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Hydrodynamic Model of Quantum Mechanics

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An analysis of typical microphysical systems is presented in the hydrodynamic formulation of quantum mechanics. The emphasis is on the physical peculiarities appearing in the hydrodynamic picture, and on the mathematical treatment of the nonlinear quantum-hydrodynamic field equations. Further, quantum-hydrodynamic uncertainty relations are derived, which relate the minimum uncertainty products to the interior quantum stresses.

I. INTRODUCTION

THE theoretical description of microphysical systems is generally based on the wave mechanics of Schrödinger,[1] the matrix mechanics of Heisenberg,[2] or the path-integral mechanics of Feynman.[3] Another approach is the hydrodynamic formulation of quantum mechanics due to Madelung[4] and de Broglie.[5] The hydrodynamic theory of quantum mechanics has been later extended by de Broglie (idea of the "double solution")[6] and Bohm (idea of "subquantic medium")[7] and used as a preliminary theoretical scheme for quasicausal interpretations of microphysical phenomena.[6,7]

In the hydrodynamic formulation of quantum mechanics, the complex, linear Schrödinger equation is replaced by real, nonlinear differential equations for a probability density and its velocity field (formally similar to the equations of continuity and motion of ordinary hydrodynamics). The complete mathematical and physical equivalence to wave mechanics is established by means of a compatibility condition which relates a single wave function, i.e., a solution of the Schrödinger equation, to each hydrodynamic solution in a unique way.

The physical particularities of the quantum-hydrodynamic model and the mathematical procedure for the solution of the nonlinear quantum-hydrodynamic differential equations are discussed. In this connection, concrete static, dynamic, and time-dependent microscopic quantum systems are considered: the particle in a box, the harmonic oscillator, the hydrogenlike atom, and the nonstationary particle motion. Fundamental uncertainty relations are derived which associate the product of the variances of conjugate particle variables to the intrinsic quantum stresses.

II. HYDRODYNAMIC FORMULATION

The wave field $\psi(\mathbf{r},t)$ of a particle of mass m_0 in a given force field with the potential $U(\mathbf{r},t)$ is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=U\psi-\frac{\hbar^2}{2m_0}\nabla^2\psi. \tag{1}$$

From this fundamental equation, a nonrelativistic quantum hydrodynamics can be derived as follows.

1. Hydrodynamic Field Equations

Equation (1) is a linear, but complex, differential equation. It is separable by means of the general complex statement

$$\psi(\mathbf{r},t)=R(\mathbf{r},t)e^{iS(\mathbf{r},t)}. \tag{2}$$

In the resulting real and imaginary part of Eq. (1), the real variables $R(\mathbf{r},t)$ and $\nabla S(\mathbf{r},t)$ are replaced by the velocity field $\mathbf{v}(\mathbf{r},t)$ and density field $\rho(\mathbf{r},t)$ in accordance with

$$\mathbf{v}=(\hbar/m_0)\nabla S \tag{3}$$

and

$$\rho=R\cdot R. \tag{4}$$

[Note that Eq. (3) has a physical meaning only at points where $\rho\neq 0$.] Thus, the field equations of quantum mechanics in the hydrodynamic picture result in the form[4,5]

$$\frac{\partial}{\partial t}(m_0\rho\mathbf{v})+\nabla\cdot(m_0\rho\mathbf{v}\mathbf{v})=-\rho\nabla\left(U-\frac{\hbar^2}{2m_0}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) \tag{5}$$

and

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{v})=0. \tag{6}$$

The wave function of $\psi(\mathbf{r},t)$ is invariant with respect to a change of its phase $S(\mathbf{r},t)$ by an integer multiple of 2π [Eq. (2)]. Hence, Eq. (3) gives

$$\oint m_0\mathbf{v}\cdot d\mathbf{r}=\hbar\oint dS=2\pi\hbar m,\quad m=0,\pm1,\pm2,\ldots \tag{7}$$

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[1] E. Schrödinger, *Collected Papers on Wave Mechanics* (W. M. Deans, London, 1928).

[2] H. S. Green, *Matrix Mechanics* (P. Noordhoff Ltd., Groningen, 1965).

[3] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[4] E. Madelung, Z. Physik **40**, 322 (1926). For a more recent discussion of the hydrodynamic formulation of wave mechanics, see T. Takabayasi, Progr. Theoret. Phys. (Kyoto) **8**, 143 (1952); **9**, 187 (1953); **14**, 283 (1955).

[5] L. de Broglie, Compt. Rend. **183**, 447 (1926); **184**, 273 (1927); **185**, 380 (1927).

[6] L. de Broglie, Compt. Rend. **233**, 641 (1951); **234**, 265 (1952); **235**, 557 (1952); **235**, 1345 (1952); **235**, 1435 (1952); **236**, 1453 (1953); **237**, 441 (1953); **239**, 737 (1954); **241**, 345 (1955); **244**, 529 (1957); **264**, 1041 (1967).

[7] D. Bohm, Phys. Rev. **85**, 166, 180 (1952); D. Bohm and D. Pines, *ibid.* **85**, 338 (1952); **92**, 609 (1953); D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).

as a condition of compatibility between quantum hydrodynamics and wave mechanics. The path of integration of Eq. (7) may not pass through regions where $\rho=0$ (i.e., where $\mathbf{v}$ is possibly singular), but is otherwise arbitrary.

Equations (5) and (6) represent a complete system of differential equations for the fields $\rho(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$. Equation (7) relates each hydrodynamic solution $(\rho,\mathbf{v})_m$ to a wave solution ψ in a unique way.

The field $\rho(\mathbf{r},t)$ has the meaning of a probability distribution [Eqs. (2) and (4)], i.e., the probability for finding the particle in the vicinity $d\mathbf{r}$ of the point $\mathbf{r}$ is, at time t,

$$dW=\rho\, d\mathbf{r}\,,\iiint \rho d\mathbf{r}=1\,. \tag{8}$$

(Spatial integrations extend over the entire region of the system.) Any variation of the probability density $\rho(\mathbf{r},t)$ with t is accompanied by a probability density flow $\rho\mathbf{v}$ into or away from the respective field point $\mathbf{r}$ [Eq. (6)].

