay rate. The $K_{\mu3}^\pm$ radiative corrections are small, averaging only a few tenths of a percent except near the boundaries of the Dalitz plot. Moreover, because both positive and negative values are present, the integrated corrections to the spectra and the decay rate are even smaller because of cancellations. In Fig. 1, we present a sample of the results for the radiative corrections to the $K_{\mu3}^\pm$ Dalitz plot. We shall postpone the discussion of these results until after similar results for the radiative corrections to $K_{\mu3}^\pm$ decays have been given (see Sec. IV).

III. $K_{\mu3}^0$ RADIATIVE CORRECTION

The calculation of the radiative corrections to the $K_{\mu3}^0$ Dalitz plot is based on expressions given in Ref. 2, and we shall adhere to the notation of that paper with some minor exceptions. We repeat the caveat of Sec. II: The notation in this section applies only to the decays of neutral kaons.

As in the case of $K_{\mu3}^\pm$ decays, we split the radiative corrections to the $K_{\mu3}^0$ Dalitz plot into three parts, virtual, infrared, and real inner bremsstrahlung. [The zero-order energy correlation is given by the same expression, Eq. (1), provided masses and from factors appropriate to neutral kaon decays are used.]

The virtual contribution[10] is, to lowest order in α,

$$\begin{aligned}\Gamma_v(E_\mu,E_\pi)&=(2\pi)^{-3}|f_+|^2\{[2m_KE_\mu E_\nu-m_K^2(W_\pi-E_\pi)]\,\mathrm{Re}A\\&\quad-\tfrac{1}{2}m_\mu^2E_\nu\,\mathrm{Re}[(A^*+B)(1-\xi)]\\&\quad+\tfrac{1}{4}m_\mu^2(W_\pi-E_\pi)|1-\xi|^2\,\mathrm{Re}B\}\,,\quad(15)\end{aligned}$$

where

$$A=(\alpha/\pi)[\tfrac{3}{2}\ln(\Lambda/m_\mu)-1+t_1-\tfrac{1}{2}m_\mu^2(1-\xi)t_2/h^2]\,,\quad(16)$$

$$B=(\alpha/\pi)[-\tfrac{3}{2}\ln(\Lambda/m_\mu)-(7/4)+t_1-2t_2/(1-\xi)]\quad(17)$$

and

$$\begin{aligned}t_1=\frac{a}{\Delta}\bigg[&\tfrac{2}{3}\pi^2+\mathrm{Li}_2\left(\frac{a+2m_\pi^2-\Delta}{a+2m_\pi^2+\Delta}\right)+\mathrm{Li}_2\left(\frac{a+2m_\mu^2-\Delta}{a+2m_\mu^2+\Delta}\right)\\&+\frac{1}{4}\left(\ln\frac{a+2m_\pi^2-\Delta}{a+2m_\pi^2+\Delta}\right)^2+\frac{1}{4}\left(\ln\frac{a+2m_\mu^2-\Delta}{a+2m_\mu^2+\Delta}\right)^2\\&+\left(1+\ln\frac{h^2}{\Delta}\right)\ln\left(\frac{a+\Delta}{2m_\mu m_\pi}\right)\bigg]+i\,\mathrm{Im}t_1\,,\quad(18)\end{aligned}$$

$$t_2=-\ln\left(\frac{m_\pi}{m_\mu}\right)+\left(\frac{a+2m_\pi^2}{\Delta}\right)\left(i\pi-\ln\frac{a+\Delta}{2m_\mu m_\pi}\right),\quad(19)$$

with a and Δ as given in Ref. 2. The imaginary part of t_1 does not contribute to any physical process (to this order in α) and once again we have omitted the infrared-divergent terms which cancel out in the final result.

The infrared contribution to the radiative corrections to the $K_{\mu3}^0$ Dalitz plot has the same form as Eq. (8) with, however, a different expression[11] for $I_0(E_\mu,E_\pi)$, namely,

$$\begin{aligned}I_0(E_\mu,E_\pi)=\frac{a}{\Delta}\bigg\{&\left[2\ln\left(\frac{w_{\max}}{w_{\min}}\right)-\ln\left(\frac{w_0}{2m_\mu m_\pi}\right)\right]\ln\left(\frac{a+\Delta}{2m_\mu m_\pi}\right)-\ln\left(\frac{w_{\max}+2\Delta}{2\Delta}\right)\ln\frac{(w_{\max}+2\Delta)\Delta}{2m_\mu^2m_\pi^2}+2\,\mathrm{Li}_2\left(-\frac{a-\Delta}{2\Delta}\right)\\&-2\,\mathrm{Li}_2\left(-\frac{w_{\max}}{a+\Delta}\right)-2\,\mathrm{Li}_2\left(-\frac{a-\Delta}{w_{\max}+2\Delta}\right)+\tfrac{1}{2}\,\mathrm{Li}_2\left(-\frac{a+\Delta}{w_0}\right)-\tfrac{1}{2}\,\mathrm{Li}_2\left(-\frac{a-\Delta}{w_0}\right)\bigg\}+\ln\frac{(H_\pi^2-m_\mu^2)(H_\mu^2-m_\pi^2)}{x_{\max}^2}\\&-\ln\frac{m_\pi}{m_\mu}+\left(\ln\frac{E_\pi+p_\pi}{m_\pi}+\ln\frac{E_\mu+p_\mu}{m_\mu}\right)^2-\left(\ln\frac{a+\Delta}{2m_\mu m_\pi}\right)^2,\quad(20)\end{aligned}$$

[9] Note that Eq. (19) of Ref. 1 should read, in part, $c_{-1,0}=c_{1-2,}=-2$. See E. S. Ginsberg, Phys. Rev. **187**, 2280(E) (1969).
[10] Figures 1(b)–1(f) of Ref. 2.
[11] The expression given in Ref. 2 is marred by misprints. See E. S. Ginsberg, Phys. Rev. **187**, 2280(E) (1969).

