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Specific Heat of a Classical, Plane, Rigid, Dipole Rotator in Electromagnetic Zero-Point Radiation

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A semiquantitative calculation is performed for the specific heat of a plane, rigid, electric-dipole rotator having classical interactions with thermal electromagnetic radiation including electromagnetic zero-point radiation. The calculation is termed semiquantitative because of the introduction of an unevaluated parameter to account for the failure (at low frequencies) of a familiar approximation. The calculation seems of interest for three reasons: (i) It provides a rough classical understanding of qualitative behavior which is usually attributed to quantum interactions. (ii) It again emphasizes that in statistical thermodynamics a consistent classical theory including electromagnetism must include electromagnetic zero-point radiation. (iii) It is another step in a general program attempting to use electromagnetic zero-point radiation as an alternative hypothesis to quanta. From the calculation, the specific heat of the rotator is found to vanish with vanishing slope as the temperature $T \to 0$, and to go smoothly over to the traditional classical value $\frac{1}{2}k$ at high temperatures. Moreover, the rotator probability distribution with frequency, which departs from the traditional classical theory at absolute zero, becomes the usual Boltzmann distribution in the limits of high temperature or large moment of inertia. These results are in contrast to the classical calculation of Fokker in 1914 involving the Planck spectrum without zero-point radiation, which found behavior in complete contradiction with experiment—including infinite slope for the specific heat at $T=0$ and a failure to approach $\frac{1}{2}k$ at high temperatures. Despite some qualitative features in agreement with experiment, it is suggested that the new results cannot be compared directly with quantum theory or with molecular specific heats, because the calculation involving classical radiation damping does not assure a separation of variables from the internal molecular variables, which would also be influenced by electromagnetic zero-point radiation.

I. INTRODUCTION

IT is presently a familiar suggestion in physics that the earliest indications of quantum phenomena are to be found in connection with thermal phenomena. Maxwell was apparently worried as early as 1869 about the decrease in molecular specific heats at low temperatures,[1] and, of course, Planck[2] introduced the idea of quanta in order to explain the spectrum of thermal blackbody radiation. However, the early quantum calculations were mixtures of classical ideas with quantum hypotheses superimposed. Thus Planck calculated the average energy of an oscillator in a radiation field using classical electromagnetic theory, and then he quantized the energy levels of the oscillator. Although Einstein[3] rushed on to assign quantum aspects to electromagnetic radiation, Planck held back hoping to preserve classical wave properties for radiation. It was a calculation of Fokker[4] in 1914 which convinced Planck[5] that it was impossible to assume classical properties for the interaction of radiation and matter. Fokker showed that the specific heat of a classical plane, rigid, electric-dipole rotator in equilibrium with the Planck spectrum of electromagnetic radiation was in direct contradiction with experiment. In the present paper, we are repeating the calculation of Fokker with the additional assumption that the universe contains fluctuating temperature-independent (zero-point) electromagnetic radiation with a Lorentz-invariant spectrum. We now find that the use of a classical interaction between radiation and matter is after all in rough qualitative agreement with experimental observations.

In a series of recent papers,[6–8] it has been suggested that the hypothesis of classical electromagnetic zero-point radiation in the universe might provide a basis for a classical explanation of many phenomena which are presently accounted for in terms of quanta. Assuming the Lorentz invariance of the radiation spectrum, one may derive,[6] up to a multiplicative constant, the energy spectrum $\frac{1}{2}\hbar\omega$ per normal mode. This radiation has been used to give a derivation of the Planck blackbody spectrum,[6] an explanation of photon statistics,[7] and an understanding of the third law of thermodynamics[8] entirely within the realm of classical electromagnetic theory. Moreover, it has been suggested that this fluctuating zero-point radiation might lead to a

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[1] See the historical description given in the introductory text by R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1963), Vol. I, pp. 40-8, 40-9.

[2] M. Planck, Verhandl. Deut. Phys. Ges. **2**, 237 (1900).

[3] A. Einstein, Ann. Physik **17**, 132 (1905); **20**, 199 (1906).

[4] A. D. Fokker, Ann. Physik **43**, 810 (1914).

[5] M. Planck, Sitzber. Deutsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech. 512 (1915). Fokker's calculation carried such conviction for Planck precisely because it gave a connection between oscillator and rotator specific heats which involved only electromagnetic theory, and was independent of the dubious hypothesis of statistical mechanics.

The work of Fokker and Planck in Refs. 4 and 5, respectively, came to the writer's attention from the historical account by E. T. Wittaker, *A History of the Theories of Aether and Electricity: The Modern Theories, 1900–1926* (Philosophical Library, Inc., New York, 1954).

[6] T. H. Boyer, Phys. Rev. **182**, 1374 (1969).

[7] T. H. Boyer, Phys. Rev. **186**, 1304 (1969).

[8] T. H. Boyer, Phys. Rev. D **1**, 1526 (1970).

Brownian motion in accord with that used by Nelson[9] to give a derivation of the time-dependent and -independent Schrödinger equations from Newtonian mechanics. Within the context of the new hypothesis and also in the light of the historical situation mentioned in the first paragraph, a calculation of the specific heat of a classical, plane, rigid, dipole rotator represents a natural further step.

To date, all of the calculations[10] using electromagnetic zero-point radiation have given results in exact accord with the results of quantum theory, though now the results are obtained with a classical understanding. There seem, however, to be qualitative differences in the predictions of the two theories for systems of large volume at low temperatures. However, systems of this sort (such as liquid helium) are so complex as to be momentarily insoluble in either a zero-point classical or a quantum context. Coming to the present work, there is rough qualitative agreement in the high- and low-temperature limits between the present calculation and quantum theory, but there is a discrepancy in the intermediate-temperature range. We will explain later that we do not believe this discrepancy represents a true departure between the theories, but rather involves a distinction as to what is a rigid rotator system. In particular, we do not believe that the hydrogen molecule for which rotational specific heats have been measured at low temperature can be represented as the rigid, electric-dipole rotator used in the zero-point classical calculation here.

II. CALCULATION OF ROTATOR AVERAGE ENERGY

1. Outline—Use of Fokker's Calculation

Our evaluation of the interaction of a classical, rigid, electric-dipole rotator with electromagnetic radiation follows the outline provided by Fokker[4] more than half a century ago. We will find, however, that the introduction of electromagnetic zero-point radiation not only alters the results in Fokker's work, but also brings some important changes in the allowed calculational approximations.

We consider an ensemble of plane, rigid, electric-dipole rotators in classical blackbody radiation at temperature T including electromagnetic zero-point radiation. The spectral distribution of energy is thus[6]

$$\rho(\omega,T)=\frac{\omega^2}{\pi^2c^3}\left(\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}+\tfrac{1}{2}\hbar\omega\right), \tag{1}$$

with the thermal energy density

$$u=\int_{\omega=0}^{\infty}[\rho(\omega,T)-\rho(\omega,0)]d\omega=\sigma T^4.$$

In equilibrium, the probability $W(\omega,T)$ that a rotator has the frequency ω satisfies the time-independent equation

$$W(\omega,T)f(\omega)\tau-W(\omega,T)\langle R\rangle+\frac{1}{2}\frac{\partial}{\partial\omega}[W(\omega,T)\langle R^2\rangle]=0. \tag{2}$$

This equation is now known as the Fokker-Planck equation. Here $f(\omega)$ is the effective angular velocity damping coefficient of the rotator due to emission of radiation, $\langle R\rangle$ is the average (angular) impulse acquired by the rotator in time τ, and $\langle R^2\rangle$ is the mean-square impulse. Once the probability distribution $W(\omega,T)$ has been obtained from (2), the average energy E of a rotator is

$$E(T)=\int_{\omega=0}^{\infty}\tfrac{1}{2}I\omega^2W(\omega,T)d\omega, \tag{3}$$

where I is its moment of inertia. The specific heat then follows as

$$C=\frac{\partial E(T)}{\partial T}. \tag{4}$$

The entire problem devolves into the evaluation of the quantities $f(\omega)$, $\langle R\rangle$, and $\langle R^2\rangle$ by the use of the classical electromagnetic interactions of a rigid, electric-dipole rotator and electromagnetic radiation. The analogous calculation for the case of a dipole oscillator is now familiar from Planck's analysis.[11] Thus Planck found that an oscillator of frequency ω had an average energy

$$E_{\text{osc}}(\omega,T)=(\pi^2c^3/\omega^2)\rho(\omega,T), \tag{5}$$

where $\rho(\omega,T)$ is the energy per normal mode in the electromagnetic field. Actually, Planck's fully classical calculation does not take the route through Eq. (2). Hence Fokker[4] repeated the classical oscillator calculations using (2) with the instantaneous oscillator energy as the parameter of interest; he found that the analysis again led to Planck's result (5).