The velocity field $\mathbf{v}(\mathbf{r},t)$, Eq. (5), of the position probability changes with $\mathbf{r}$ and t like that of a hydrodynamic fluid in the force fields of an exterior potential $U(\mathbf{r},t)$ and interior potential $Q(\mathbf{r},t)$:

$$Q=-\left(\frac{\hbar^2}{2m_0}\right)\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\,. \tag{9}$$

The quantum fluid (in the sense of a statistical ensemble of particles) exhibits, however, an essential difference compared to an ordinary fluid: In a rotating motion, $\mathbf{v}(\mathbf{r},t)$ decreases (increases) with increasing (decreasing) distance from the center [Eq. (7)].

The expectation values for the velocity field $\mathbf{v}=(\hbar/m_0)\nabla S$ of quantum hydrodynamics (QH) and the velocity operator $\mathfrak{v}=-i(\hbar/m_0)\nabla$ of wave mechanics (WM) are equal:

$$\langle\mathbf{v}\rangle_{\mathrm{QH}}\equiv\iiint\rho\mathbf{v}\, d\mathbf{r}=\iiint\psi^*\mathfrak{v}\psi d\mathbf{r}\equiv\langle\mathfrak{v}\rangle_{\mathrm{WM}}\,. \tag{10}$$

Similar identities do not hold for the higher-order expectation values, $\langle\mathbf{v}^n\rangle_{\mathrm{QH}}\neq\langle\mathfrak{v}^n\rangle_{\mathrm{WM}}$, $|n|>2$. The expectation value for the quantum force vanishes at all times (theorem of Ehrenfest),[8] i.e.,

$$\langle-\nabla Q\rangle=\iiint\rho(-\nabla Q)d\mathbf{r}=\mathbf{0}\,, \tag{11}$$

since

$$\iiint\rho\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)d\mathbf{r}=\tfrac{1}{2}\oint(\rho\nabla\nabla\ln\rho)\cdot d\mathbf{f}=\mathbf{0}\,. \tag{12}$$

From the mathematical point of view, Eqs. (11) and (12) may be looked upon as integral equations for the variables $Q(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$, respectively.

2. Stationary Quantum States

The time-independent states of a quantum system are special solutions $\{\rho(\mathbf{r}),\mathbf{v}(\mathbf{r})\}$ of Eqs. (5)–(7). Two types of stationary states are distinguished.

(a) *Dynamic States*

For $\partial/\partial t=0$ and $\mathbf{v}\neq\mathbf{0}$, Eqs. (5) and (6) give

$$\nabla\left(\tfrac{1}{2}m_0\mathbf{v}^2+U-\frac{\hbar^2}{2m_0}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)=\mathbf{0}\,, \tag{13}$$

$$\nabla\cdot(\rho\mathbf{v})=0\,; \tag{14}$$

i.e.,

$$\tfrac{1}{2}m_0\mathbf{v}^2+U-\frac{\hbar^2}{2m_0}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}=E\,, \tag{15}$$

$$\rho\mathbf{v}=\nabla\times\mathbf{F}\,. \tag{16}$$

In a dynamic system, the inertia $(m_0\mathbf{v}\cdot\nabla\mathbf{v})$, exterior forces $(-\nabla U)$, and interior forces $(-\nabla Q)$ are in balance at every field point [Eq. (13)]. The sum of the kinetic $(\tfrac{1}{2}m_0\mathbf{v}^2)$, exterior potential (U), and interior potential (Q) energies is invariant, i.e., equal to the integration constant $E\neq E(\mathbf{r})$ [Eq. (15)]. $E\equiv\langle E\rangle$ represents the total energy of the dynamic system. The probability flow density $\rho\mathbf{v}$ has no sources [Eq. (14)]; i.e., its stream lines are closed [Eq. (16)].

(b) *Static States*

For $\partial/\partial t=0$ and $\mathbf{v}=\mathbf{0}$, Eqs. (5) and (6) give

$$\nabla\left(U-\frac{\hbar^2}{2m_0}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)=\mathbf{0}\,, \tag{17}$$

i.e.,

$$U-\frac{\hbar^2}{2m_0}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}=E\,. \tag{18}$$

In a static system, the exterior force $(-\nabla U)$ is balanced by the interior force $(-\nabla Q)$ at any field point [Eq. (17)]. The sum of the exterior (U) and interior (Q) potential energies is invariant, i.e., equal to the integration constant $E\neq E(\mathbf{r})$ [Eq. (18)]. $E\equiv\langle E\rangle$ represents the total energy of the static system.

Equations (15), (16), and (18) are the differential equations describing the dynamic and static system (with given boundary conditions), respectively, and also the eigenvalue equations for the respective total energies E. The compatibility condition [Eq. (7)] is a boundary condition for the dynamic system; it is satisfied in a trivial way for a static system, since in the latter, $\mathbf{v}\equiv\mathbf{0}$ and $m\equiv0$.

The stationary states are either stable (e.g., the ground state of an atom) or unstable (e.g., an excited

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[8] P. Ehrenfest, Z. Physik **45**, 455 (1927).

state of an atom) with respect to small, arbitrary perturbations (proper stationary and quasistationary states, respectively).

III. MICROSCOPIC QUANTUM SYSTEMS

As an illustration to the hydrodynamic formalism, individual static, dynamic, and time-dependent quantum systems are analyzed with respect to the physical and mathematical aspects of the hydrodynamic model.