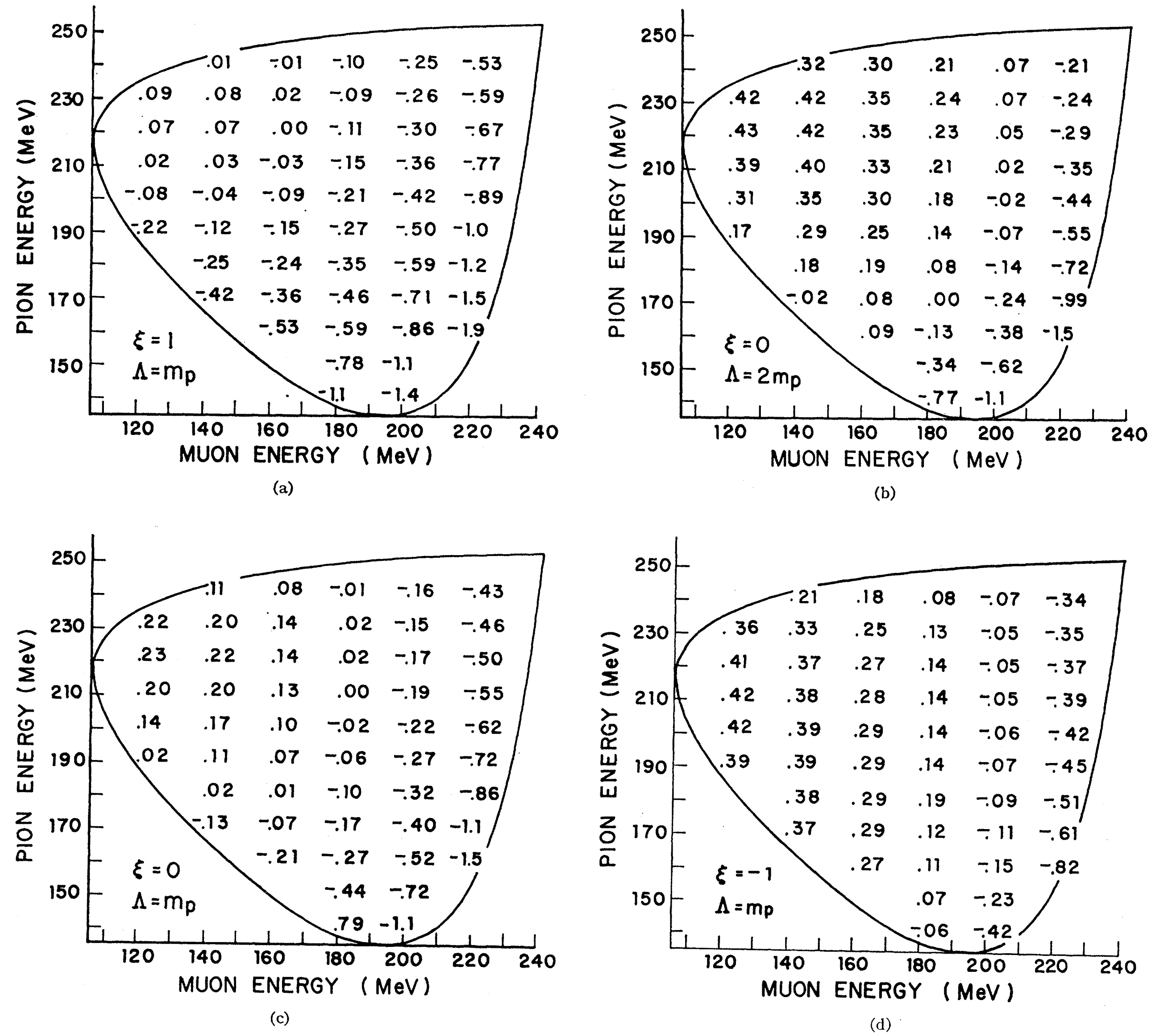

FIG. 1. Fractional radiative correction in % at various points in the $K_{\mu 3}^{\pm}$ Dalitz plot (indicated by the corresponding decimal point) for several values of ξ and Λ.

where

$$w_{\max}=2(E_\mu+p_\mu)(E_\pi+p_\pi)-a-\Delta, \tag{21}$$

$$w_{\min}=2m_K(a+\Delta)\Delta^{-3/2} \times[(a-b)(W_\mu-E_\mu)(W_\pi-E_\pi)]^{1/2}, \tag{22}$$

$$w_0=w_1+[w_1^2-4m_\pi^2m_\mu^2]^{1/2}, \tag{23}$$

$$w_1=(a-b)^{-1}[ab-4m_\mu^2m_\pi^2], \tag{24}$$

and

$$b=m_\pi^2\left(\frac{W_\pi-E_\pi}{W_\mu-E_\mu}\right)+m_\mu^2\left(\frac{W_\mu-E_\mu}{W_\pi-E_\pi}\right). \tag{25}$$

Lastly, the real inner bremsstrahlung contribution to the radiative corrections to the $K_{\mu 3}^0$ Dalitz plot can be written

$$\Gamma_{\mathrm{RIB}}(E_\mu,E_\pi)=\frac{\alpha}{4\pi}\frac{|f_+|^2}{(2\pi)^3}\frac{1}{m_K}\sum_{i=0}^{7}V_i, \tag{26}$$

where the expressions for the V_i's are given in Appendix C. As before, all the integrals implied in our earlier expression for the real inner bremsstrahlung contribution have been evaluated analytically and are given in Appendix A.

The sum of Eqs. (8), (15), and (26) [using Eq. (20) for $I_0(E_\mu,E_\pi)$] constitutes the radiative corrections to the $K_{\mu 3}^0$ Dalitz plot. This has been evaluated on a computer for several values of the parameter ξ and for two different cutoffs. A sampling of these results is shown in Figs. 2(a)–2(d), where the percent of radiative correction [i.e., $100\times\Gamma_{\mathrm{RC}}(E_\mu,E_\pi)/\Gamma_0(E_\mu,E_\pi)$] is