2. Angular-Velocity–Dependent Damping

The evaluation of the parameter $f(\omega)$ is straightforward and will be obtained first. A classical rigid dipole rotating in the xy plane with angular velocity ω emits

[9] E. Nelson, Phys. Rev. **150**, 1079 (1966).

[10] See Refs. 6 and 7. There is also a series of calculations initiated by H. B. G. Casimir which, although phrased in terms of quantum electromagnetic zero-point energy, have an immediate reinterpretation in terms of classical electromagnetic zero-point radiation by use of the stress-energy tensor technique given by T. H. Boyer [Phys. Rev. **174**, 1631 (1968)]. See H. B. G. Casimir, Proc. Koninkl. Ned. Akad. Wetenschap. **51**, 793 (1948); J. Chim. Phys. **46**, 407 (1949); T. H. Boyer, Phys. Rev. **174**, 1764 (1968); **180**, 19 (1969); **185**, 2039 (1969); Ann. Phys. (N. Y.) **56**, 474 (1970). See also T. W. Marshall, Nuovo Cimento **38**, 206 (1965).

[11] M. Planck, *Theory of Heat Radiation* (Dover, New York, 1959).

radiation in the form[12]

$$\mathbf{E}=-\frac{\omega^2}{c^2r}\frac{\mathbf{r}\mathbf{r}\cdot\mathbf{p}-\mathbf{p}r^2}{r^2}, \tag{6}$$

$$\mathbf{B}=\frac{\omega^2}{c^2r}\frac{\mathbf{r}\times\mathbf{p}}{r}, \tag{7}$$

where

$$\mathbf{p}=p(\hat{\imath}\sin\omega t+\hat{\jmath}\cos\omega t). \tag{8}$$

Computing the Poynting vector $(c/4\pi)\mathbf{E}\times\mathbf{B}$ and integrating over all space, the energy radiated per unit time is

$$\mathcal{E}_{\rm rad}=\tfrac{2}{3}p^2\omega^4/c^3. \tag{9}$$

The energy radiated can be regarded as associated with a retarding torque

$$f(\omega)=\mathcal{E}_{\rm rad}/\omega=\tfrac{2}{3}p^2\omega^3/c^3. \tag{10}$$

3. Fluctuating Impulse Due to Fluctuating Radiation

The evaluation of the mean impulse $\langle R\rangle$ and the mean-square impulse $\langle R^2\rangle$ experienced by the oscillator during a time interval τ is considerably more complicated. Indeed, our calculation is incomplete on one fundamental point.

The impulse received can be separated into two basic parts, one depending upon the average angular velocity ω of the oscillator during the time interval, and a second contribution independent of ω which becomes important at low angular velocity. The mean impulse $\langle R\rangle$ will vanish as $\omega\to 0$ since the fluctuating impulse which is independent of ω will be as often positive as negative. However, the mean-square impulse $\langle R^2\rangle$ does not vanish as $\omega\to 0$. Thus at high frequency of rotation, there are strong frequency-dependent impulses, whereas at low frequencies of rotation, the frequency has little influence on the random impulses.

In the calculation to follow, we are using the approximations of Fokker's work, and so are able to evaluate only the high-frequency contribution to $\langle R^2\rangle$. We will show later that whereas the approximation does not affect Fokker's conclusions, it omits an additional contribution in the presence of electromagnetic zero-point radiation which is crucial to understanding the rotator specific heat at low temperatures.

The fluctuating radiation in which the rotator is immersed can be written as a superposition of transverse plane waves

$$\mathbf{E}(\mathbf{x},t)=\sum_{\lambda=1}^{\lambda}\int d^3k\,\hat{\epsilon}(\mathbf{k},\lambda)\mathfrak{h}(\Omega_{\mathbf{k}},T)\times\cos[\Omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x}+\psi(\mathbf{k},\lambda)], \tag{11}$$

$$\mathbf{B}(\mathbf{x},t)=\sum_{\lambda=0}^{\lambda}\int d^3k\,\frac{\mathbf{k}\times\hat{\epsilon}(\mathbf{k},\lambda)}{k}\mathfrak{h}(\Omega_{\mathbf{k}},T)\times\cos[\Omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x}+\psi(\mathbf{k},\lambda)], \tag{12}$$

where

$$\hat{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{k}=0,\quad \hat{\epsilon}(\mathbf{k},\lambda)\cdot\hat{\epsilon}(\mathbf{k},\lambda')=\delta_{\lambda\lambda'},\quad \Omega_{\mathbf{k}}=c|\mathbf{k}|. \tag{13}$$

The random phase $\psi(\mathbf{k},\lambda)$ is used[11] because of the fluctuating character of the radiation. The energy per normal mode is given by the Planck radiation spectrum with zero-point radiation

$$\rho(\Omega,T)=\frac{\Omega^2}{c^3}\mathfrak{h}^2(\Omega,T)=\frac{\Omega^2}{\pi^2c^3}\left(\frac{\hbar\Omega}{e^{\hbar\Omega/kT}-1}+\tfrac{1}{2}\hbar\Omega\right). \tag{14}$$

The rotator is assumed to be an electric dipole of moment $\mathbf{p}$ and moment of inertia I, free to rotate about the z axis in the xy plane. The equation of motion for the dipole is

$$\frac{d(I\omega)}{dt}=(\mathbf{p}\times\mathbf{E})_z=-pE_x\sin\theta+pE_y\cos\theta, \tag{15}$$

where

$$\mathbf{p}=p(\hat{\imath}\cos\theta+\hat{\jmath}\sin\theta) \tag{16}$$

and

$$\omega=\frac{d\theta}{dt}. \tag{17}$$

At large angular velocities of the rotator, it is useful to separate out the average angular velocity ω during a short time interval τ and write

$$\theta=\theta_0+\omega t+\sigma,\quad 0\leq t\leq\tau. \tag{18}$$

The equation of motion (15) can then be rewritten as

$$\frac{Id^2\sigma}{dt^2}=-p\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}(\Omega,T)\cos(\Omega t+\psi)[\epsilon_x\sin(\theta_0+\omega t+\sigma)-\epsilon_y\cos(\theta_0+\omega t+\sigma)] \tag{19}$$

or

$$\frac{Id^2\sigma}{dt^2}=+p\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}(\Omega,T)\big[((\cos\sigma)\tfrac{1}{2}\epsilon_x\{\sin[(\Omega-\omega)t+\psi-\theta_0]-\sin[(\Omega+\omega)t+\psi+\theta_0]\}$$
$$-(\sin\sigma)\tfrac{1}{2}\epsilon_x\{\cos[(\Omega-\omega)t+\psi-\theta_0]+\cos[(\Omega+\omega)t+\psi+\theta_0]\})+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\big]. \tag{20}$$

[12] Unrationalized cgs units are used. Our results differ from those of Fokker by factors of 4π.