1. Particle in a Box

One of the simplest quantum systems is a particle enclosed in a one-dimensional box by infinite potential walls at $x=0$ and $x=a$,

$$U(x)=0,\quad 0<x<a;\quad U(x\leq 0)=\infty;\quad U(x\geq a)=\infty. \tag{19}$$

In the stationary state, $\rho(x)\mathbf{v}(x)=\mathbf{c}$ by Eq. (14), where either of the boundary conditions $\rho(0)=0$ or $\rho(a)=0$ specifies that $\mathbf{c}=\mathbf{0}$. Hence,

$$\mathbf{v}(x)=\mathbf{0}. \tag{20}$$

Accordingly, Eq. (18) applies, which leads [under consideration of Eq. (19)] to the linear boundary-value problem

$$\frac{d^2\sqrt{\rho}}{dx^2}=-\left(\frac{2m_0E}{\hbar^2}\right)\sqrt{\rho},\quad 0<x<a,$$

$$\rho(0)=0,\quad \rho(a)=0. \tag{21}$$

A solution $\rho=\rho_n$ exists if the total energy of the system assumes certain eigenvalues $E=E_n$. One finds[9,10]

$$\rho_n=\frac{2}{a}\sin^2\left(\frac{n\pi}{a}x\right),\quad E_n=\frac{\hbar^2}{2m_0}\left(\frac{n\pi}{a}\right)^2,$$

$$n=1,2,3,\ldots. \tag{22}$$

Equations (20) and (21) indicate that the energy eigenvalues are potential energies of the particle in the quantum potential,

$$E_n=Q_n,\quad \tfrac{1}{2}m_0\mathbf{v}_n{}^2=0.$$

2. Harmonic Oscillator

The one-dimensional harmonic oscillator is a particle in a spatially symmetric potential energy field of the form

$$U(x)=\tfrac{1}{2}m_0\omega^2x^2,\quad -\infty\leq x\leq+\infty. \tag{23}$$

In the stationary state, $\rho(x)\mathbf{v}(x)=\mathbf{c}$ by Eq. (14), where either of the boundary conditions $\rho(-\infty)=0$ or $\rho(+\infty)=0$ specifies that $\mathbf{c}=\mathbf{0}$. Hence,

$$\mathbf{v}(x)=\mathbf{0}. \tag{24}$$

Accordingly, Eq. (18) applies, which leads [under consideration of Eq. (23)] to the linear boundary-value problem

$$\frac{d^2\sqrt{\rho}}{dx^2}=-\left[\left(\frac{2m_0E}{\hbar^2}\right)-\left(\frac{m_0\omega}{\hbar}\right)^2x^2\right]\sqrt{\rho},\quad -\infty<x<+\infty,$$

$$\rho(x=-\infty)=0,\quad \rho(x=+\infty)=0. \tag{25}$$

A solution $\rho=\rho_n$ exists if the total energy of the system assumes certain eigenvalues, $E=E_n$. One finds[9,10]

$$\rho_n=\frac{(m_0\omega/\hbar)^{1/2}}{\pi^{1/2}2^nn!}\left[H_n\left(\frac{m_0\omega x^2}{\hbar}\right)^{1/2}\right]^2e^{-(m_0\omega/\hbar)x^2},$$

$$E_n=\hbar\omega(n+\tfrac{1}{2}),\quad n=0,1,2,\ldots. \tag{26}$$

Equations (24) and (25) indicate that the energy eigenvalues are potential energies of the particle in the exterior and quantum potential,

$$E_n=U+Q_n,\quad \tfrac{1}{2}m_0\mathbf{v}_n{}^2=0.$$

The particle in a box and the harmonic oscillator are examples for static systems (the generalization to the three-dimensional case is trivial). In each case, the particle appears to be at rest ($\mathbf{v}=\mathbf{0}$, $\langle\mathbf{v}\rangle=\mathbf{0}$), in the former because of the absence of forces ($-\nabla U=\mathbf{0}$, $-\nabla Q=\mathbf{0}$), and in the latter because of the balance between the forces ($-\nabla U-\nabla Q=\mathbf{0}$). In the hydrodynamic picture, a stationary solution of either of these systems with $\mathbf{v}\neq\mathbf{0}$ is not feasible because $\mathbf{v}\neq\mathbf{0}$ would imply a discontinuity of the probability density flow at the boundaries. This result is explained as the average statistical behavior of the particle in the final stationary state.[11]

3. Hydrogenlike Atom

A quantum system exhibiting both static and dynamic states is the hydrogenlike atom. In this case, an electron ($e=-|e|$) is bound by the Coulomb field of a nucleus ($Z|e|$), so that the potential energy is

$$U(r)=-Ze^2/r,\quad 0\leq r\leq\infty. \tag{27}$$

According to Eqs. (15), (16), and (27), the stationary state of the hydrogenlike atom corresponds to the boundary-value problem

$$\nabla^2\sqrt{\rho}=-\frac{2m_0}{\hbar^2}\left(E-\tfrac{1}{2}m_0v^2+\frac{Ze^2}{r}\right)\sqrt{\rho},\quad \nabla\cdot(\rho\mathbf{v})=\mathbf{0},$$

$$0<r<\infty,\quad 0\leq\delta\leq\pi,\quad 0\leq\phi\leq2\pi; \tag{28}$$

$$\rho(r=\infty,\delta,\phi)=0,$$

$$\rho(r,\delta,\phi)_{\delta\leq\pi/2}=\rho(r,\delta+\tfrac{1}{2}\pi,\phi).$$

The probability currents can only flow in closed lines. Because of the azimuthal symmetry of the system, the

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[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Vols. I and II.

[10] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955).