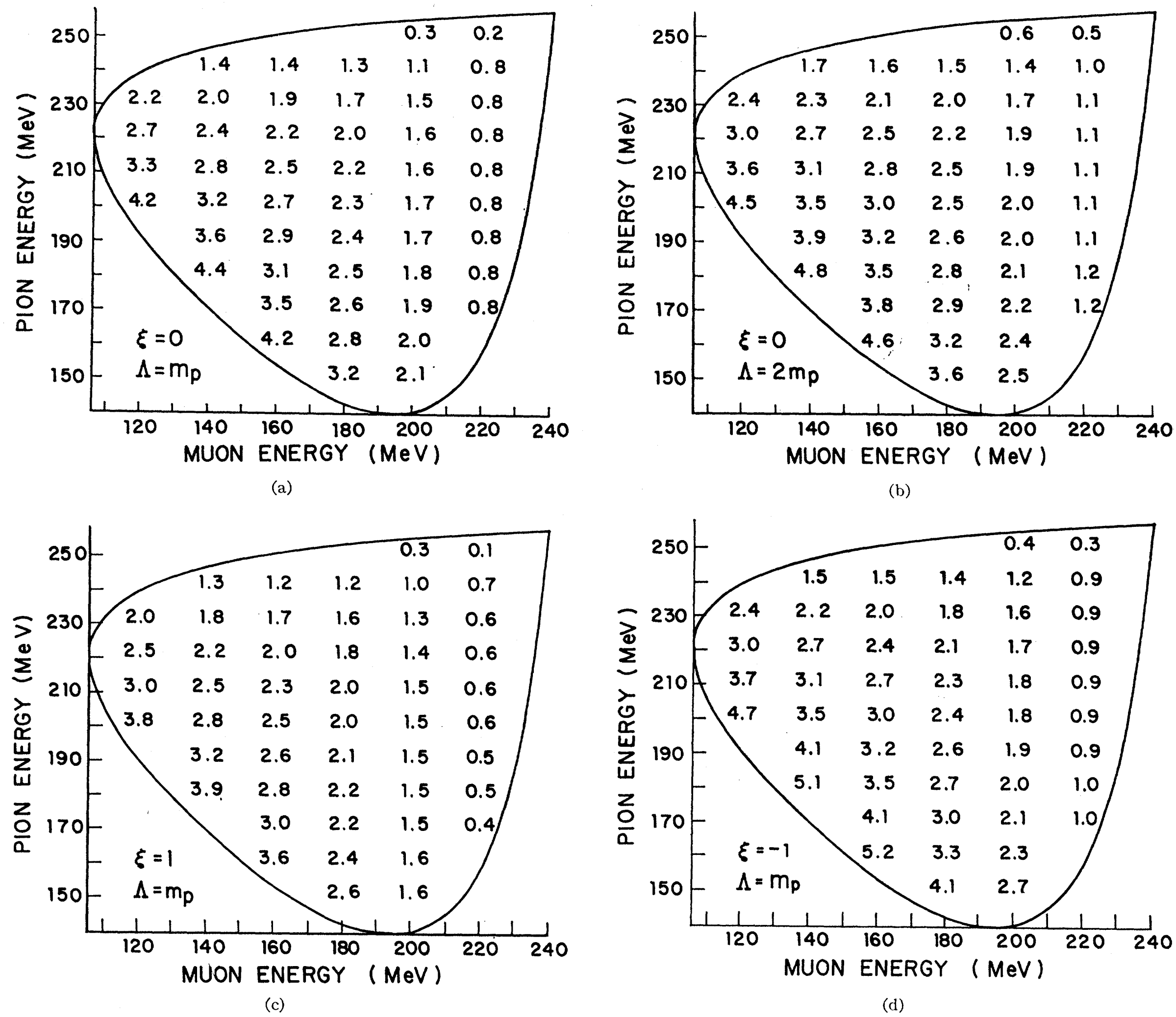

FIG. 2. Fractional radiative correction in % at various points in the $K_{\mu 3}{}^0$ Dalitz plot (indicated by the corresponding decimal point) for several values of ξ and Λ.

plotted for various points in the Dalitz plot. The actual form of $\Gamma_{\mathrm{RC}}(E_\mu,E_\pi)$ and $\Gamma_0(E_\mu,E_\pi)$ is indicated in Figs. 3 and 4 for one particular muon energy and various values of ξ and Λ. The radiative corrections to the energy spectrum of either the muon or pion can be calculated by numerical integration. As an example, we show the radiative corrections to the muon spectrum in Fig. 5 for the particular choice of $\xi=0$. Finally, the radiative corrections to the decay rate (or the fractional change in lifetime $\Delta\tau/\tau=-\Gamma_{\mathrm{RC}}/\Gamma_0$) can be calculated by numerically integrating over both the pion and muon energies. We present some illustrative numbers in Table I.

IV. DISCUSSION AND SUMMARY OF RESULTS

In this section we shall make some brief comments on the results of the computations described in Secs. II and III. First, it can be seen from Fig. 1 that the radiative corrections to the $K_{\mu 3}{}^{\pm}$ Dalitz plot are only a fraction of a percent, too small to affect the results of present experiments. The radiative corrections to the $K_{\mu 3}{}^0$ Dalitz plot, on the other hand, are an order of magnitude greater, probably within the range of accuracy of recent experiments.[12] The reason is the presence of an electromagnetic final-state interaction between the charged muon and pion in the neutral kaon decay, which is of course absent in the decays of charged kaons. The Coulomb part of the electromagnetic final-state interaction contributes to the π^2 term in Eq. (18) and has the characteristic form $\pi\alpha/v$, where $v=\Delta/a$ is the relative speed of one of the particles when the other is at rest.

The Coulomb part of the final-state interaction is expected to dominate the radiative corrections when

[12] See C. Rubbia, Ref. 6.

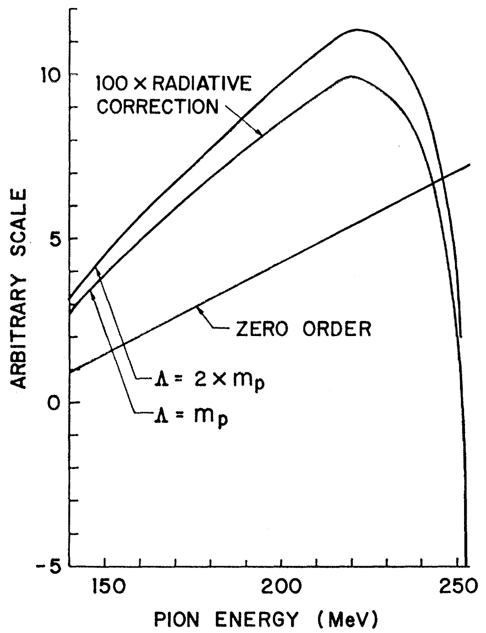

FIG. 3. Zero-order $K_{\mu 3}{}^0$ Dalitz plot and radiative corrections for $E_\mu = 190$ MeV, $\xi = 0$, and $\Lambda = m_p$ and $2m_p$.