In the approximation, valid for large ω, that we wish to obtain only the part of the impulse dependent upon ω, we may ignore the nonlinear character of the differential equation (20). We may approximate $\cos\sigma\cong 1$ and $\sin\sigma\cong 0$ during the time interval τ. Integrating (20) twice with respect to time, keeping terms only to first order in t, and recalling from (18) that $\sigma=0$ at $t=0$, we have

$$\sigma\cong-\frac{p}{I}\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}(\Omega,T)\left[\frac{\epsilon_x}{2}\left(\frac{\sin[(\Omega-\omega)t+\psi-\theta_0]-\sin(\psi-\theta_0)}{(\Omega-\omega)^2}-\frac{\sin[(\Omega+\omega)t+\psi+\theta_0]-\sin(\psi+\theta_0)}{(\Omega+\omega)^2}\right)+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\right].\quad(21)$$

In this same approximation, it is appropriate to compute the angular impulse R during the time interval τ by inserting this value of σ into the right-hand side of (20) with the approximations $\cos\sigma\sim 1$, $\sin\sigma\sim\sigma$. Thus,

$$R\cong\int_{t=0}^{t=\tau}dt\,p\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}(\Omega,T)\Bigg\{\left(\frac{\epsilon_x}{2}\{\sin[(\Omega-\omega)t+\psi-\theta_0]-(\omega\to-\omega,\,\theta_0\to-\theta_0)\}+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\right)$$
$$+[\tfrac{1}{2}\epsilon_x\{\cos[(\Omega-\omega)t+\psi-\theta_0]+(\omega\to-\omega,\,\theta_0\to-\theta_0)\}+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)]\times\frac{p}{I}\sum_{\lambda'=1}^{2}\int d^3k'\,\mathfrak{h}(\Omega',T)$$
$$\times\left[\frac{\epsilon_x}{2}\left(\frac{\sin[(\Omega'-\omega)t+\psi'-\theta_0]-\sin(\psi'-\theta_0)}{(\Omega-\omega)^2}-(\omega\to-\omega,\,\theta_0\to-\theta_0)\right)+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\right]\Bigg\}.\quad(22)$$

In evaluating $\langle R\rangle$, we must average over the random phases $\psi(\mathbf{k},\lambda)$ and $\psi(\mathbf{k}',\lambda')$, over all directions $\mathbf{k}$, $\mathbf{k}'$, and over the random starting angle θ_0. With these considerations in mind, we have

$$\langle R\rangle=\int_{t=0}^{t=\tau}dt\frac{p^2}{I}\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2}\int d^3k\int d^3k'\,\mathfrak{h}(\Omega,T)\mathfrak{h}(\Omega',T)\left[\frac{\epsilon_x\epsilon_x'}{4}\left(\cos[(\Omega-\omega)t+\psi-\theta_0]\times\frac{\sin[(\Omega'-\omega)t+\psi'-\theta_0]-\sin(\psi'-\theta_0)}{(\Omega'-\omega)^2}-(\omega\to-\omega,\,\theta_0\to-\theta_0)\right)+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\right].\quad(23)$$

The time integral involves

$$\int_{t=0}^{t=\tau}dt\,\{-\cos[(\Omega-\omega)t+\psi-\theta_0]\sin(\psi'-\theta_0)+\tfrac{1}{2}\sin[(\Omega-\omega)t+\psi-\theta_0+(\Omega'-\omega)t+\psi'-\theta_0]+\tfrac{1}{2}\sin[(\Omega-\omega)t+\psi-\theta_0-(\Omega'-\omega)t-\psi'+\theta_0]\}.\quad(24)$$

After integrating, and expanding the arguments of the sine and cosine functions, we average specifically in ψ as

$$\langle\sin\psi(\mathbf{k},\lambda)\sin\psi(\mathbf{k}',\lambda')\rangle=\langle\cos\psi(\mathbf{k},\lambda)\cos\psi(\mathbf{k}',\lambda')\rangle=\tfrac{1}{2}\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'},\quad(25)$$
$$\langle\sin\psi(\mathbf{k},\lambda)\cos\psi(\mathbf{k}',\lambda)\rangle=0.$$

The expression then collapses to

$$\langle R\rangle=\frac{p^2}{4I}\sum_{\lambda=1}^{2}\int d^3k\,\mathfrak{h}^2(\Omega,T)(\epsilon_x^2+\epsilon_y^2)\times\left(\frac{\sin^2[\frac{1}{2}(\Omega-\omega)\tau]}{(\Omega-\omega)^3}-\frac{\sin^2[\frac{1}{2}(\Omega+\omega)\tau]}{(\Omega+\omega)^3}\right).\quad(26)$$

Now integrating over all direction $\mathbf{k}$, and substituting for $\mathfrak{h}^2(\Omega,T)$ from (14), we have

$$\langle R\rangle=4\pi\frac{p^2}{4I}\int_{\Omega=0}^{\infty}d\Omega\,\tfrac{2}{3}\rho(\Omega,T)\times\left(\frac{\sin^2[\frac{1}{2}(\Omega-\omega)\tau]}{(\Omega-\omega)^3}-\frac{\sin^2[\frac{1}{2}(\Omega+\omega)\tau]}{(\Omega+\omega)^3}\right).\quad(27)$$

The evaluation of the frequency-dependent contribution to $\langle R^2\rangle$ up to terms first order in τ involves the approximation $\sigma=0$ on the right-hand side of Eqs. (19) and (20). Thus, taking the terms in the first large bracket in (22),

$$R\cong\sum_{\lambda=1}^{\lambda}\int d^3k\,\mathfrak{h}(\Omega,T)\times\left\{\frac{\epsilon_x}{2}\left(\frac{\cos(\psi-\theta_0)-\cos[(\Omega-\omega)\tau+\psi-\theta_0]}{\Omega-\omega}-(\omega\to-\omega,\,\theta_0\to-\theta_0)\right)+(\epsilon_x\to\epsilon_y,\,\theta_0\to\theta_0-\tfrac{1}{2}\pi)\right\}.\quad(28)$$

Accordingly forming R^2 and then averaging over ψ, ψ', and θ_0,

$$\langle R^2\rangle\cong4\pi\frac{p^2}{4I}\int_{\Omega=0}^{\infty}d\Omega\,\tfrac{4}{3}\rho(\Omega,T)\times\left(\frac{\sin^2[\frac{1}{2}(\Omega-\omega)\tau]}{(\Omega-\omega)^2}+\frac{\sin^2[\frac{1}{2}(\Omega+\omega)\tau]}{(\Omega+\omega)^2}\right).\quad(29)$$

At this point, we must evaluate the integrals in Eqs. (27) and (29). Only terms first order in τ are of interest for use in Eq. (2). For $\omega>0$ and $\omega\tau\gg 2\pi$, the functions

$$\frac{\sin^2[\frac{1}{2}(\Omega-\omega)\tau]}{(\Omega-\omega)^2} \quad \text{and} \quad \frac{\sin^2[\frac{1}{2}(\Omega-\omega)\tau]}{(\Omega-\omega)^3}$$

are sharply peaked at $\Omega=\omega$. Thus the integral in (29) picks out values of $\rho(\Omega,T)$ near $\Omega=\omega$. Correspondingly, the integral in (27) picks out $\partial\rho(\Omega,T)/\partial\Omega$ near $\Omega=\omega$, since $(\sin^2 x\tau)/x^3$ is an odd function of x. It is convenient for this latter consideration to expand $\rho(\Omega,T)$ near $\Omega=\omega$,

$$\rho(\Omega,T)=\rho(\omega,T)+\frac{\partial\rho(\omega,T)}{\partial\omega}(\Omega-\omega)+\cdots. \tag{30}$$

The basic integral needed is

$$\int_{-\infty}^{\infty}\frac{\sin^2 x\tau}{x^2}dx=\pi\tau. \tag{31}$$

The resulting values for $\langle R\rangle$ and $\langle R^2\rangle$ are thus

$$\langle R\rangle\cong\frac{\pi^2p^2}{3I}\frac{\partial\rho(\omega,T)}{\partial\omega}\tau, \tag{32}$$

$$\langle R^2\rangle\cong(2\pi^2p^2/3I)\rho(\omega,T)\tau. \tag{33}$$

These are the results[12] obtained by Fokker in 1914. Although the integrals (27) and (29) indeed have the contributions in (32) and (33), the evaluations become inaccurate as $\omega\rightarrow 0$ when τ is held fixed. Also the high-frequency convergence must be considered. These considerations become particularly important for the electromagnetic spectrum of (1) in which electromagnetic zero-point radiation has been included.