[11] M. Born and W. Ludwig, Z. Physik **150**, 106 (1958).

velocity potential is of the form $S=m\phi$ [Eq. (3)], corresponding to a velocity field independent of ϕ:

$$\mathbf{v}=\frac{\hbar}{m_0}\frac{\mathbf{e}_\phi}{r\sin\delta}m\,,\quad m=0,\pm1,\pm2,\ldots. \tag{29}$$

This statement is in accord with the compatibility condition, Eq. (7),

$$\oint m_0\mathbf{v}\cdot d\mathbf{r}=\hbar m\oint\frac{\mathbf{e}_\phi\cdot d\mathbf{r}}{r\sin\delta}=\hbar m\int_0^{2\pi}d\phi=2\pi\hbar m. \tag{30}$$

By substitution of Eq. (29), Eq. (28) is reduced to the linear differential equation

$$\nabla^2\sqrt{\rho}=-\frac{2m_0}{\hbar^2}\left[E-\tfrac{1}{2}m_0\left(\frac{\hbar}{m_0}\right)^2\frac{m^2}{r^2\sin^2\delta}+\frac{Ze^2}{r}\right]\sqrt{\rho}. \tag{31}$$

ρ is independent of ϕ, i.e., $\rho=\rho(r,\delta)$ since $\mathbf{v}\|\mathbf{e}_\phi$, and $\nabla\cdot(\rho\mathbf{v})=\mathbf{v}\cdot\nabla\rho=0$ [Eq. (29)]. By means of the statement

$$\sqrt{\rho}=f(r)g(\delta)\,, \tag{32}$$

Eq. (31) can be separated into differential equations with respect to r and δ, respectively (λ = separation parameter):

$$\frac{r^2}{f}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)+\frac{2m_0}{\hbar^2}\left(E+\frac{Ze^2}{r}\right)f\right]=\lambda \tag{33}$$

and

$$\lambda=-\frac{1}{g}\left[\frac{1}{\sin\delta}\frac{d}{d\delta}\left(\sin\delta\frac{dg}{d\delta}\right)-\frac{m^2}{\sin^2\delta}g\right]. \tag{34}$$

Solutions to Eqs. (33) and (34) exist if the constants E and λ assume the eigenvalues[9,10]

$$\lambda_l=l(l+1)\,,\quad l=0,1,2,\ldots \tag{35}$$

and

$$E_n=-\frac{\frac{1}{2}Ze^2}{\hbar^2/m_0Ze^2}n^{-2},\quad n=1,2,3,\ldots. \tag{36}$$

As is known,[9,10] the solutions of Eq. (33) are $f(r)\propto\xi^l\exp(-\frac{1}{2}\xi)L_{n+l}^{2l+1}(\xi)$, where $\xi=2r/na$ and

$$a=\hbar^2/m_0Ze^2\,, \tag{37}$$

and the solutions of Eq. (34) are $g(\delta)\propto P_l^m(\cos\delta)$. Thus one finds after normalization that[9,10]

$$\rho_{nlm}=C_{nlm}\left(\frac{2r}{na}\right)^{2l}e^{-2r/na}\left[L_{n+l}^{2l+1}\left(\frac{2r}{na}\right)P_l^m(\cos\delta)\right]^2,$$
$$C_{nlm}=\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}\times\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}. \tag{38}$$

For physical reasons, only the following combinations of quantum numbers are acceptable:

$$0\leq l\leq n-1,\quad -l\leq m\leq +l.$$

Equations (29) and (38) represent the complete solution $\{\rho,\mathbf{v}\}_{nlm}$ of the quantum hydrodynamics of the hydrogenlike atom.

By means of the recurrence relations for the associated Laguerre and Legendre polynomials,[10] one shows from Eq. (38) that [Eq. (9)]

$$Q=-\frac{\hbar^2}{2m_0}\left(\frac{1}{a^2n^2}-\frac{2}{ar}+\frac{m^2}{r^2\sin^2\delta}\right), \tag{39}$$

while [Eq. (27)]

$$U(r)=-\frac{\hbar^2/m_0}{ar} \tag{40}$$

and [Eq. (29)]

$$\tfrac{1}{2}m_0\mathbf{v}^2=\tfrac{1}{2}m_0\left(\frac{\hbar}{m_0}\right)^2\frac{m^2}{r^2\sin^2\delta}. \tag{41}$$

Equations (39)–(41) exhibit singularities at $r=0$, and Eqs. (39) and (41) also for $\delta=0$ and $\delta=\pi$. These singularities have no physical implications, since the expectation values of Q, U, and $\frac{1}{2}m_0\mathbf{v}^2$ remain finite.

By comparison of Eqs. (39)–(41), it is seen that the quantum potential energy Q overcompensates the Coulomb energy U and the kinetic energy $\frac{1}{2}m_0\mathbf{v}^2$ at any field point (r,δ,ϕ). The remaining energy is finite and represents the observable energy of the system,

$$\tfrac{1}{2}m_0\mathbf{v}^2+U+Q=-\hbar^2/2m_0a^2n^2\equiv E.$$

The states with $m=0$ are static states ($\mathbf{v}=\mathbf{0}$) and the states with $m\neq0$ are dynamic states ($\mathbf{v}\neq\mathbf{0}$) [Eqs. (29), (39), and (41)]. In any state $m\neq0$, the rotational motion decreases with increasing distance r, i.e., for a given direction δ, $v_\phi\propto1/r$ [Eq. (29)].

It is remarkable that no centrifugal forces ($m_0\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{0}$) appear for $m=0$ states, in which the Coulomb and quantum forces balance each other alone, $-\nabla U-\nabla Q=\mathbf{0}$. The quantum-hydrodynamic picture thus leads to conceptions completely different from the classical Bohr model.

4. Free Motion of a Particle

A simple example of a time-dependent quantum phenomenon is the free motion of a particle satisfying prescribed initial values. According to Eqs. (5)–(7), this problem is described in the one-dimensional case by

$$m_0\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial x}\left(-\frac{\hbar^2}{2m_0}\frac{\partial^2\sqrt{\rho}}{(\sqrt{\rho})\partial x^2}\right), \tag{42}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0. \tag{43}$$

The initial state of the particle is specified by a discrete velocity (determined by measurement) and a Gaussian position distribution (distribution parameter α):

$$v(x,t=0)=v_0\,, \tag{44}$$

$$\rho(x,t=0)=(1/\pi^{1/2}\alpha)e^{-(x/\alpha)^2}=\rho_0(x). \tag{45}$$

This means that at $t=0$, the center of the distribution $\rho(x)$ is at $\langle x\rangle_0=0$ and has the velocity $\langle v\rangle_0=v_0$. The boundary conditions are

$$\rho(x=+\infty,t)=0,\quad \rho(x=-\infty,t)=0; \tag{46}$$

$$v(x=v_0t,\,t)=v_0. \tag{47}$$

At any $t\gtrless 0$, one has $\langle(-\partial Q/\partial x)\rangle=0$ [Eqs. (11) and (12)]. This suggests that Eq. (42) can be separated into ($\langle x\rangle=v_0t$ because of absence of exterior forces)[12]