one of the charged particles is moving slowly relative to the other, corresponding to $v \longrightarrow 0$. In this limit, each of the dilogarithms in Eq. (18) contributes $\frac{1}{6}\pi^2$, which, when added to the first term, results in the so-called Coulomb correction. Using this as a basis for an approximation to the radiative corrections to the $K_{\mu 3}{}^0$ Dalitz plot yields

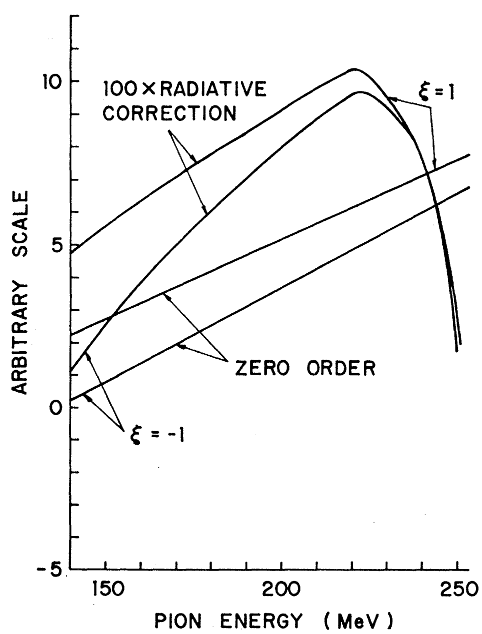

FIG. 4. Zero-order $K_{\mu 3}{}^0$ Dalitz plot and radiative corrections for $E_\mu = 190$ MeV, $\xi = \pm 1$, and $\Lambda = m_p$.

$$\Gamma_{\mathrm{Coulomb}}(E_\mu, E_\pi) = (\pi\alpha/v)\Gamma_0(E_\mu, E_\pi). \tag{27}$$

Except for a small kinematic range near $v=0$, the Coulomb correction is very nearly a constant multiple of the zero-order energy correlation. For the values of energy shown in Figs. 3 and 4, for example, the Coulomb correction is almost a constant fraction (2.38%) of $\Gamma_0(E_\mu, E_\pi)$, whereas the shape of the complete model-dependent radiative correction is quite different. Moreover, Eq. (27) is always positive. The Coulomb correction to the muon spectrum varies between 2.31% and 2.95% of the zero-order spectrum depending on E_μ, which can be compared to the behavior shown in Fig. 5. The Coulomb correction to the lifetime is $(\Delta\tau/\tau)_{\mathrm{Coulomb}} = -2.53\%$, which is fairly close to the complete results given in Table I.

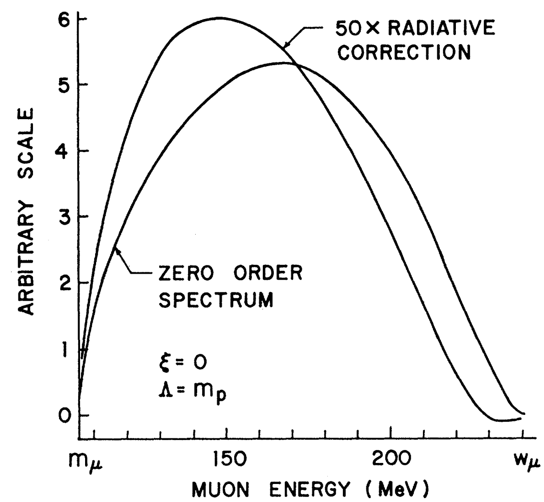

FIG. 5. Zero-order $K_{\mu 3}{}^0$ muon spectrum and radiative corrections for $\xi = 0$ and $\Lambda = m_p$.

We have chosen to base the numerical estimates for the radiative corrections on a cutoff of one proton mass, as is customary in nuclear β decay. There is probably no theoretical justification for this procedure, but if the numerical results are found to be relatively insensitive to reasonable variations in the cutoff, then they may be of some use. We have chosen to measure the sensitivity to the cutoff by comparing the numerical results for cutoffs of one and two proton masses, respectively. It can be seen from Figs. 2(a) and 2(b) that doubling the cutoff alters the numerical estimates by a few tenths of a percent (on the average, about 0.2%). For the $K_{\mu 3}{}^0$ radiative corrections this is not a significant change but in the $K_{\mu 3}{}^\pm$ case the change is of the same order of magnitude as the radiative correction. (This is true for all values of ξ between -1 and $+1$.) Therefore, the numerical estimates for the radiative corrections to $K_{\mu 3}^{\pm}$ decays are meaningful only as regards their general order of magnitude.[13]

{We remark, parenthetically, that we have evaluated the cutoff dependence of the standard divergent integral, retaining only the leading terms in m^2/Λ^2, where m is the mass of a charged particle entering into the decay.[14] This is customary in nuclear β decay also. The terms of order m^2/Λ^2 add a contribution to the standard divergent integral of the form

$$F(m,\Lambda)=\ln\left(1+\frac{m^2}{\Lambda^2}\right)+\frac{m^2}{\Lambda^2}\left(1+\frac{m^2}{\Lambda^2}\right)^{-1}. \tag{28}$$

This would add the following extra terms to the leading terms already included in the radiative corrections:

$$(\alpha/\pi)[F(M,\Lambda)-\tfrac{1}{4}(m_\mu,\Lambda)] \quad \text{to the } A \text{ term}$$

and

$$(\alpha/\pi)[F(M,\Lambda)-(7/4)F(m_\mu,\Lambda)] \quad \text{to the } B \text{ term},$$

where M is the mass of the appropriate charged meson. For $K_{\mu 3}^0$ decays, $M=m_\pi$ and the extra terms are negligible. For $K_{\mu 3}^{\pm}$ decays, $M=m_K$ and the effect is larger, but still less than half the amount associated with doubling the cutoff from m_p to $2m_p$. If $\Lambda=2m_p$, the extra terms are negligible even for $M=m_K$. Thus, the limitations due to the cutoff procedure are not altered by ignoring such terms.}

A brief comment concerning the numerical accuracy of our results: We checked the expressions for the radiative corrections to $K_{\mu 3}$ decays against our previous results for K_{e3} decays in two independent ways. First, we performed the limit $m_\mu \to m_e \to 0$ analytically after doing all the integrations indicated in the Appendices. Second, we checked the expression numerically, by computer, for $m_\mu=m_e$ up to the fifth decimal place. Terms which do not contribute in the limit $m_\mu \to m_e$ were checked in the following manner. Some of these are apparently divergent since the variables $\beta_i^{\max}$ can be zero for kinematically allowed energies. By explicit expansion up through third order, these terms canceled exactly as required. Thus, there is some justification for confidence in the numerical results.