4. Discussion of Approximations in Fokker's Calculations

It is crucial for the low-temperature behavior of the rotator that we consider the results obtained by Fokker in the light of the approximation employed. At the very outset it was assumed that the rotator Brownian motion could be treated in a "linearized" approximation in which the change in the rotator position during the time interval τ was computed by integrating an expression for the torque which was valid only when the rotator experienced a vanishingly small torque. That more complicated nonlinear behavior may become important is evident when we substitute the radiation spectrum (1) into the integrals (27) or (29), and notice that the integrals are divergent for large frequency. Of course, for any physical rotator, the finite size will provide a high-frequency cutoff. However, it is nevertheless apparent that the frequency-dependent results (32) and (33) do not represent the entire contribution to the Brownian motion. The region in which the results (32) and (33) are most susceptible to error is in the limit of low frequencies, $\omega\rightarrow 0$. We notice that in this limit, $\langle R\rangle\rightarrow 0$ independent of $\rho(\omega,T)$, corresponding to the subtraction in the integrand of (27). This is indeed appropriate since at $\omega=0$ there is no preferred direction of rotation. On the other hand, $\langle R^2\rangle$ does not vanish at $\omega\rightarrow 0$, but rather still includes the divergence provided by the electromagnetic zero-point energy. At low frequency, we expect that nonlinear behavior of the Brownian motion becomes crucial.

In the case of Fokker's work involving the Planck radiation spectrum without zero-point energy, the failure of the approximations at low frequencies is immaterial. In the first place, we notice that the integrals in (27) and (29) are absolutely convergent. It is true that for any temperature T, the probability density $W(\omega)$ at low frequencies is in error owing to the failure of the approximations to $\langle R\rangle$ and $\langle R^2\rangle$. However, what causes the failure of the approximations is the further fluctuation effects of higher-frequency electromagnetic waves which, in the absence of zero-point energy, involve only a finite density of electromagnetic energy. However, the energy of the oscillator goes as $\frac{1}{2}I\omega^2$, and hence the low frequencies do not contribute significantly to the average energy. When the temperature T is decreased so that frequencies ω' which were formerly unimportant now are contributing the major portion of the energy, the accuracy of the values for $\langle R\rangle$ and $\langle R^2\rangle$ at ω' in (32) and (33) has also improved because the general level of radiation has been reduced and the corrections at these frequencies became proportionately smaller. At no point do we find a temperature for which the average rotator energy calculated using (32) and (33) is entirely erroneous.

The presence of fluctuating radiation at zero temperature entirely alters our understanding of the Brownian motion of the rotator at low frequencies of rotation. The level of zero-point radiation is fixed independent of T, and hence the approximations in (32) and (33) fail at a fixed value of frequency which is independent of the temperature. Now when the temperature T decreases so that the thermal energy in the spectrum falls at frequencies below this cutoff frequency, the value using (32) and (33) for the thermal contribution to the rotator energy is entirely erroneous. If we are to be able to calculate the average rotator energy and specific heat at low temperatures, then we must introduce corrections into the calculations for $\langle R\rangle$ and $\langle R^2\rangle$.

The calculation of the corrections seems to represent the major failure of this work. We have not yet been able to handle the nonlinear aspects of the Brownian motion equation (19). At the moment, we will provide only a heuristic argument. In agreement with the vanishing of the integral (27) at $\omega=0$, we expect that there is no frequency-independent contribution to $\langle R\rangle$; any random force which is independent of ω will not show any preference in direction. On the other hand, we interpret the divergence of $\langle R^2\rangle$ in (29) as symptomatic of a need to handle the nonlinear aspects of the mean-square impulse. At low frequencies, we expect a contribution to $\langle R^2\rangle$ which is independent of frequency. If we assume that $\langle R^2\rangle$ is proportional to the dipole moment p squared, then the available constants require that

$$\langle R^2\rangle|_{\omega=0}=\lambda(2p^2\hbar^4/3I^4c^3) \qquad (34)$$

with λ an unknown numerical constant. This corresponds to a cutoff frequency in the influence of the zero-point spectrum in (33) at

$$\omega_c\sim\hbar/I. \qquad (35)$$

A cutoff frequency of this magnitude is precisely what we would speak of in the *quantization* of the angular momentum of the rotator. Also in line with the heuristic connections to quantum theory, the divergence in the value for $\langle R^2\rangle$ in the linear approximation corresponds to $\Delta\omega\to\infty$ for $\Delta\theta=0$, in agreement with the Heisenberg uncertainty relation $\Delta\theta\Delta\omega I\sim\hbar$. Thus we believe that the determination of the Brownian motion corrections to $\langle R^2\rangle$ in (33) brings us up against the fundamental understanding of quantum particle motion as due to electromagnetic zero-point radiation.

In any event, we believe that it is not unreasonable to expect that $\langle R^2\rangle\to$ const as $\omega\to 0$. This is the only requirement in order to understand the qualitative behavior of the specific heat of a classical rigid rotator as $T\to 0$. Thus we will return to Fokker's equation (2) with the radiation spectrum (1), the value for $f(\omega)$ in (10), that for $\langle R\rangle$ in (32), and that for $\langle R^2\rangle$ given by

$$\langle R^2\rangle=\frac{2\pi^2p^2}{3I}\rho(\omega,T)\tau+\lambda\frac{2p^2\hbar^4}{3I^4c^3}\tau, \qquad (36)$$

with λ an unknown parameter.

III. ROTATOR ENERGY AS FUNCTIONAL OF CLASSICAL RADIATION SPECTRUM

1. Stationary Probability Distribution in Frequency

The Fokker equation (2) provides the differential equation for the stationary distribution of the rotation frequencies. Thus substituting (10), (32), and (36), the

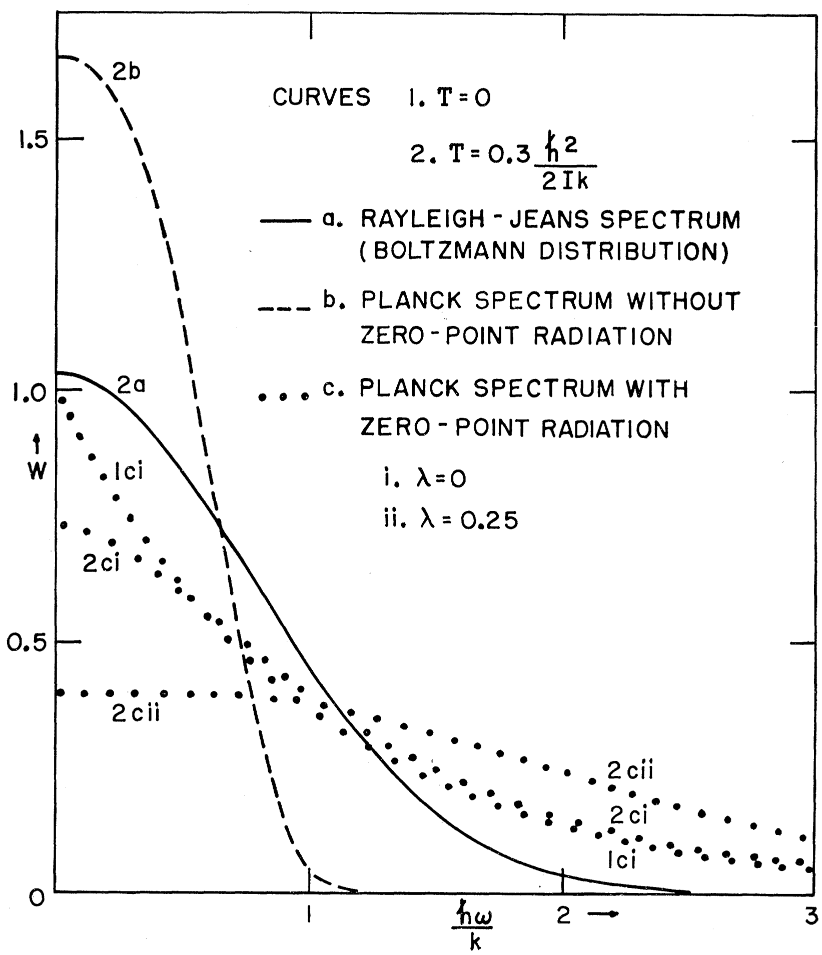

FIG. 1. Frequency distribution of a plane, rigid, electric-dipole rotator in equilibrium via classical electromagnetic interactions with classical radiation for various spectra of electromagnetic radiation.

