$$K_3^\epsilon = [a_{n'}/4\mu^4(\mu^2-\Delta^2)](\mu^2-1)^{1/2} \times \{\mu^2(\mu^2-\Delta^2)-\mu(\mu^2-\Delta^2)^{1/2}-(\mu^2-a_n^2) \times [(\mu^2-1)(\mu^2-\Delta^2-1)]^{1/2}\},$$

$$K_4 = [a_n a_{n'} a_{n-1}/4\mu^4(\mu^2-\Delta^2)](\mu^2-1)(\mu^2-\Delta^2-1)^{1/2}, \quad \Delta^2 = 2H/H_q,$$

$$K_5^\epsilon = [a_{n+1}/4\mu^4(\mu^2+\Delta^2)]\{\mu^2(\mu^2+1)-\epsilon(\mu^2-1) \times [(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\}(\mu^2+\Delta^2-1)^{1/2},$$

$$K_6^\epsilon = [a_n(\mu^2-1)^{1/2}/4\mu^4(\mu^2+\Delta^2)] \times \{\mu^2(\mu^2+\Delta^2)+\mu[\mu^2+\Delta^2]^{1/2} - \epsilon[(\mu^2-1)(\mu^2+\Delta^2-1)(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\},$$

$$K_7 = [a_{n'}(\mu^2-1)^{1/2}/4\mu^4(\mu^2+\Delta^2)] \times \{\mu^2(\mu^2+\Delta^2)-\mu[\mu^2+\Delta^2]^{1/2} - (\mu^2-a_n^2)[(\mu^2-1)(\mu^2+\Delta^2-1)]^{1/2}\},$$

$$K_8 = [a_n a_{n'} a_{n+1}/4\mu^4(\mu^2+\Delta^2)](\mu^2-1)(\mu^2+\Delta^2-1)^{1/2}.$$

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Classification of Paraparticles*

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Hartle and Taylor have shown that the cluster law imposes certain restrictions on the allowed symmetries of first-quantized systems of identical particles. We extend their results to show that all particles which are neither bosons nor fermions and which obey the cluster law can be divided into two classes, those of finite order, and those of infinite order. For every positive integer p there are two types of finite-order particle, which we call parabosons and parafermions of order p. Ordinary bosons and fermions can be fitted into this scheme as particles of order 1. We conjecture that the finite-order particles can be identified with the parafermions and parabosons of the second-quantized, parafield theory. Infinite-order particles would seem to have no analog in the second-quantized theory, as presently formulated.

I. INTRODUCTION

IN this paper we establish a simple classification of the possible types of paraparticle (that is, particles which are neither bosons nor fermions). Our results, which extend the recent work of Hartle and Taylor,[1] are based on a quantum-mechanical (that is, first-quantized) point of view.[2]

II. CLUSTER LAW AND CONSEQUENT RESTRICTIONS ON ALLOWED SYMMETRY TYPES

The cluster law requires that two isolated experiments which are sufficiently well separated must not interfere. For example, an experiment involving n particles localized in London should be completely unaffected by the presence or absence of m more particles localized on the moon, and vice versa.

Hartle and Taylor apply the cluster law to a system of $n+1$ particles, one of which is far removed and isolated from the rest. By requiring that any observation localized near the n particles should see some allowed n-particle state, they establish the following result.

(a) If a particle has $(n+1)$-particle states corresponding to the irreducible representation (IR) $D(n+1,\lambda)$ of the permutation group[3] S_{n+1}, then it must have n-particle states corresponding to *all* IR $D(n,\nu)$ of S_n whose Young diagrams can be obtained from that of $D(n+1,\lambda)$ by removing one square.

We first generalize the result (a) so as to apply to systems of $n+m$ particles divided into two isolated groups of n and m particles. To this end we note that (a) can be rephrased to say that there must be n-particle states for all $D(n,\nu)$ of S_n which are obtained when $D(n+1,\lambda)$ is restricted to S_n by holding one variable fixed.[4] Applied to $(n+m)$-particle systems, the generalization of this result can be shown to be the following.