The radiative correction to the ratio of the decay rates for $K_{\mu 3}^0$ and $K_{\mu 3}^{\pm}$ can be obtained easily from the numbers given in Table I. Following arguments previously given in Ref. 2, we find

TABLE I. Fractional change in lifetime because of radiative corrections to the $K_{\mu 3}$ Dalitz plot.

	ξ	Λ	$\Delta\tau/\tau$
$K_{\mu 3}^{\pm}$	0	m_p	0.06%
$K_{\mu 3}^0$	0	m_p	−2.02%
	0	$2m_p$	−2.31%
	+1	m_p	−1.85%
	−1	m_p	−2.15%

$$\frac{\Gamma_{\exp}(K_{\mu 3}^0)}{\Gamma_{\exp}(K_{\mu 3}^+)}=\frac{\Gamma_0(K_{\mu 3}^0)}{\Gamma_0(K_{\mu 3}^+)}(1+\delta), \tag{29}$$

where

$$\delta=\left(\frac{\Delta\tau}{\tau}\right)_{K_{\mu 3}^+}-\left(\frac{\Delta\tau}{\tau}\right)_{K_{\mu 3}^0}\simeq 2.1\%. \tag{30}$$

The value given in Eq. (30) is relatively insensitive to the cutoff because the cutoff dependence is of the same form in both $K_{\mu 3}^+$ and $K_{\mu 3}^0$ and, to first order in α and neglecting electromagnetic mass differences for the mesons, cancels out in the ratio. Making the plausible assumption that the $\Delta S=\Delta Q$ and $\Delta S=-\Delta Q$ amplitudes have the same Lorentz-covariant form, using the TCP theorem, and keeping only the lowest term in the CP-violating parameter ϵ, one finds[2]

$$\frac{\Gamma_{\exp}(K_L\to\pi^{\mp}\mu^{\pm}\nu)}{\Gamma_{\exp}(K^+\to\pi^0\mu^+\nu)}=\frac{\Gamma_0(K_{\mu 3}^0)}{\Gamma_0(K_{\mu 3}^+)}|1+x|^2(1+\delta), \tag{31}$$

where x is the ratio of the $\Delta S=\pm\Delta Q$ amplitudes and δ is given by Eq. (30). If the $\Delta I=\frac{1}{2}$ rule were valid, the ratio of the zero-order rates in Eq. (31) would be 2, except for phase-space corrections.

ACKNOWLEDGMENTS

The author is grateful to Professor Paul Singer and the Physics Department at the Technion for hospitality during the summer and for a generous grant of computer time.

APPENDIX A

In this appendix we give expressions for the nontrivial integrals appearing in the real inner bremsstrahlung contributions to the radiative corrections to $K_{\mu 3}$ decays. We follow the notation of the Appendix in Ref. 1 for the invariant integrals $I_{m,n}$, etc., with i, $j=\mu$, π denoting muon or pion variables, respectively.

$$J_1(i)=\int_0^{x_{\max}}\frac{dx}{\beta_i}\ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=2[\mathrm{Li}_2(\gamma_i^2\zeta_i)+\mathrm{Li}_2(\zeta_i^{-1})-\mathrm{Li}_2(\gamma_i^2)-\mathrm{Li}_2(1)], \tag{A1}$$

[13] This is consistent with the earlier estimate of the radiative correction to the $K_{\mu 3}^{\pm}$ lifetime in Ref. 5, which, however, applies to different experimental conditions.

[14] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), p. 181.

$$J_2(i)=\int_0^{x_{\max}} \frac{x\,dx}{\beta_i} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=m_i^2(\gamma_i^2+1)J_1(i)+2m_i^2(\gamma_i^2-1)[L_3(i)-2L_4(i)]+2x_{\max}+4\beta_i^{\max}L_1(i)\,, \tag{A2}$$

$$J_3(i)=\int_0^{x_{\max}} \frac{x^2dx}{\beta_i} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=2m_i^2(\gamma_i^2+1)J_2(i)-m_i^4(\gamma_i^4+1)J_1(i)+m_i^4(\gamma_i^4-1)[2L_4(i)-L_3(i)]-2m_i^2x_{\max}$$
$$+\tfrac{1}{2}x_{\max}{}^2-4(\alpha_i^{\max}+m_i^2)\beta_i^{\max}L_1(i)\,, \tag{A3}$$

$$J_4(i)=\int_0^{x_{\max}} \frac{x\,dx}{\beta_i^2}$$
$$=\gamma_i^{-1}(\gamma_i^2+1)L_2(i)-4\ln[\tfrac{1}{2}m_i^2(\gamma_i^2-1)/|\beta_i^{\max}|]\,, \tag{A4}$$

$$J_5(i)=m_i^4\int_0^{x_{\max}} \frac{dx}{\beta_i^3} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=2\gamma_i^{-2}\{L_1(i)+L_3(i)+2L_5(i)+m_i^2(\beta_i^{\max}\zeta_i)^{-1}L_1(i)+2(\gamma_i^2-1)^{-1}[L_3(i)-L_4(i)+L_5(i)+\tfrac{1}{2}\gamma_iL_2(i)]\}\,, \tag{A5}$$

$$J_6(i)=m_i^2\int_0^{x_{\max}} \frac{x\,dx}{\beta_i^3} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=(\gamma_i^2+1)J_5(i)-4m_i^2L_1(i)/\beta_i^{\max}-8(\gamma_i^2-1)^{-1}[L_3(i)-L_4(i)+L_5(i)+\tfrac{1}{2}\gamma_iL_2(i)]\,, \tag{A6}$$

$$J_7(i)=m_i^6\int_0^{x_{\max}} \frac{dx}{\beta_i^4}$$
$$=\gamma_i^{-2}[m_i^2(\alpha_i^{\max}+m_i^2)(\beta_i^{\max})^{-2}-2(\gamma_i^2+1)(\gamma_i^2-1)^{-2}-\tfrac{1}{2}\gamma_i^{-1}L_2(i)]\,, \tag{A7}$$

$$J_8(i)=m_i^4\int_0^{x_{\max}} \frac{x\,dx}{\beta_i^4}$$
$$=(\gamma_i^2+1)J_7(i)+8(\gamma_i^2-1)^{-2}-2(m_i^2/\beta_i^{\max})^2\,, \tag{A8}$$