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\sqrt{\rho}}{(\sqrt{\rho})\partial x^2}\right)=\frac{2}{a(t)^2}(x-v_0t) \tag{48}$$

and

$$m_0\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=\frac{\hbar^2/m_0}{a(t)^2}(x-v_0t). \tag{49}$$

Integration of Eq. (48) gives, under consideration of the boundary conditions, Eq. (46), a solution quadratic in $(x-v_0t)$:

$$\rho=[\pi a(t)]^{-1/2}e^{-(x-v_0t)^2/a(t)}. \tag{50}$$

This function indeed satisfies the initial condition, Eq. (45), if the initial value of $a(t)$ is chosen as

$$a(t=0)=\alpha^2. \tag{51}$$

Insertion of Eq. (50) into the equation of continuity, Eq. (43), indicates that for $x=v_0t$,

$$\frac{1}{2a}\frac{da}{dt}=\left(\frac{\partial v}{\partial x}\right)_{x=v_0t}. \tag{52}$$

Under consideration of this relation, the differential equation for $a(t)$ obtains by performing the operation $(\partial/\partial x)_{x=v_0t}$ on Eq. (49):

$$a\frac{d^2a}{dt^2}-\frac{1}{2}\left(\frac{da}{dt}\right)^2=2\left(\frac{\hbar}{m_0}\right)^2. \tag{53}$$

The solution of Eq. (53) with the initial condition, Eq. (51) [satisfying the requirement that $\rho(x,t)$ be real, which is met by the symmetry condition $a(+t)=a(-t)$, Eq. (50)] is

$$a(t)=\alpha^2+(\hbar t/m_0\alpha)^2. \tag{54}$$

According to Eqs. (50) and (54), the probability density is Gaussian in form with a time-dependent distribution parameter $a(t)$, and spreads with the classical particle velocity v_0:

$$\rho=\frac{\pi^{-1/2}}{[\alpha^2+(\hbar t/m_0\alpha)^2]^{1/2}}\exp\left[-\frac{(x-v_0t)^2}{\alpha^2+(\hbar t/m_0\alpha)^2}\right]. \tag{55}$$

Similarly, integration of Eq. (49) gives with the initial condition, Eq. (44), and the boundary condition, Eq. (47), for the submicroscopic velocity field of the particle

$$v=\frac{v_0\alpha^2+(\hbar/m_0\alpha)^2tx}{\alpha^2+(\hbar t/m_0\alpha)^2}. \tag{56}$$

Equations (55) and (56) represent the quantum-hydrodynamic solution of the free motion of a particle. While the observable motion of the particle is uniform, $\langle v\rangle=v_0$, the associated submicroscopic motions are inhomogeneous in x and t space owing to the action of the interior quantum force:

$$-\frac{\partial Q}{\partial x}=\frac{\hbar^2/m_0}{[\alpha^2+(\hbar t/m_0\alpha)^2]^2}(x-v_0t).$$

The quantum forces on the semispaces $-\infty\leq x\leq\langle x\rangle$ and $\langle x\rangle\leq x\leq+\infty$, $\langle x\rangle=v_0t$, compensate each other,

$$\langle m_0dv/dt\rangle_{-\infty}{}^{\langle x\rangle}=-\langle m_0dv/dt\rangle_{\langle x\rangle}{}^{+\infty}.$$

This means that the particle in free motion polarizes the vacuum behind itself, $x\leq v_0t$, and ahead of itself, $x\geq v_0t$, in such a form that the resulting interior quantum forces are distributed symmetrically with respect to the plane through the observable particle position $\langle x\rangle=v_0t$ at any time t.

IV. UNCERTAINTY PRINCIPLE

In conventional quantum theory, the impossibility of determining the variances of the position coordinate Δx_i and of the conjugate momentum component Δp_i ($\mathfrak{p}_i=-i\hbar\nabla_i$) with arbitrary accuracy is commonly attributed to the unavoidable perturbation exerted on the particle by the measuring process.[13] Since the measuring device is generally not defined quantitatively and its perturbation may eventually be very large, the uncertainty relation is formulated as a larger-than-or-equal-to equation[13]:

$$\Delta p_i\Delta x_i\geq\tfrac{1}{2}\hbar.$$

In connection with this interpretation, it is strange that definitive nonzero variances Δx_i and Δp_i obtain for quantum systems which are not exposed to a measuring device. This has been demonstrated by means of so-called negative-result experiments.[14] Further, it should be noted that one obtains theoretically the (nonzero) variances Δx_i and Δp_i of quantum systems without including in the analysis perturbations from or presence of a measuring device at all.

In the hydrodynamic model of quantum mechanics, the uncertainty relations result in a quite natural way from the momentum perturbations associated with the interior quantum stresses. The quantum potential Q has an effect on the particle similar to a hydrodynamic pressure with a driving force $-\nabla Q$. Introducing the identity $-m_0\rho\nabla Q=\nabla\cdot[\rho(\tfrac{1}{2}\hbar)^2\nabla\nabla\ln\rho]$, Eq. (9), and the

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[12] In Eq. (48), the x-independent proportionality factor is written for convenience in the form $2/a^2$.

[13] W. Heisenberg, *The Physical Principles of Quantum Mechanics* (Dover, New York, 1952); N. Bohr, Nature **121**, 580 (1928).