(b) If a particle has $(n+m)$-particle states with IR $D(n+m,\lambda)$, then it must have n-particle states corresponding to *all those* $D(n,\nu)$ and m-particle states corresponding to *all those* $D(m,\mu)$ such that the outer Kronecker product $D(n,\nu)\times D(m,\mu)$ is contained in the restriction of $D(n+m,\lambda)$ to $S_n\times S_m$.

In both of the results (a) and (b) one starts from the assumed existence of some $(n+m)$-particle symmetry and deduces the existence of certain symmetries for

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[1] J. B. Hartle and J. R. Taylor, Phys. Rev. **178**, 2043 (1969).

[2] By this we mean that we describe a system of n identical particles with certain of the wave functions $f(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ of the Hilbert space $L^2(R^{3n})$. The alternative is the second-quantized approach, in which one uses the vectors generated by the action of field operators on a vacuum state. We emphasize this distinction because, even though the two points of view are known to be equivalent for ordinary bosons and fermions, their relationship is not well understood for the general paraparticle. See, for example, Ref. 9 of A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **138**, B1155 (1965), and also the concluding remarks of this paper.

[3] As is well known—see Ref. 1 or A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**, B248 (1964)—every pure state of a system of n identical particles is associated with some IR of the permutation group S_n.

[4] Indeed, this is the form in which the result is first proved in Ref. 1.

n- and m-particle systems. This reasoning can be inverted. If we suppose that a given particle has n-particle states of symmetry $D(n,\nu)$ and m-particle states of symmetry $D(m,\mu)$, then according to the cluster law there exists an $(n+m)$-particle state which consists of n particles in London with symmetry $D(n,\nu)$ and m particles on the moon with symmetry $D(m,\mu)$. It follows from the same arguments as lead to (b) that this $(n+m)$-particle state must correspond to one of the IR $D(n+m,\lambda)$ of S_{n+m} which contain $D(n,\nu)\times D(m,\mu)$ when restricted to $S_n\times S_m$. That is, the following obtains.

(c) If a particle has n-particle states of symmetry $D(n,\nu)$ and m-particle states of symmetry $D(m,\mu)$, then it must have $(n+m)$-particle states corresponding to *at least one of those* IR $\mathrm{D}(n+m,\lambda)$ of S_{n+m} which contain $D(n,\nu)\times D(m,\mu)$ when restricted to $S_n\times S_m$.

We shall discuss below how the representations with this property can be determined in practice. Meanwhile we describe the application of results (a)–(c).

III. CLASSIFICATION

Associated with every species of particle is a family of Young diagrams corresponding to the allowed multiparticle symmetry types of the particle in question. For example, the allowed states of an ordinary fermion correspond to all diagrams with just one column, those of a boson to all diagrams with just one row. We now use the three results above to classify all possible families of allowed symmetries, starting with a definition: A particle will be said to have *finite column order* p if it has states associated with some Young diagram of p columns, but no states associated with any diagram of more than p columns.

We make a corresponding definition for a particle of *finite row order*. With these definitions it is clear, for example, than an ordinary fermion has column order 1 and a boson row order 1, and that these are the only particles of order 1.

If a particle has neither finite row order nor finite column order, then it must have states with diagrams of arbitrarily many rows and states with diagrams of arbitrarily many columns. We say that such a particle has *infinite order*.

For particles of finite order, we can prove the following result.

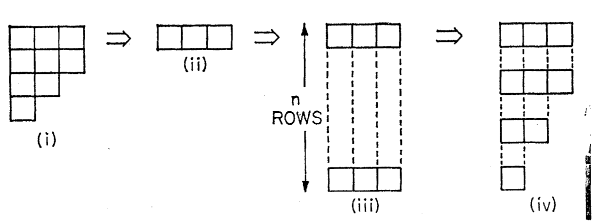

FIG. 1. Diagrams illustrating the chain of reasoning used to establish the result (d) (for the case $p=3$).