$$J_9(i)=m_i^8\int_0^{x_{\max}} \frac{dx}{\beta_i^5} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=\tfrac{2}{3}\gamma_i^{-2}\{\gamma_i^2(\gamma_i^2-1)^{-1}[J_7(i)-4(\gamma_i^2-1)^{-2}]+(m_i^2/\beta_i^{\max})^2[\gamma_i^2(\gamma_i^2-1)^{-1}+(\alpha_i^{\max}+m_i^2)L_1(i)/|\beta_i^{\max}|]$$
$$+8(\gamma_i^2-1)^{-3}[L_3(i)-L_4(i)+L_5(i)+\tfrac{1}{2}\gamma_iL_2(i)]+4\gamma_i^{-2}[(\gamma_i^2-1)^{-2}(L_3(i)-L_4(i)+L_5(i))-\tfrac{1}{4}\gamma_i(\gamma_i^2-1)^{-1}L_2(i)$$
$$-\tfrac{1}{2}m_i^2\zeta_i^{-1}L_1(i)/\beta_i^{\max}+\ln|\tfrac{1}{2}(\alpha_i^{\max}+\beta_i^{\max})/\beta_i^{\max}|]\}\,, \tag{A9}$$

$$J_{10}(i)=m_i^6\int_0^{x_{\max}} \frac{x\,dx}{\beta_i^5} \ln\left(\frac{\alpha_i+\beta_i}{\alpha_i-\beta_i}\right)$$
$$=(\gamma_i^2+1)J_9(i)-\tfrac{4}{3}\{\gamma_i^2(\gamma_i^2-1)^{-1}J_7(i)+(m_i^2/\beta_i^{\max})^2[m_i^2L_1(i)/\beta_i^{\max}+(\gamma_i^2-1)^{-1}]$$
$$+8(\gamma_i^2-1)^{-3}[L_3(i)-L_4(i)+L_5(i)-\tfrac{1}{2}+\tfrac{1}{4}\gamma_i^{-1}(\gamma_i^2+1)L_2(i)]\}\,, \tag{A10}$$

where

$$L_1(i)=\ln[(\alpha_i^{\max}+\beta_i^{\max})/(\alpha_i^{\max}-\beta_i^{\max})]\,, \tag{A11}$$

$$L_2(i)=\ln\left[\left(\frac{m_i^2(\gamma_i+1)^2-x_{\max}}{m_i^2(\gamma_i-1)^2-x_{\max}}\right)\left(\frac{\gamma_i-1}{\gamma_i+1}\right)^2\right]\,, \tag{A12}$$

$$L_3(i)=\ln(x_{\max}/m_i^2)\,, \tag{A13}$$

$$L_4(i)=\ln(\gamma_i^2-1)\,, \tag{A14}$$

$$L_5(i)=\ln(\tfrac{1}{2}m_i^2/|\beta_i^{\max}|)\,, \tag{A15}$$

$$\gamma_i^2 = H_j^2/m_i^2\,, \tag{A16}$$

$$\zeta_i = (E_i + p_i)(m_K - E_j + p_j)^{-1}\,, \tag{A17}$$

$$\alpha_i^{\max} + m_i^2 = E_i(m_K - E_j) - p_i p_j\,, \tag{A18}$$

$$\beta_i^{\max} = p_i(m_K - E_j) - E_i p_j\,. \tag{A19}$$

In addition, the following integrals are necessary for the $K_{\mu 3}{}^{\pm}$ real inner bremsstrahlung contribution:

$$R_1 = \tfrac{1}{2} m_K \int_0^{x_{\max}} \frac{dx}{\beta_K} \ln\left(\frac{\alpha_K + \beta_K}{\alpha_K - \beta_K}\right)$$

$$= E_\nu \ln\left(\frac{4E_\nu^2}{x_{\max}}\right) - (p_\mu - p_\pi) \ln\left(\frac{E_\nu + p_\mu - p_\pi}{E_\nu - p_\mu + p_\pi}\right), \tag{A20}$$

$$R_2 = m_K^2 \int_0^{x_{\max}} \frac{dx}{\beta_K^2}\left[1 - \frac{\alpha_K}{2\beta_K} \ln\left(\frac{\alpha_K + \beta_K}{\alpha_K - \beta_K}\right)\right]$$

$$= \ln\left(\frac{4E_\nu^2}{x_{\max}}\right) - \left(\frac{E_\nu}{p_\mu - p_\pi}\right) \ln\left(\frac{E_\nu + p_\mu - p_\pi}{E_\nu - p_\mu + p_\pi}\right), \tag{A21}$$

$$R_3 = \frac{1}{2m_K} \int_0^{x_{\max}} dx\, \beta_K \ln\left(\frac{\alpha_K + \beta_K}{\alpha_K - \beta_K}\right)$$

$$= \frac{1}{3}\left[E_\nu x_{\max} + E_\nu^3 \ln\left(\frac{4E_\nu^2}{x_{\max}}\right) - (p_\mu - p_\pi)^3 \ln\left(\frac{E_\nu + p_\mu - p_\pi}{E_\nu - p_\mu + p_\pi}\right)\right]. \tag{A22}$$

APPENDIX B

In this appendix we give expressions for the terms which enter into the real inner bremsstrahlung contribution to the radiative corrections to $K_{\mu 3}{}^{\pm}$ decays of Eq. (14). The notation in this appendix refers only to the decays of charged kaons (see Sec. II).