[14] W. Renninger, Z. Physik **158**, 417 (1960).

momentum conservation equation (5) gives

$$\nabla\cdot[\rho(m_0\mathbf{v}m_0\mathbf{v})]=\nabla\cdot[\rho(\tfrac{1}{2}\hbar)^2\nabla\nabla\ln\rho]+\cdots.$$

Accordingly, the interior quantum stresses are, in their possible effects, potentially equivalent to momentum stresses p_ip_j imparted to the hydrodynamic fluid associated with the particle:

$$\mathbf{pp}=-(\tfrac{1}{2}\hbar)^2\nabla\nabla\ln\rho=-(\tfrac{1}{2}\hbar)^2\left(\frac{\nabla\nabla\rho}{\rho}-\frac{\nabla\rho}{\rho}\frac{\nabla\rho}{\rho}\right). \quad (57)$$

This is similar to an ordinary liquid, in which a pressure gradient produces a fluid motion as soon as the constraints or forces balancing the negative pressure gradient are changed or removed.

The expectation values of the momentum stresses, $\langle p_ip_j\rangle$, represent the observable momentum stresses of the particle. According to Eq. (57),

$$\langle\mathbf{pp}\rangle=(\tfrac{1}{2}\hbar)^2\int\!\!\int\!\!\int\rho\,\nabla\ln\rho\,\nabla\ln\rho\,d\mathbf{r},$$

since

$$\int\!\!\int\!\!\int\nabla\nabla\rho\,d\mathbf{r}=\oint\nabla\rho\,d\mathbf{f}=\overleftrightarrow{0}.$$

It is recognized that the momentum stresses p_ip_j, Eq. (57), are due to unobservable (first term) and observable (second term) quantum stresses. The observable momentum stresses are given by the dyad

$$\mathbf{qq}=(\tfrac{1}{2}\hbar)^2\nabla\ln\rho\,\nabla\ln\rho,\quad\langle\mathbf{qq}\rangle\neq\overleftrightarrow{0}. \quad (58)$$

They determine the observable uncertainties or variancies Δx_{ij} of the conjugate components of the position tensor $\mathbf{rr}$ of the particle. Thus one finds from Eq. (58) the fundamental relation

$$\langle q_iq_j\rangle(\Delta x_{ij})^2=(\tfrac{1}{2}\hbar)^2\epsilon_{ij}(s)^2, \quad (59)$$

where

$$\langle q_iq_j\rangle=(\tfrac{1}{2}\hbar)^2\int\!\!\int\!\!\int\rho\,\nabla_i\ln\rho\,\nabla_j\ln\rho\,d\mathbf{r}, \quad (60)$$

$$(\Delta x_{ij})^2=\langle(x_i-\langle x_i\rangle)(x_j-\langle x_j\rangle)\rangle. \quad (61)$$

$\epsilon_{ij}(s)$ is a function of the set of quantum numbers specifying the state of the quantum system. For systems with a separable distribution function $\rho(\mathbf{r},t)=\rho_1(x_1,t)\rho_2(x_2,t)\rho_3(x_3,t)$, the nondiagonal variances vanish: $\Delta x_{ij}=0$ for $i\neq j$. In this case, Eqs. (59)–(61) reduce to

$$\langle q_i^2\rangle^{1/2}\Delta x_i=\tfrac{1}{2}\hbar\epsilon_i(s), \quad (62)$$

where

$$\langle q_i^2\rangle=(\tfrac{1}{2}\hbar)^2\int\!\!\int\!\!\int\rho(\nabla_i\ln\rho)^2d\mathbf{r}, \quad (63)$$

$$(\Delta x_i)^2=\langle(x_i-\langle x_i\rangle)^2\rangle. \quad (64)$$

Equation (59) is the tensorial formulation of the quantum-hydrodynamic uncertainty principle. The uncertainty relation for the diagonal components, Eq. (62), is formally similar to that of wave mechanics for the conjugate variables of momentum and position.

The application of the hydrodynamic uncertainty relation to concrete quantum systems and the evaluation of the function of quantum states is demonstrated in the following examples.

(a) By means of the solution for the particle in the field-free (one-dimensional) box, Eq. (22), one verifies that

$$\langle q_x^2\rangle=4(\tfrac{1}{2}\hbar)^2(n\pi/a)^2,\quad\langle(x-\langle x\rangle)^2\rangle=\tfrac{1}{12}a^2.$$

Accordingly, the uncertainty relation, Eq. (62), becomes, for the particle in the box,

$$(\langle q_x^2\rangle)^{1/2}\Delta x=\tfrac{1}{2}\hbar(\pi/\surd3)n,\quad n=1,2,3,\ldots. \quad (65)$$

The function of states for this system is

$$\epsilon(n)=(\pi/\surd3)n\geq\pi/\surd3.$$

(b) By means of the solution for the harmonic (one-dimensional) oscillator, Eq. (26), one verifies that

$$\langle q_x^2\rangle=2(\tfrac{1}{2}\hbar)^2(m_0\omega/\hbar)(1+2n),$$
$$\langle(x-\langle x\rangle)^2\rangle=(\tfrac{1}{2}\hbar/m_0\omega)(1+2n).$$

Accordingly, the uncertainty relation, Eq. (62), becomes, for the harmonic oscillator,

$$\langle q_x^2\rangle^{1/2}\Delta x=\tfrac{1}{2}\hbar(1+2n),\quad n=0,1,2,\ldots. \quad (66)$$

The function of states for this system is $\epsilon(n)=1+2n\geq1$.

(c) By means of the solution for the particle in the spherically symmetric Coulomb field, Eq. (38), one verifies that[15]

$$\langle q_r^2\rangle=\left(\frac{\hbar}{na}\right)^2\left[1-2\frac{l}{n}\frac{l+1}{2l+1}\right],$$

and

$$\langle(r-\langle r\rangle)^2\rangle=(\tfrac{1}{2}a)^2[n^2(n^2+2)-l^2(l+1)^2].$$

Accordingly, the uncertainty relation, Eq. (62), becomes, for the r components of the dynamical variables of the electron in the hydrogenlike atom,

$$\langle q_r^2\rangle^{1/2}\Delta r=\tfrac{1}{2}\hbar\epsilon_r(n,l),\quad n=1,2,3,\ldots,\quad l\leq n-1 \quad (67)$$

where

$$\epsilon_r(n,l)=\left\{\left(1-2\frac{l}{n}\frac{l+1}{2l+1}\right)\left[(n^2+2)-\frac{l^2}{n^2}(l+1)^2\right]\right\}^{1/2}.$$

The function of states for this case is $\epsilon_r(n,l)\geq\surd3$; in particular,

$$\epsilon_r(n,l)=(2+n^2)^{1/2}\qquad\text{for } l=l_{\min}=0$$
$$=\left(\frac{1+2n}{2n-1}\right)^{1/2}\qquad\text{for } l=l_{\max}=n-1.$$

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[15] The integrals over the associated Laguerre polynomials are readily evaluated by means of their generating functions (Ref. 10).