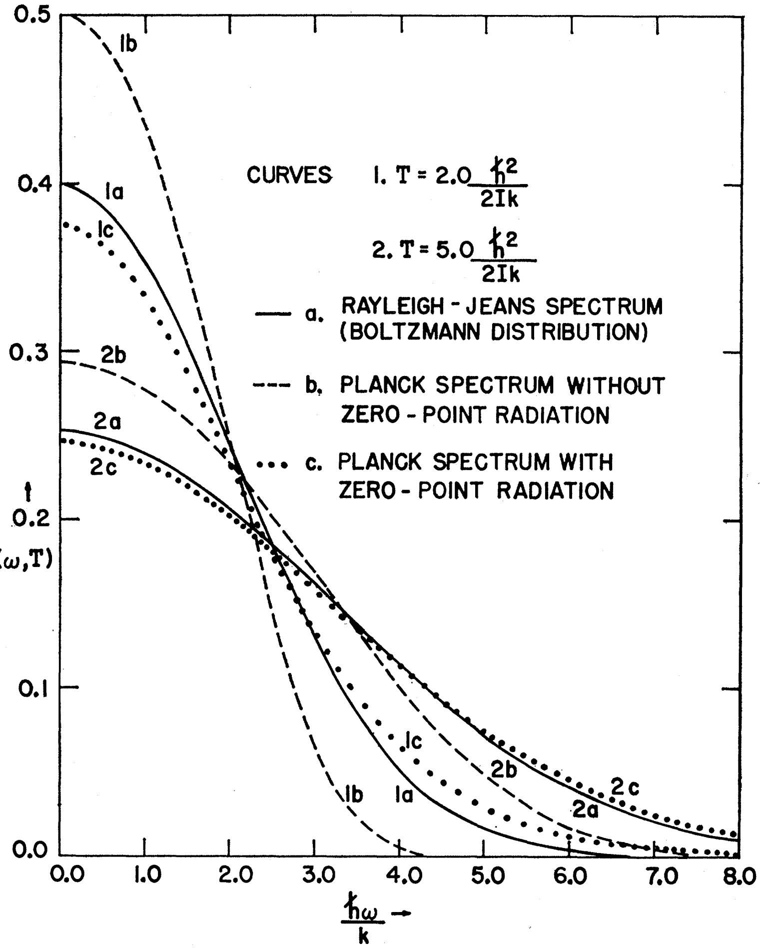

FIG. 2. Frequency distribution of the rotator is found to approach the Boltzmann distribution at high temperature provided electromagnetic zero-point radiation is present. In the absence of zero-point radiation, the discrepancy in the distribution remains proportionately large.

equation is

$$W\frac{2p^2\omega^3}{3c^3}\tau - W\frac{2\pi^2p^2}{3I}\frac{\partial\rho}{\partial\omega}\tau + \frac{1}{2I}\left[W\frac{4\pi^2p^2}{3}\frac{\partial\rho}{\partial\omega}\tau + \left(\frac{4\pi^2p^2}{3}\rho + \lambda\frac{4p^2\hbar^4}{3I^3c^3}\right)\frac{\partial W}{\partial\omega}\tau\right]=0. \quad (37)$$

Solving the first-order differential equation for $W(\omega,T)$, we find

$$W(\omega,T) = \text{const}\,\exp\left(\int_{\omega'=0}^{\omega'=\omega}\frac{-I\omega'^3 d\omega'}{\pi^2c^3[\rho(\omega',T)+\lambda\hbar^4/\pi^2I^3c^3]}\right). \quad (38)$$

At this point, we see that the choice of the thermal radiation spectrum $\rho(\omega',T)$ determines the probability distribution $W(\omega,T)$.

a. Rayleigh-Jeans Radiation Spectrum

As a point of comparison, it is of interest to introduce the Rayleigh-Jeans spectrum ρ_{RJ} into Eq. (38). This corresponds to $\lambda=0$ and

$$\rho_{\mathrm{RJ}}(\omega,T) = (\omega^2/\pi^2c^3)kT, \quad (39)$$

giving the Boltzmann distribution W_{B}:

$$W_{\mathrm{B}}(\omega,T) = \text{const}\,\exp(-\tfrac{1}{2}I\omega^2/kT). \quad (40)$$

(See Fig. 1, curve 2a.) This is precisely the traditional classical energy distribution falling off with the Boltzmann factor $\exp(-\text{energy}/kT)$.

Indeed, Fokker's calculation might also be reversed. Starting at the traditional classical rotator energy distribution (40), we arrive via use of (37) at the Rayleigh-Jeans radiation law (39). Thus the Rayleigh-Jeans law can be derived using classical theory from the Boltzmann distribution function for a rotator. In 1914, this was interpreted as another blow against classical theory. However, recently it has been pointed out the ideas of energy equipartition fail for any particles having electromagnetic interactions. Thus the traditional arguments which lead to the Boltzmann distribution involve the assumption that the rotator does not radiate as it rotates. A careful classical treatment[7] of statistical equilibrium including radiation actually leads to the Planck blackbody spectrum including electromagnetic zero-point radiation.

b. *Planck Spectrum without Zero-Point Radiation—Fokker's Result*

The early quantum derivations of the Planck radiation spectrum ρ_P did not include zero-point energy, but rather were of the form

$$\rho_P(\omega,T)=\frac{\omega^2}{\pi^2c^3}\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}. \quad (41)$$

Assuming that the radiation is classical, this spectrum can be inserted in Eq. (39) to obtain Fokker's result

$$W_F(\omega,T)=\text{const}\exp\left(-\frac{IkT}{\hbar^2}e^{\hbar\omega/kT}+\frac{I\omega}{\hbar}\right). \quad (42)$$

(See Fig. 1, curve 2b.) At high frequencies, the distribution falls to zero very rapidly, falling as the exponential of an exponential.

c. *Planck Spectrum with Zero-Point Radiation*

In earlier work,[6] it has been pointed out that classical electromagnetic theory leads naturally to the Planck radiation spectrum including electromagnetic zero-point radiation. Thus this is the radiation spectrum with which we hope to obtain agreement with experiment. Inserting the spectrum (1) into Eq. (38), we obtain

$$W_{PZ}^{(\lambda)}(\omega,T)=\text{const}$$
$$\times\exp\left(\int_{\omega'=0}^{\omega'=\omega}\frac{-I\omega'^3d\omega'}{\frac{1}{2}\hbar\omega'^3\coth(\hbar\omega'/2kT)+\frac{1}{2}\lambda\hbar^4/I^3}\right). \quad (43)$$

(See Fig. 1, curves 1ci, 2ci, and 2cii.) Only in the limit $\lambda\rightarrow 0$ have we obtained an analytic expression for $W_{PZ}^{(\lambda)}$,

$$W_{PZ}^{(0)}(\omega,T)=\text{const}[\cosh(\hbar\omega/2kT)]^{-4IkT/\hbar^2}. \quad (44)$$

(See Fig. 1, curves 1ci and 2ci.) In the limit $T\rightarrow 0$, this is

$$W_{PZ}^{(0)}(\omega,0)=\text{const}\exp(-2I\omega/\hbar). \quad (45)$$

(See Fig. 1, curve 1ci.)