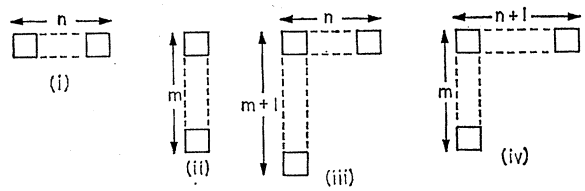

FIG. 2. A particle of infinite order must include all of these diagrams among its allowed symmetries.

(d) Any particle of column (or row) order p must in fact have multiparticle states corresponding to *every* Young diagram with p or fewer columns (or rows).

This result means that there is actually only one possible family of allowed symmetries for a particle of column (or row) order p—namely, that one containing *all* Young diagrams of up to p columns (or rows). We can refer to the unique type of particle with column order p as a *parafermion of order* p; that of row order p we can call a *paraboson of order* p.

We prove the result (d) for a particle of column order p as follows.

The particle has states corresponding to some diagram with p columns. Consider, for example, diagram (i) of Fig. 1 for the case $p=3$. By removing squares we can reduce this diagram to a single row of p squares [see Fig. 1 (ii)]. Thus from the result (a) above, our particle must have states associated with this latter diagram, which corresponds to the totally symmetric p-particle representation $D(p,s)$. We next note that according to the result (c) above, the particle must have $(n\times p)$-particle states (with n an arbitrary integer) associated with at least one of the representations containing

$$D(p,s)\times D(p,s)\times\cdots\times D(p,s)\quad(n\text{ factors}).$$

However, as we show in the Appendix, the only such diagram with no more than p columns is the rectangular diagram of p columns and n rows. [See Fig. 1 (iii).] Accordingly, the particle must have states associated with this diagram. Finally, by removing squares from this diagram we establish that the particle must have states associated with any diagram of p or fewer columns. [See Fig. 1 (iv).] Q.E.D.

It remains to consider particles of infinite order. These are not as easily classified as those of finite order. They must include among their allowed diagrams both single rows and single columns of arbitrary length, that is, $D(n,s)$ and $D(m,a)$ with n and m arbitrary. [See Figs. 2 (i) and 2 (ii).] The only representations which contain $D(n,s)(\times Dm,a)$ are the "L-shaped" diagrams of

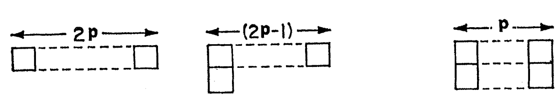

FIG 3. Diagrams of those IR of S_{2p} which contain $D(p,s)\times D(p,s)$.

Figs. 2 (iii) and 2 (iv). (See Appendix.) Since n and m are arbitrary we can conclude the following.

(e) A particle of infinite order must include among its allowed symmetries *all* symmetric, *all* totally antisymmetric, and *all* "L-shaped" diagrams.

We have not attempted to classify further the particles of infinite order. Since the product of two L-shaped representations is always contained in an L-shaped representation, one possible infinite-order particle is one which allows the symmetries of (e) but no others. There are certainly other possibilities. At the other extreme, for example, is a particle with states corresponding to *all* IR.

IV. CONCLUSIONS

We have seen that all particles of finite order can be classified into parafermions and parabosons of orders $p=1, 2, \ldots$. This scheme closely resembles that of the particles of second-quantized, parafield theory. Indeed, Kamefuchi and Ohnuki[5] have shown that the n-particle wave functions associated with a parafermi (or parabose) field of order p support a representation of S_n containing every IR with p or fewer columns (or rows). This suggests that the first- and second-quantized paraparticles can be identified in the same way that first- and second-quantized fermions and bosons can. We plan to show in a later paper that this is so.[6]

The infinite-order paraparticles apparently do not correspond in any natural way to the particles of parafield theory.