Similarly, one shows that $\epsilon_\delta \neq 0$, $\epsilon_{ij}=0$ for $i\neq j$, and $\epsilon_\phi=0$.

(d) By means of the solution for the time-dependent (one-dimensional) motion of a free particle, Eq. (55), one verifies that

$$\langle q_x^2\rangle = 2(\tfrac{1}{2}\hbar)^2/a(t)\,, \quad \langle(x-\langle x\rangle)^2\rangle = \tfrac{1}{2}a(t)\,.$$

Accordingly, the uncertainty relation, Eq. (62), becomes, for the particle in free motion,

$$\langle q_x^2\rangle^{1/2}\Delta x = \tfrac{1}{2}\hbar\,. \tag{68}$$

Finally, a momentum component (i) could be defined by the expression

$$(2m_0Q_i)^{1/2}, \quad Q_i = -(\hbar^2/2m_0)(\nabla_i^2\sqrt{\rho})/\sqrt{\rho}\,,$$

to form the uncertainty relation

$$\langle 2m_0Q_i\rangle^{1/2}\Delta x_i = \tfrac{1}{2}\hbar\epsilon_i(s)\,. \tag{69}$$

This uncertainty relation corresponds to that for the diagonal conjugate variables, Eq. (62).

Equation (58) indicates that the momentum transfer responsible for the indeterminacy phenomenon is given by the quantum momentum

$$\mathbf{q} = \tfrac{1}{2}\hbar\nabla \ln\rho\,. \tag{70}$$

According to the quantum-hydrodynamic considerations presented, the minimum uncertainty products result from the interior quantum stresses of the microphysical system. Perturbations due to the measuring process (which have neither been defined nor taken into account) are not essential for the indeterminacy phenomenon. Since $Q\propto\hbar^2$, the responsible mechanism is obviously a nonlinear interior interaction. As a model for this nonlinear quantum interaction, the conception of a polarized vacuum reacting back on the polarizing particle is at hand.[16] Similarly, a hidden turbulence (excited by the presence of the particle) kicking the particle to and fro in a random manner could lead to an explanation of the nonlinear quantum force.[17] In both models, the mechanism giving rise to the uncertainty phenomenum in quantum systems would be similar to that in classical stochastic systems,[18] e.g., a colloidal particle in Brownian motion (for which $\Delta p\Delta x = MD$; M = particle mass, D = diffusion coefficient).[19]

V. DISCUSSION

The hydrodynamic approach to quantum mechanics is equivalent to wave mechanics. For static systems, and for dynamic systems with simple symmetry properties, the nonlinear hydrodynamic equations can be reduced to (not complex) linear differential equations. For nonstationary systems, a similar reduction to (not complex) linear differential equations is evidently not feasible even in simple cases. The quantum-hydrodynamic considerations have led to (1) an original insight into the physics of microphysical systems, (2) a different conception of the origin of the indeterminacy phenomenon, and (3) a novel formulation of the uncertainty principle.

In connection with the results presented, the following further mathematical and physical aspects of the hydrodynamic model of quantum mechanics are to be discussed.

(a) For nonstationary quantum systems, the hydrodynamic fields $\rho(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ can be represented by series involving the eigensolutions $\rho_q(\mathbf{r})$ and $\mathbf{v}_q(\mathbf{r})$ for the stationary states (q) of energy E_q. The corresponding series representation of the nonstationary wave function is[10]

$$\psi(\mathbf{r},t) = \sum_q c_q u_q(\mathbf{r}) e^{-iE_qt/\hbar}\,, \tag{71}$$

in accordance with the linearity of the wave equation. [The expansion coefficients c_q are determined by the initial value of the wave function $\psi(\mathbf{r},0)$.] By Eq. (3), the wave functions $u_q(\mathbf{r})$ for the stationary states are related to their hydrodynamic fields $\rho_q(\mathbf{r})$ and $\mathbf{v}_q(\mathbf{r})$:

$$u_q(\mathbf{r}) = [\rho_q(\mathbf{r})]^{1/2}\exp\left(\frac{im_0}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}_q(\mathbf{r})\cdot d\mathbf{r}\right). \tag{72}$$

The constant $\mathbf{r}_0$ is the position vector of an arbitrary, nonsingular fixed point [the corresponding constant phase factor does not change the physical content of $u_q(\mathbf{r})$]. By substituting Eq. (71), with Eq. (72), into the general relations

$$\rho = \psi^*\psi \quad \text{and} \quad \rho\mathbf{v} = -(i\hbar/2m_0)[\psi^*\nabla\psi - \psi\nabla\psi^*]\,,$$

one obtains the desired series representations for the nonstationary hydrodynamic fields [$\neq$ indicates summation over terms $p\neq q$]:

$$\rho(\mathbf{r},t) = \sum_p |c_p^2|\rho_p(\mathbf{r}) + \sum_{p,q}^{\neq} |c_p||c_q|[\rho_p(\mathbf{r})\rho_q(\mathbf{r})]^{1/2} \times\cos(\omega_{pq}t + \Omega_{pq}(\mathbf{r}) + \phi_{pq}) \tag{73}$$