2. Behavior of Probability Distributions $W(\omega,T)$

The probability distributions $W(\omega,T)$ for the various assumed electromagnetic radiation spectra are presented in Figs. 1 and 2. At the absolute zero of temperature, the probability distributions for thermal radiation without zero-point radiation are δ-function distributions at $\omega=0$. Thus the Boltzmann and Fokker distributions should be imagined as along the W axis in Fig. 1. However, in the presence of zero-point radiation, the rotator is subject to fluctuations even at the absolute zero. Thus for the Lorentz-invariant zero-point spectrum of (1), the rotator probability distribution is as curve

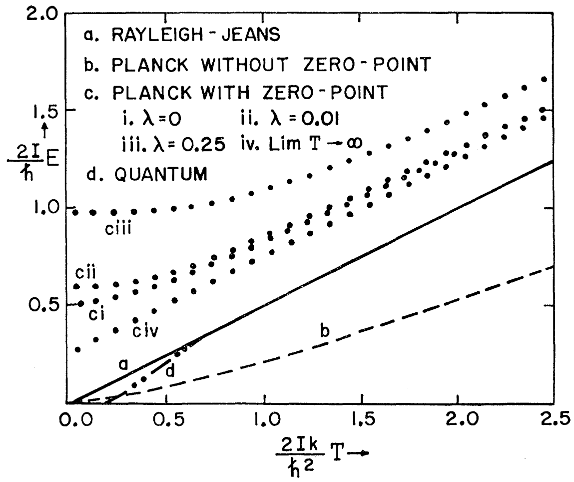

FIG. 3. Average energy of the dipole rotator as a function of temperature for various electromagnetic radiation spectra.

1ci provided we neglect the correction at small frequencies by taking $\lambda=0$.

The presence of thermal radiation displaces the probability distributions from their $T=0$ values. Thus the Boltzmann curve *a* and Fokker curve *b* spread out from the W axis. The Fokker curve drops much more sharply than the Boltzmann curve because of the exponential of an exponential falloff. In the presence of zero-point radiation, the increased temperature modifies only the low-frequency shape of the distribution. Curves 1ci and 2ci indicate the change in shape when no correction is made for the failure of our approximations at low frequencies, $\lambda=0$. The effect of making the correction at low frequencies is indicated by curve 2cii. Here we have used a large value of λ $(=0.25)$ to show clearly that the probability distribution will be flattened at low frequencies and that low temperatures affecting only the low-frequency portion of the curve will not bring any significant alteration in the shape of the probability distribution from that actually holding at $T=0$. It is this suppression of the influence of thermal radiation at low temperatures which we find accounts for the vanishing derivative of the rotator specific heat at low temperatures.

The arguments of traditional classical theory are expected to be valid at high temperatures and at large moments of inertia. Thus it is of considerable interest to follow the distribution curves to higher temperatures as in Fig. 2. Since the importance of low frequencies becomes negligible for high temperatures, it is possible to ignore the low-frequency correction given by the parameter λ and to consider the curves for $\lambda=0$. Comparing curve 2ci of Fig. 1 and curves 1c and 2c of Fig. 2, we find that the rotator probability distribution in the presence of thermal radiation including zero-point radiation goes over very rapidly to a Boltzmann distribution with increasing temperature or increasing rotator moment of inertia. This is just as it should be to give contact with traditional classical theory. The need for electromagnetic zero-point radiation with the assumed spectrum (41) is demonstrated by following the curves labeled *b* which give Fokker's results. The Fokker curve does come nearer to the Boltzmann curve, but the proportional discrepancy in the abscissa remains large. Indeed, in Fig. 3 we find that the average energy given by the *b* curve does not approach the traditional classical $\frac{1}{2}kT$. If the traditional classical arguments are to hold at high temperatures, then a consistent classical theory requires the presence of electromagnetic zero-point radiation.

3. Rotator Average Energy and Specific Heat

Once having obtained the frequency spectrum $W(\omega,T)$, it is an easy matter to obtain the rotator average energy as

$$\langle E(T)\rangle=\int_{\omega=0}^{\infty}\tfrac{1}{2}I\omega^2W(\omega,T)d\omega\bigg/\int_{\omega=0}^{\infty}W(\omega,T)d\omega. \quad (46)$$

We have evaluated the necessary integrals analytically only in the case $W_{PZ}^{(0)}(\omega,0)$, finding

$$\langle E_{PZ}^{(0)}(0)\rangle=\hbar^2/4I. \quad (47)$$

However, the average energy $\langle E(T)\rangle$ was obtained for other values of T and other electromagnetic spectra by numerical integrations. The results are plotted in Fig. 3. The rotator specific heat is obtained as the slope of the specific-heat curve and is plotted in Fig. 4.

In Fig. 3, the straight line *a* shows the traditional classical average energy $\frac{1}{2}kT$. The result *b* obtained by Fokker using the Planck spectrum without zero-point radiation falls increasingly far below this line as the

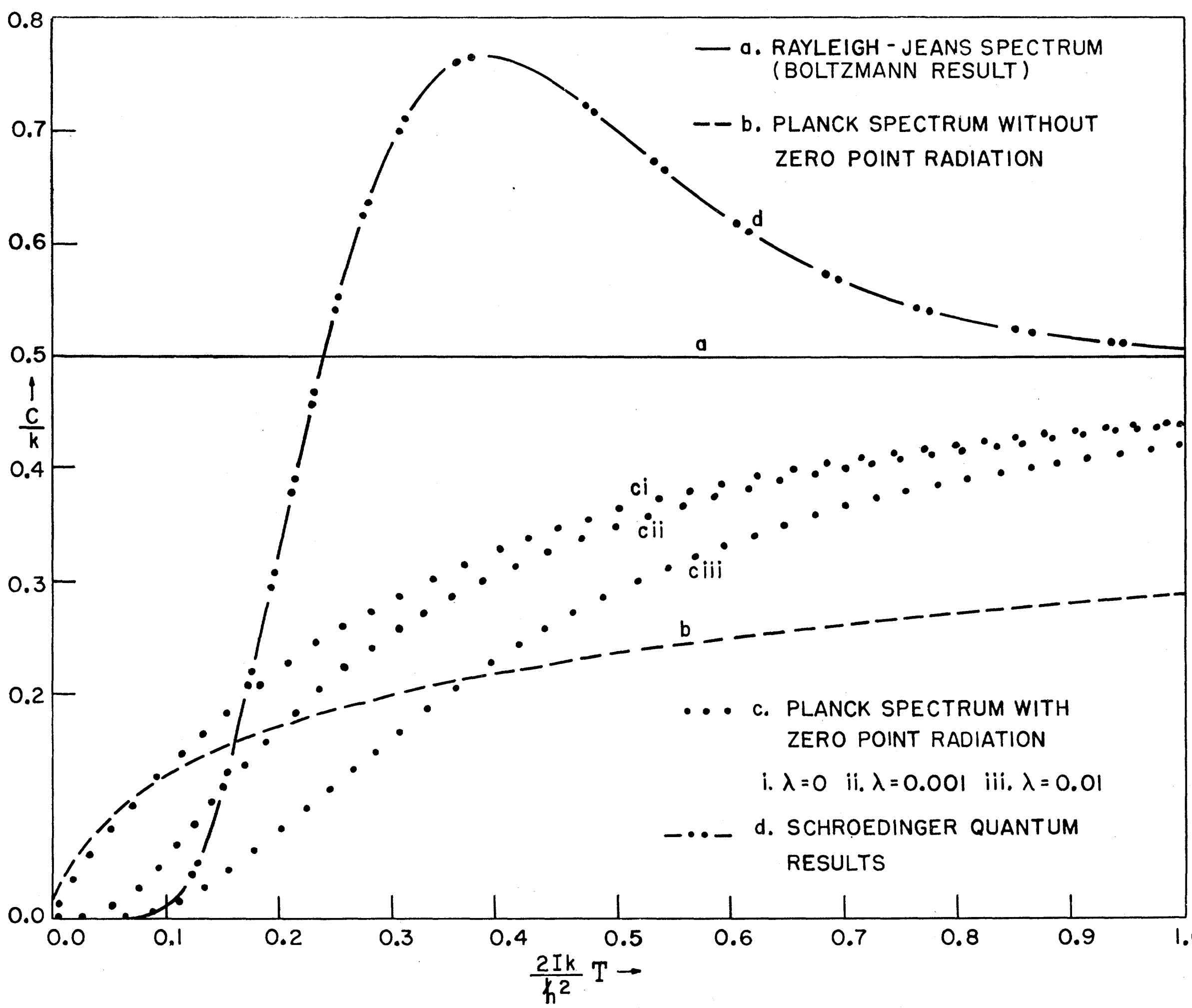

FIG. 4. Specific heat of the dipole rotator as a function of temperature. These curves are the derivatives of the curves in Fig. 3.