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APPENDIX: PROOFS

We have to determine which representations $D(n+m,\lambda)$ of S_{n+m} contain, when restricted to $S_n\times S_m$, a given product representation $D(n,\nu)\times D(m,\mu)$. The simplest procedure is to utilize a prescription given in Hamermesh[7] for determining which $D(n+m,\lambda)$ occur in a certain reducible representation of S_{n+m} which Hamermesh calls the *outer-product representation.* This representation is defined as follows: If we have n particles in the IR $D(n,\nu)$ and m more particles in the IR $D(m,\mu)$, then as regards permutations of the n particles among themselves, and of the m particles among themselves, the total system gives the representation $D(n,\nu)\times D(m,\mu)$ of $S_n\times S_m$. However, if we consider permutations which mix the n particles with the m particles, then the associated representation is a representation of S_{n+m}, and in general is reducible. This reducible representation is denoted by Hamermesh as the outer-product representation. Those IR $D(n+m,\lambda)$ of S_{n+m} which are contained in the outer-product representation are found by a simple prescription which is given by Hamermesh[7] and which we state below (for the relevant case). Now, some simple algebra shows (what is perhaps intuitively clear) that $D(n+m,\lambda)$ is contained in the outer-product representation formed from $D(n,\nu)$ and $D(m,\mu)$ if and only if $D(n,\nu)\times D(m,\mu)$ is contained in the restriction of $D(n+m,\lambda)$ to $S_n\times S_m$.[8] Thus we can immediately apply the prescription of Hamermesh to the problem at hand.

We consider the case where the second representation contains only one row; that is, $D(m,\mu)=D(m,s)$. [This case is sufficient for the proof of the results (d) and (e).] For this case, the prescription tells us that the representations which contain $D(n,\nu)\times D(m,s)$ are those whose diagrams can be obtained by adding the squares of $D(m,s)$ to the diagram of $D(n,\nu)$ in any way such that no two squares are added to the same column. [See Fig. 3 for the case $D(n,\nu)=D(m,\mu)=D(p,s)$.]

In the proof of the result (d) we need to determine those representations which contain

$$D(p,s)\times D(p,s)\times\cdots\times D(p,s)\quad (n\text{ factors}).$$

From repeated application of the above prescription it is clear that none of these diagrams can have more than n rows, and hence that all must have p or more columns. In particular, the only diagram with just p columns is the $n\times p$ rectangle of Fig. 1 (iii). This completes the proof of the result (d).

In exactly the same way we can prove that the only IR containing $D(n,s)\times D(m,a)$ are the L-shaped representations of Figs. 2 (iii) and 2 (iv), and thus complete the proof of the result (e).

Finally, it is a straightforward matter to use the general prescription of Hamermesh to prove that the product of two L-shaped representations is contained in an L-shaped diagram, as asserted at the end of Sec. III.

[5] S. Kamefuchi and Y. Ohnuki, Ann. Phys. (N. Y.) **51**, 337 (1969).

[6] R. H. Stolt and J. R. Taylor, Nucl. Phys. (to be published). It is tempting to claim that the identification is immediately obvious. Of the various obstacles which actually stand in its way we mention two: (1) In the first-quantized approach states are represented by many-dimensional subspaces, or generalized rays—a redundancy which does not appear in the second-quantized approach. (2) The permutation operators of the two theories are not the same. As we shall show, both of these difficulties can be overcome—the first using the result of Ref. 1 that there is an equivalent formulation of the first-quantized theory without the redundancy of the generalized ray, the second using the two kinds of permutation introduced by P. V. Landshoff and H. P. Stapp, Ann. Phys. (N. Y.) **45**, 72 (1967).

[7] M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962), p. 249.

[8] This is actually a special case of the Frobenius reciprocity theorem.