and

$$\begin{aligned}\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) = {} & \sum_p |c_p^2|\rho_p(\mathbf{r})\mathbf{v}_p(\mathbf{r}) \\ & +\tfrac{1}{2}\sum_{p,q}^{\neq}|c_p||c_q|[\rho_p(\mathbf{r})\rho_q(\mathbf{r})]^{1/2}[\mathbf{v}_p(\mathbf{r})+\mathbf{v}_q(\mathbf{r})] \\ & \times\cos(\omega_{pq}t+\Omega_{pq}(\mathbf{r})+\phi_{pq}) + \frac{\hbar}{2m_0}\sum_{p,q}^{\neq}|c_p||c_q| \\ & \times\{[\rho_p(\mathbf{r})]^{1/2}\nabla[\rho_q(\mathbf{r})]^{1/2} - [\rho_q(\mathbf{r})]^{1/2}\nabla[\rho_p(\mathbf{r})]^{1/2}\} \\ & \times\sin(\omega_{pq}t+\Omega_{pq}(\mathbf{r})+\phi_{pq})\,,\end{aligned} \tag{74}$$

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where

$$\omega_{pq}=(E_p-E_q)/\hbar\,, \tag{75}$$

$$\Omega_{pq}(\mathbf{r})=-\frac{m_0}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}}[\mathbf{v}_p(\mathbf{r})-\mathbf{v}_q(\mathbf{r})]\cdot d\mathbf{r}\,, \tag{76}$$

$$\phi_{pq}=(1/2i)\ln(c_p{}^*c_q/c_pc_q{}^*)\,. \tag{77}$$

The series in Eqs. (73) and (74) are nonlinear and transcendental in accordance with the nonlinearity of the hydrodynamic field equations (5) and (6). According to Eqs. (73) and (74), a nonstationary quantum system corresponds to a stationary mass and current density ($p=q$ terms) which are superimposed by oscillations ($p\neq q$ terms) at the quantum frequencies ω_{pq} [Eq. (75)] of the transitions between states p and q. The phase of the oscillations is $\Delta_{pq}=\Omega_{pq}(\mathbf{r})+\phi_{pq}$ [Eqs. (76) and (77)], i.e., changes with position.

In simple cases, the hydrodynamic fields of nonstationary quantum systems can be obtained by a straightforward solution of the nonlinear field equations (5) and (6), as has been shown by means of the example of the particle in free motion. A more systematic mathematical approach would consist in applying the similarity methods of classical mechanics.[20]

(b) The quantum-hydrodynamic uncertainty relations are necessarily different from those of wave mechanics because $\langle(m_0v_i)^2\rangle_{\mathrm{QH}}\neq\langle\mathfrak{p}_i{}^2\rangle_{\mathrm{WM}}$. Actually, an uncertainty relation analogous to that of wave mechanics does not exist at all. For example, in the case of the motion of a free particle [Eqs. (55) and (56)], there results

$$\Delta(m_0v)\cdot\Delta x=\tfrac{1}{2}m_0(\hbar/m_0\alpha)^2t\,;\quad \text{i.e., } \Delta(m_0v)\cdot\Delta x\rightarrow 0 \quad \text{for } t\rightarrow 0\,.$$

The minimum uncertainty products, Eqs. (59) and (62), have been shown to be a direct consequence of the interior quantum stresses. These quantum stresses are, in their possible effects, equivalent to momentum stresses, since the momentum is conserved in any process [Eq. (5)]. The expectation values of the hydrodynamic momentum m_0v_i and of the wave-mechanical momentum operator $\mathfrak{p}_i$ are equal, i.e., $\langle m_0v_i\rangle=\langle\mathfrak{p}_i\rangle$ [Eq. (10)]. Accordingly, $\langle m_0v_i\rangle$ is observable, since $\langle\mathfrak{p}_i\rangle$ (ordinary wave-mechanical momentum) is observable.

According to Eq. (59) or Eq. (62), the expectation values of the quantum stresses can be determined experimentally from the measurement of the minimum product of the corresponding conjugate momentum and position uncertainties of any microphysical system in any quantum state (s). Thus, it is seen that the expectation values of the interior quantum stresses exhibit themselves in the experimentally observable minimum indeterminacy in microphysical measurements.

In this connection, it should be noted that the momentum stresses p_ip_j [Eq. (57)], are caused by two types of quantum stresses,

$$\mathbf{pp}=\overleftrightarrow{\sigma}+\mathbf{qq}\,.$$

The stresses σ_{ij} and q_iq_j differ in an essential respect since they have a vanishing and nonvanishing expectation value, respectively:

$$\overleftrightarrow{\sigma}=-(\tfrac{1}{2}\hbar)^2\nabla\nabla\rho/\rho\,,\quad \langle\overleftrightarrow{\sigma}\rangle=\overleftrightarrow{0}.$$

The stresses q_iq_j with nonvanishing expectation value [Eq. (58)] are responsible for the momentum uncertainties in an observation, while the stresses σ_{ij} cannot produce an observable momentum change of the particle. In a submicroscopic theory, it should be possible to relate the stresses σ_{ij} to "the particle itself" and the stresses q_iq_j to "a hidden medium."

The quantum-hydrodynamic uncertainty relations for the diagonal conjugate variables [Eq. (62)] are comparable with the uncertainty relations of wave mechanics. It can be shown that the minimum quantum-hydrodynamic uncertainty products [determined in accordance with Eq. (62)] are numerically equal to the minimum wave-mechanical uncertainty products (Δp_i and Δx_i determined by the methods of wave mechanics[10]) for any quantum system, as it should be (otherwise, at least one of the two formulations of the indeterminacy principle would not agree with experiment).

In conclusion, it is noted that the hydrodynamic approach to quantum mechanics has been applied successfully to Bose system at low temperatures.[21] [Equation (7), e.g., describes the peculiar quantized rotation in superfluids[22] similar to the quantized rotation in the hydrogenlike atom.] Further, the quantum-hydrodynamic approach seems to be also promising in other areas, e.g., for the hydrodynamic description of high-energy processes,[23] nuclear-force models,[24] and atomic-electron structures.[25]

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