temperature T increases. On the other hand, in the presence of zero-point radiation, the rotator has a zero-point energy at $T=0$, and then increases to eventually meet the asymptote $\frac{1}{2}kT+\hbar^2/4I$ shown as curve civ,

$$\langle E_{\mathrm{PZ}}^{(\lambda)}(T)\rangle \sim \tfrac{1}{2}kT+\hbar^2/4I \quad \text{as } T\rightarrow\infty\,. \tag{48}$$

The approximation parameter λ changes the zero-point energy but does not affect the asymptotic limit as shown in the curves ci, cii, and ciii.

The behavior of the specific heat is given in Fig. 4. The line a corresponds to the traditional classical value $\frac{1}{2}k$. Again we find that the Planck spectrum without zero-point radiation cannot give behavior in accord with experiment. The specific-heat curve b has an infinite slope at temperature $T=0$, and falls far below the value $\frac{1}{2}k$ at high temperatures. In the presence of electromagnetic zero-point radiation, the specific heat of the rotator is given as in curves ci, cii, and ciii. The curve ci makes no correction for the erroneous approximation at low frequencies and shows an infinite slope at $T=0$. The presence of the zero slope at $T=0$ depends specifically on a nonzero value for the parameter λ, corresponding to our qualitative arguments in Sec. II 4. At high temperatures, all of these c curves goes over to the traditional classical $\frac{1}{2}k$.

It seems relevant to comment further upon the infinite slopes at $T=0$ for the specific-heat curves b and ci. In the case of Fokker's calculation using the Planck spectrum without zero-point radiation, this infinity corresponds to the δ-function singularity of the probability distribution $W_{\mathrm{F}}(\omega,T)$ as $T\rightarrow 0$. In the presence of zero-point radiation, the infinite slope arises from an erroneous approximation at *low* frequencies. The singular behavior corresponds to a discontinuous change in the form of the probability distribution at $T=0$ for $\lambda=0$. Thus, as seen in curves 1ci and 2ci of Fig. 1, the probability distribution at $\omega=0$ has slope -1 for $T=0$ but slope 0 for any $T>0$. The correction of the approximation ($\lambda>0$) gives the probability distribution slope 0 even at $T=0$. Unfortunately, however, we have not yet been able to evaluate the parameter λ.

The behavior of the average energy or specific-heat curves at high temperatures is also of considerable interest. It is seen that the use of the Planck spectrum without zero-point radiation does not give the average energy $\frac{1}{2}kT$ or specific heat $\frac{1}{2}k$ at high temperatures where we expect the traditional classical values to be appropriate. Thus the presence of the zero-point energy is crucially important in bringing the high-temperature region into agreement with experiment. This situation is in contrast to that for a harmonic oscillator. In this latter case, the absence of the zero-point radiation made no difference in the specific heat or, except for an additive constant, in the oscillator average energy. In part this difference is characteristic of the fact that a rotator is a more delicate system than an oscillator. Thus, an oscillator responds at its own natural frequency and samples the radiation field only at this frequency, whereas the rotator has no natural frequency and hence responds to the radiation at all frequencies. In the case of an oscillator, zero-point radiation represents a single additive energy underneath the thermal energy at the fixed oscillator frequency. For a rotator, zero-point radiation provides a varying amount of energy at various frequencies underneath the thermal radiation at these frequencies. The agreement at high temperatures of the rotator specific heat with the traditional classical $\frac{1}{2}kT$ is a further confirmation of the appropriateness of the hypothesis of electromagnetic zero-point radiation corresponding to a Lorentz-invariant spectrum, $\frac{1}{2}\hbar\omega$ per normal mode.

IV. RELATIONSHIP OF CALCULATION TO QUANTUM THEORY AND TO EXPERIMENT

The foregoing calculation seems of interest even in the approximate form presented because it suggests that classical electromagnetic interactions will give a qualitative picture of an electric-dipole rotator specific heat which gives the traditional $\frac{1}{2}k$ at high temperatures and falls to zero at low temperatures. However, we would like to do far more than this. Ideally, we would like to obtain the specific heat of an arbitrary plane, rigid rotator and then compare the calculation with the quantum-mechanical treatment of a plane, rigid rotator, and also with relevant experimental data. Indeed, Fokker argued that at equilibrium the specific heat of a plane, rigid rotator was exactly the same whether or not it was also a rigid electric dipole. We should notice in particular that nowhere in our results for the dipole rotator probability distributions or energies do we find the electric-dipole moment p. In equilibrium, the dipole moment disappears from the equations because there is a balance between the absorption and emission of energy, both of which processes depend upon p. However, the fact that the system in the calculation was an electric dipole was crucial in determining the radiation damping behavior of the system, and it is by no means clear that this damping behavior is not crucially different at low temperatures from that holding for some other electromagnetic system. We will find that a naive comparison of our calculation with quantum theory and with experiment is unsatisfactory on several levels.

In quantum theory,[13] a plane, rigid rotator satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}\psi(\phi)=E\psi(\phi)\,, \tag{49}$$

[13] P. Fong, *Elementary Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1962), pp. 339–341.

and accordingly has the energy eigenvalues

$$E_n=(\hbar^2/2I)n^2, \quad n=0, \pm1, \pm2, \ldots. \quad (50)$$

The average energy for a rotator at temperature T is then given by quantum statistical mechanics as

$$E(T)=\sum_{-\infty}^{\infty} E_n e^{-E_n/kT} \Big/ \sum_{-\infty}^{\infty} e^{-E_n/kT}, \quad (51)$$

and the specific heat follows from differentiation with respect to T. Figures 3 and 4 include curves giving the quantum average energy and specific heat. The curves are qualitatively different from the results of the zero-point classical calculations reported in the previous sections of this paper. In particular, the specific-heat curve d of Fig. 4 shows a very high maximum which is totally absent (within the approximations used) from the zero-point classical curves ci, cii, and ciii.

The discrepancy becomes all the more disturbing from the point of view of using zero-point energy as an alternative hypothesis to quanta when we consult the experimental data. Hydrogen in three forms H_2, D_2, and HD remains gaseous down to sufficiently low temperatures to show the effects of the rotational specific heat.[14] In the case of HD, the quantum maximum in the specific heat (which persists in a more subdued form in a three-dimensional rigid rotator) seems to be found experimentally.[15]

However, the experimental work on H_2 and D_2 provides us with a crucial reminder. The specific-heat data on H_2 and D_2 cannot be made to fit the quantum curve d.[14] Rather, it is found that there are ortho and para forms of hydrogen neither of which satisfies the rigid rotator spectrum (50). The internal structure of the molecule must be accounted for before separating out the rotator motion. Thus we are reminded that there is no such thing in nature as a rigid rotator. Atomic systems are made up of electrons and nuclei, and it is to these particles that the Schrödinger equation applies. In the separation of variables in the equation it may be found that the system behaves to some approximation as if it were a rigid rotator.