$$U_0 = \int_0^{x_{\max}} dx (c_{0,0} I_{0,0} + c_{1,1} I_{1,1} + c_{0,-1} I_{0,-1} + c_{-1,0} I_{-1,0} + c_{0,2} I_{0,2})$$

$$= x_{\max}[\tfrac{1}{4} x_{\max} + m_K(6E_\mu + 3E_\pi - W_\pi) + 2[H_\mu^2 - \tfrac{1}{4} m_\mu^2 |1+\xi|^2]\{(E_\mu/p_\mu) \ln[(E_\mu + p_\mu)/m_\mu] - 1\}\,, \tag{B1}$$

$$U_1 = \int_0^{x_{\max}} dx\, c_{-1,1} I_{-1,1}$$

$$= (E_\mu + E_\pi)[(E_\nu^2 + p_\pi^2 - p_\mu^2) R_1 - E_\nu (p_\mu^2 - p_\pi^2) R_2 - R_3 - (E_\nu + 2E_\mu) x_{\max}]\,, \tag{B2}$$

$$U_2 = \int_0^{x_{\max}} dx\, c_{0,1} I_{0,1}$$

$$= 2\{(m_K + 2E_\mu)[m_K(m_K - 2E_\nu) - \tfrac{1}{4} m_\mu^2 |1-\xi|^2] + (m_K + E_\mu) E_\nu^2 - (E_\mu + E_\pi)(H_\mu^2 - \tfrac{1}{4} m_\mu^2 |1+\xi|^2)\} R_1 - 2(m_K + E_\mu) R_3\,, \tag{B3}$$

$$U_3 = \int_0^{x_{\max}} dx\, c_{-1,2} I_{-1,2}$$

$$= x_{\max}(p_\pi^2 - p_\mu^2 - E_\nu^2 + \tfrac{1}{2} x_{\max}) + (E_\nu + 2E_\mu) R_3 - E_\nu (p_\pi^2 - p_\mu^2) R_1 - [(m_K - E_\nu)^2 - \tfrac{1}{4} m_\mu^2 |1-\xi|^2][(E_\nu + 2E_\mu) R_1 + (p_\pi^2 - p_\mu^2) R_2 - x_{\max}]\,, \tag{B4}$$

$$U_4=\int_0^{x_{\max}} dx\; c_{1,0}I_{1,0}$$
$$=(2m_KE_\mu+m_\mu^2)[m_K(2E_\nu-m_K)+H_\mu^2-m_\mu^2\,\mathrm{Re}(\xi)]J_1(\mu)$$
$$-\tfrac{1}{2}(H_\pi^2+m_\mu^2)[H_\mu^2-\tfrac{1}{4}m_\mu^2|1+\xi|^2]J_1(\mu)-m_KE_\mu J_2(\mu)\,, \quad \text{(B5)}$$

$$U_5=\int_0^{x_{\max}} dx\; c_{1,-1}I_{1,-1}$$
$$=m_K(m_K^2-3m_KE_\mu-m_KE_\pi-m_\mu^2)\{E_\nu\gamma_\mu^{-1}(\gamma_\mu^2-1)L_2(\mu)-(E_\nu+2E_\mu)J_4(\mu)-2E_\mu J_1(\mu)$$
$$+\tfrac{1}{2}E_\mu(\gamma_\mu^2-1)^2J_5(\mu)-[E_\mu+E_\nu+\tfrac{1}{2}E_\mu(\gamma_\mu^2+1)]J_6(\mu)\}\,, \quad \text{(B6)}$$

$$U_6=\int_0^{x_{\max}} dx\; c_{2,-1}I_{2,-1}$$
$$=m_K[2m_KE_\pi-2H_\mu^2+m_\mu^2\,\mathrm{Re}(1+\xi)][4E_\mu(L_4(\mu)+L_5(\mu))$$
$$-2(E_\nu+E_\mu)\gamma_\mu^{-1}L_2(\mu)+\tfrac{1}{2}E_\nu(\gamma_\mu^2-1)J_5(\mu)-\tfrac{1}{2}(E_\nu+2E_\mu)J_6(\mu)]\,, \quad \text{(B7)}$$

$$U_7=\int_0^{x_{\max}} dx(c_{2,-2}I_{2,-2}+c_{1,-2}I_{1,-2})$$
$$=m_K^2\{(m_\mu^2-2E_\mu^2)J_1(\mu)-x_{\max}+\gamma^{-1}[12E_\mu(E_\nu+E_\mu)+\tfrac{1}{4}m_\mu^2(\gamma_\mu^2-1)^2-3E_\mu^2(\gamma_\mu^2-1)-\tfrac{1}{2}E_\nu^2(\gamma_\mu^2+1)]L_2(\mu)$$
$$+\tfrac{1}{4}[2E_\nu(E_\nu-2E_\mu)-4p_\mu^2-m_\mu^2(\gamma_\mu^2-1)]J_4(\mu)+[3(E_\nu+2E_\mu)^2-3E_\mu E_\nu(\gamma_\mu^2-1)+\tfrac{1}{4}(2E_\mu^2-m_\mu^2)(\gamma_\mu^2-1)^2]J_5(\mu)$$
$$+[E_\mu(E_\nu+E_\mu)-E_\mu^2(\gamma_\mu^2-1)+m_\mu^2]J_6(\mu)-\tfrac{3}{4}[4E_\mu(E_\nu+E_\mu)-E_\mu^2(\gamma_\mu^2-1)-E_\nu^2](\gamma_\mu^2-1)^2J_7(\mu)$$
$$+\tfrac{3}{4}[4(E_\nu+2E_\mu)^2+(\gamma_\mu^2-1)(4E_\mu^2-E_\nu^2)-E_\mu^2(\gamma_\mu^2-1)^2]J_8(\mu)-\tfrac{3}{4}(\gamma_\mu^2-1)^2[(E_\nu+2E_\mu)^2-E_\mu E_\nu(\gamma_\mu^2-1)]J_9(\mu)$$
$$+\tfrac{3}{4}[4(E_\nu+2E_\mu)(E_\mu\gamma_\mu^2+E_\mu+E_\nu)-E_\mu(E_\nu+E_\mu)(\gamma_\mu^2-1)^2]J_{10}(\mu)\}\,. \quad \text{(B8)}$$

APPENDIX C

In this appendix we give expressions for the terms which enter into the real inner bremsstrahlung contribution to the radiative corrections to $K_{\mu3}{}^0$ decays. The notation in this appendix refers only to the decay of neutral kaons (see Sec. III).

$$V_0=\int_0^{x_{\max}} dx\; c_{1,1}(p_\mu,p_\pi)I_{1,1}(p_\mu,p_\pi)$$
$$=2(2c'+a+c'')[2(E_\mu p_\pi+p_\mu E_\pi)(L_0(\mu)+L_0(\pi))-x_{\max}-\Delta L_7]+2m_\mu^2m_\pi^2[L_7^2-(L_0(\mu)+L_0(\pi))^2]$$
$$+2(E_\mu E_\pi+p_\mu p_\pi)[E_\mu E_\pi+p_\mu p_\pi-2(E_\mu p_\pi+p_\mu E_\pi)(L_0(\mu)+L_0(\pi))]+a(\Delta L_7-\tfrac{1}{2}a)\,, \quad \text{(C1)}$$

$$V_1=\int_0^{x_{\max}} dx\; c_{1,0}(p_\mu,p_K)I_{1,0}(p_\mu,p_K)$$
$$=[2m_K^2E_\mu E_\nu+ac'-\tfrac{1}{2}(H_\pi^2+m_\mu^2)c'']J_1(\mu)+(c'-\tfrac{1}{2}a)J_2(\mu)-\tfrac{1}{2}J_3(\mu)\,, \quad \text{(C2)}$$

$$V_2=\int_0^{x_{\max}} dx\; c_{1,0}(p_\pi,p_K)I_{1,0}(p_\pi,p_K)$$
$$=[-2m_K^2E_\pi E_\nu+a(c'+c'')+\tfrac{1}{2}(H_\mu^2+m_\pi^2)c'']J_1(\pi)+(c'-\tfrac{1}{2}a+\tfrac{1}{2}c'')J_2(\pi)-\tfrac{1}{2}J_3(\pi)\,, \quad \text{(C3)}$$

$$V_3=\int_0^{x_{\max}} dx\; c_{2,-1}(p_\mu,p_K)I_{2,-1}(p_\mu,p_K)$$
$$=-m_K(m_KE_\nu+c')[4E_\mu(L_4(\mu)+L_5(\mu))-\tfrac{1}{2}(E_\nu+2E_\mu)J_6(\mu)+\tfrac{1}{2}E_\nu(\gamma_\mu^2-1)J_5(\mu)-2(E_\mu+E_\nu)L_2(\mu)\gamma_\mu^{-1}]\,, \quad \text{(C4)}$$

$$V_4=\int_0^{x_{\max}} dx\, c_{2,-1}(p_\pi,p_K)I_{2,-1}(p_\pi,p_K)$$

$$=m_K[m_K^2-H_\mu^2+m_\mu^2\,\mathrm{Re}(\xi)-c''][4E_\pi(L_4(\pi)+L_5(\pi))$$

$$-2(E_\pi+E_\nu)L_2(\pi)\gamma_\pi^{-1}+\tfrac{1}{2}E_\nu(\gamma_\pi^2-1)J_5(\pi)-\tfrac{1}{2}(E_\nu+2E_\pi)J_6(\pi)], \quad \text{(C5)}$$

$$V_5=\int_0^{x_{\max}} dx\, c_{1,-1}(p_\mu,p_K)I_{1,-1}(p_\mu,p_K)$$

$$=m_K[m_K(2E_\nu-m_K)+\tfrac{1}{2}(H_\mu^2+m_\pi^2)]\{E_\nu(\gamma_\mu^2-1)\gamma_\mu^{-1}L_2(\mu)-(E_\nu+2E_\mu)J_4(\mu)-2E_\mu J_1(\mu)+\tfrac{1}{2}E_\mu(\gamma_\mu^2-1)^2J_5(\mu)$$

$$-[E_\nu+E_\mu+\tfrac{1}{2}E_\mu(\gamma_\mu^2+1)]J_6(\mu)\}+m_K\{2(E_\nu+2E_\mu)x_{\max}-\tfrac{1}{2}m_\mu^2(\gamma_\mu^2-1)^2[(E_\nu+2E_\mu)\gamma_\mu^{-1}L_2(\mu)+\tfrac{1}{2}J_6(\mu)E_\mu]$$

$$+E_\mu J_2(\mu)+[(E_\nu+2E_\mu)(H_\pi^2+m_\mu^2)-\tfrac{1}{2}E_\nu(H_\pi^2-m_\mu^2)]J_4(\mu)+[m_\mu^2(E_\nu+E_\mu)+\tfrac{1}{2}E_\mu(H_\pi^2+m_\mu^2)]$$

$$\times[2J_1(\mu)+(\gamma_\mu^2+1)J_6(\mu)-\tfrac{1}{2}(\gamma_\mu^2-1)^2J_5(\mu)]\}, \quad \text{(C6)}$$

$$V_6=\int_0^{x_{\max}} dx\, c_{1,-1}(p_\pi,p_K)I_{1,-1}(p_\pi,p_K)$$

$$=\tfrac{1}{2}(H_\mu^2+m_\pi^2)m_K\{E_\nu(\gamma_\pi^2-1)\gamma_\pi^{-1}L_2(\pi)-(E_\nu+2E_\pi)J_4(\pi)-2E_\pi J_1(\pi)+\tfrac{1}{2}E_\pi(\gamma_\pi^2-1)^2J_5(\pi)$$

$$-[E_\nu+E_\pi+\tfrac{1}{2}E_\pi(\gamma_\pi^2+1)]J_6(\pi)\}+m_K\{2(E_\nu+2E_\pi)x_{\max}-\tfrac{1}{2}m_\pi^2(\gamma_\pi^2-1)^2[(E_\nu+2E_\pi)\gamma_\pi^{-1}L_2(\pi)+\tfrac{1}{2}J_6(\pi)E_\pi]$$

$$+E_\pi J_2(\pi)+[(E_\nu+2E_\pi)(H_\mu^2+m_\pi^2)-\tfrac{1}{2}E_\nu(H_\mu^2-m_\pi^2)]J_4(\pi)+[m_\pi^2(E_\nu+E_\pi)+\tfrac{1}{2}E_\pi(H_\mu^2+m_\pi^2)]$$

$$\times[2J_1(\pi)+(\gamma_\pi^2+1)J_6(\pi)-\tfrac{1}{2}(\gamma_\pi^2-1)^2J_5(\pi)]\}, \quad \text{(C7)}$$

$$V_7=\int_0^{x_{\max}} dx[2m_\mu^2I_{2,-2}(p_\mu,p_K)-2I_{1,-2}(p_\mu,p_K)]$$

$$\equiv U_7, \quad \text{(C8)}$$

where we have used the following abbreviations:

$$c'=m_K(E_\nu-2E_\mu)+m_\mu^2\,\mathrm{Re}(1-\xi), \quad \text{(C9)}$$

$$c''=m_K^2-\tfrac{1}{4}m_\mu^2|1-\xi|^2, \quad \text{(C10)}$$

$$L_0(i)=\ln[(E_i+p_i)/m_i], \quad i=\mu \text{ or } \pi \quad \text{(C11)}$$

$$L_7=\ln[(a+\Delta)/(2m_\mu m_\pi)]. \quad \text{(C12)}$$