It has been pointed out by an increasingly large number of writers[16] that quantum mechanics bears a strong affinity to a theory of Markov processes. In particular, the time-dependent and time-independent Schrödinger equations have been derived from classical Newtonian mechanics on which there has been superimposed a random walk. It is our heuristic idea that perhaps the origin of the random motion should be assigned to fluctuating electromagnetic zero-point radiation. At the moment, there seem to be formidable obstacles to carrying through this proposal as a mathematical proof. However, it is relevant to interpret our calculation for a rigid rotator in the light of these heuristic ideas. Just as quantum theory is required to account for the internal structure of the rotator molecule while allowing separation off of the rotator variables, so too electromagnetic zero-point radiation is expected to provide the fluctuations giving the molecular structure as well as affecting the rotator behavior. However, the analysis through zero-point radiation specifically involves the absorption and then the emission of radiation. In the previous classical calculation, we have assumed that all emission of radiation arose from the rotation of the dipole and that there was no emission connected with the charged-particle structure. It is not at all clear that a detailed analysis of atomic particles subject to zero-point radiation would have a separation of the internal and rotational motions which took the permanent dipole form assumed here. For this reason, we believe that the classical calculation does not represent the same situation as a quantum rigid rotator in its application to explaining molecular specific heats.[17]

V. CLOSING SUMMARY

Although elementary quantum-mechanics textbooks cite the experimental behavior of blackbody radiation and of specific heats of material particles at low temperatures as evidence for quantum theory, it is now not clear that this evidence is conclusive. Although quantum theory seems to provide a satisfactory theory of these phenomena, it is also true that classical theory including electromagnetic zero-point radiation allows an equally good explanation of blackbody radiation and of oscillator specific heats. In the present paper, we have pointed out that this same classical theory may go far to explain rotational specific heats. Within the approximations applied, the results of the calculation for an electric-dipole rotator are qualitatively different from those of the quantum theory of a plane, rigid rotator. However, it is proposed that actually the zero-point classical and the quantum notions of a plane, rigid rotator are distinct. Comparison with experimental molecular specific heats does not yet seem possible for the zero-point classical calculation performed here.

Aside from the crucial question of describing physical phenomena, it is also of interest to see the development of a classical theory encompassing electromagnetism and statistical thermodynamics. It is found here just as in the earlier thermodynamic analysis of an ideal

[14] R. H. Fowler, *Statistical Mechanics*, 2nd ed. (Cambridge U. P., Cambridge, England, 1966), pp. 82–89.

[15] K. Clusius and E. Bartholomé, Z. Elektrochem. **40**, 524 (1934); Z. Physik. Chem. (Leipzig) **B29**, 162 (1935).

[16] E. Nelson, Phys. Rev. **150**, 1079 (1966). This article contains a list of references. There is more recent work by, among others, T. Dankel [thesis, Princeton University, 1969 (unpublished)], and E. Santos [Nuovo Cimento **B59**, 65 (1969)]. See also T. W. Marshall, Proc. Roy. Soc. (London) **A276**, 475 (1963); Proc. Cambridge Phil. Soc. **61**, 537 (1965).

[17] After enquiring further into the relationship of electromagnetic zero-point radiation and charged-particle Brownian motion, the author hopes to return to reconsider the question of molecular rotational specific heats.

gas that a consistent classical theory which accepts the equipartition theorem for high temperatures or massive particles seems to require the idea that the universe contains electromagnetic zero-point radiation with a Lorentz-invariant spectrum.

Note added in manuscript: *Macroscopic Observation of Zero-Point Radiation.* The question has been raised as to whether the hypothesis of classical electromagnetic zero-point radiation used here may not lead to contradictions with familiar experimental observations. The observations which might be affected by zero-point radiation can be separated generally into two classes—those involving electromagnetic interactions and those involving the gravitational effects of energy concentrations.

In dealing with electromagnetic effects, there are two characteristics of zero-point radiation to keep in mind. Firstly, the radiation is completely incoherent. This means that the rapid fluctuations tend to cancel each other out and no instrument involving many electrons will measure a current due to zero-point radiation. The contrast in behavior between the coherent radiation (often in the back of one's mind in classical theory) and the incoherent zero-point radiation is analogous to the effects on a bridge of a band of men walking in step or walking out of step. Coherent radiation will give rise to net currents in a conductor just the way the band of men marching in step will bring the bridge into an over-all vibration pattern. The incoherent radiation gives rise only to fluctuations of individual particles just as the men marching out of step will give rise to local vibrations of small parts of the bridge but not to a net vibration of the entire bridge. Thus a small amount of coherent radiation will be observable in electromagnetic detectors, whereas the large energy of zero-point radiation will not be. The zero-point radiation will merely give rise to local fluctuations superimposed upon any coherent radiation.

Thermal radiation is incoherent radiation, and hence one may consider the effects of zero-point radiation upon detectors of thermal radiation. Here we must mention the second aspect of zero-point radiation which affects its electromagnetic detection. This is the assumption that zero-point radiation is homogeneous and isotropic in the universe. Therefore, one cannot create a net transfer of zero-point energy from one region of space to another. Since the spectrum of zero-point radiation is Lorentz invariant, one cannot observe the radiation by changing to a relatively moving inertial frame. The delicate balance in the average energy in various regions of space and in various coordinate frames may be compared to the great uniformity of the average pressure on all of the walls of a box containing Avogadro's number of gas molecules; this uniformity exists despite the fact that large gas-particle velocities are involved. It is noteworthy in the zero-point radiation spectrum that the energy density per unit frequency interval increases with the increase in frequency of the electromagnetic waves. This is analogous to an increase in the energy of particles in an ideal gas coupled with an increase in the number of particles, so that the energy fluctuations per unit volume do not increase in proportion to the energy.

From these considerations it appears that zero-point radiation will not affect the ordinary detectors of classical electromagnetic radiation. Rather, the fluctuations in radiation will lead to fluctuations of the granular structure of matter which are averaged out in macroscopic observations. In the future, we hope within appropriate approximations to show quantitatively that the fluctuations of electromagnetic zero-point radiation may be regarded as the source of the fluctuations used by Nelson[16] and others to derive from Newtonian mechanics the Schrödinger equation applicable to atomic physics.

The gravitational effects of the energy in electromagnetic zero-point radiation remain obscure. However, it does not seem reasonable at this time to reject the idea of zero-point radiation on this basis. After all, the electromagnetic energy singularities of point charges should also give rise to gravitational singularities, and we excuse these singularities by remarking that the gravitational aspects of elementary particles are not yet fully understood. It is noteworthy that Casimir[18] has speculated on the possibility of compensating Coulomb energy divergences in quantum field theory with the zero-point energy divergence of the electromagnetic field. Also, Wheeler[19] has noted that electromagnetic zero-point fluctuations (whether regarded as classical or quantum) will lead to negative energies of gravitational attraction. He suggests: "Individually the components of the vacuum energy are enormous and collectively they compensate."

ACKNOWLEDGMENT

I wish to thank Professor Martin Klein for his interest in this work and for his comments[20] on the historical aspects of quantum theory.

[18] H. B. G. Casimir, Physica **19**, 846 (1953).

[19] J. A. Wheeler, in *Battelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), pp. 269–270.

[20] Professor Klein has pointed out that a derivation of the blackbody spectrum from semiclassical ideas appears in the work of A. Einstein and O. Stern [Ann. Physik **40**, 551 (1913)]. These authors assign zero-point *energy* $\hbar\omega$ (no factor of $\frac{1}{2}$) to a harmonic oscillator of frequency ω, use classical electromagnetic interactions, and find that the analysis of A. Einstein and L. Hopf [Ann. Physik **33**, 1105 (1910)] is modified so as to give the Planck spectrum *without* zero-point energy. The procedure is actually inconsistent because the classical oscillator must have the same average energy as the field normal mode of frequency ω, and will radiate away the additional zero-point energy; the inconsistencies appear in troublesome factors of two throughout the calculation. Einstein and Stern did not turn to the idea of zero-point *radiation*. Their work differs from the derivations of Refs. 6 and 7 in the force analysis, in the differential equation for the radiation spectrum, and in the thermodynamics involved.