Transverse Electrical Conductivity of a Relativistic Gas in an Intense Magnetic Field

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The transverse electrical conductivity is computed for a system of degenerate relativistic electrons in a strong magnetic field $H \approx 10^{10}$–10^{13} G. As suggested by pulsar models, such fields exist in nature in collapsed bodies, like neutron stars. The present computation is valid in the outer regions of the star where the scatterers are not degenerate, while the electrons are taken to be at zero temperature. The scattering mechanism is assumed to be the electron-ion Coulomb scattering. Numerical values of the transverse conductivity are given in the range $10^9 \leq H \leq 10^{13}$ G, and a comparison is made with the longitudinal and zero-field conductivities. It is found that for densities $10^7 \leq \rho \leq 5 \times 10^7$, $\sigma_{11} \approx 20\sigma_0$ and $\sigma_0 \approx 3\sigma_\perp$. As the density increases, both $\sigma_\perp$ and σ_{11} tend to σ_0.

I. INTRODUCTION

THE problem of the electrical conductivity σ of a relativistic degenerate electron gas has been considered recently in two cases: one involving high density and no external magnetic field,[1] and a second case involving high density with a magnetic field parallel to the electric field.[2] In both computations, the scattering mechanism was thought to be impurity scattering, as described by a screened Coulomb electron-ion potential. The impurity system was allowed to form a sublattice with a certain ion-ion correlation strength[1] $\Gamma = 23 Z^{5/3} \rho_6^{1/3} T_6^{-1}$, where ρ_6 is the matter density in units of $10^6 \mu_e$ g/cm^3 ($\mu_e = Z/A$) and T_6 is the temperature in units of 10^6 °K. Another possible mechanism giving rise to electrical resistivity is the interaction of the electrons with the lattice vibrations.[3] The dominance of one mechanism with respect to the other is simply dictated by the temperature. As is usually done in problems of Ohmic conductivity, the electric field was treated as a perturbation. Accordingly, in Refs. 1 and 2, the conductivity was computed by making use of the Boltzmann transport equation. However, when the electric field is perpendicular to H, the Boltzmann transport equation approach cannot be applied, since the diagonal elements of the components of the velocity operators perpendicular to H vanish. This problem was recognized long ago, and the solution has been to use the density-matrix approach. A full discussion of this point is given in Ref. 4. This method is also used in the present paper. As expected on physical grounds, the presence of a magnetic field changes σ from a scalar to a tensor. While the definition of σ_{11} is straightforward, the "transverse" conductivity is defined by means of the relation[4]

$$\sigma_\perp = (\sigma_{yy}^2 + \sigma_{xy}^2)/\sigma_{yy}, \tag{1}$$

if H is along z, and E is in the y direction. Contrary to σ_{yy}, which depends solely on the scattering process, the σ_{xy} component is due entirely to the drift velocity $v_D/c = (\mathbf{E}\times\mathbf{B})/B^2$. Since this velocity is independent of the charge of the particle, there is no net current in the x direction, i.e., $\sigma_{xy} = 0$, if the ions are free to move. This situation is encountered in a plasma, while it is violated if the ions exhibit a crystalline structure.

A magnetic field along the z axis does not change the free motion of the electron in this direction, but strongly modifies the transverse motion. This results in an increase in the transverse resistivity with respect to the longitudinal case. A sizable difference is found between σ_{11}, σ_0, and $\sigma_\perp$ for $H = H_q$ and $10 \leq \rho_6 \leq 50$ (see Table VI). It is, however, to be expected that at very high densities $\rho \geq 10^{10}$–10^{14} g/cm^3, such a difference would eventually disappear. This general trend has been found to be a common feature of all the astrophysical processes whose occurrence is not linked to the existence of a magnetic field.[5]

In Sec. II the wave function and eigenvalues of an electron in crossed electric and magnetic fields are derived, as well as the matrix element of the velocity operator to be employed in the subsequent section. In Sec. III the density-matrix approach and the relevant matrix elements are examined briefly. In Sec. IV the final expression for $\sigma_\perp$ is deduced. Numerical values of $\sigma_\perp$ for a certain range of densities and magnetic fields are left for Sec. V, as well as a discussion of the possible astrophysical relevance of this computation.

II. DIRAC EQUATION IN CROSSED ELECTRIC AND MAGNETIC FIELDS

We first consider an electron of mass m and charge e in an external homogeneous magnetic field $\mathbf{H}$, directed along the z axis, and a constant electric field $\mathbf{E}$, directed along the y axis. By solving the Dirac equation,[6] in the

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gauge $A_\mu \equiv (-Hy, 0, 0, -Ey)$, we obtain the following energy eigenvalues (here $\hbar = c = 1$):

$$\epsilon_{n,p} \equiv \epsilon = \eta\gamma^{-1}[p_z^2 + m^2(1+2n\theta\gamma^{-1})]^{1/2} + \beta p_x, \quad (2)$$

where

$$n = 0, 1, 2, \ldots, \infty, \quad p \equiv (p_x, p_z), \quad \theta = H/H_q,$$
$$H_q = m^2c^3/eh = 4.414 \times 10^{13}\ \text{G}, \quad \eta = \pm 1, \quad \beta = E/H, \quad (3)$$
$$\gamma^{-2} = 1 - \beta^2.$$

When $E \to 0$, the eigenvalues reduce to

$$\epsilon_{n,p}{}^0 = \eta[p_z^2 + m^2(1+2n\theta)]^{1/2}, \quad (4)$$

which is the standard form of the energy eigenvalues of an electron in a magnetic field.[7]

To find the eigenfunctions, a complete set of commuting operators must be specified. These may be[8] $\mathcal{H}$, p_z, p_x, and $p_\sigma = -\boldsymbol{\sigma}\cdot(i\boldsymbol{\nabla} + e\mathbf{A})$. In this case the eigenfunctions are[6]

$$|n,p,s,\beta\rangle \equiv N(\Gamma_+ u + \Gamma_- \tilde{u}) e^{-\xi^2/2} e^{i(p_x x + p_z z - \epsilon t)}, \quad (5)$$

where

$$2\Gamma_\pm{}^2 = \gamma \pm 1,$$
$$\xi = (\theta\lambda_c{}^{-2}\gamma^{-1})^{1/2}[y + \lambda_c{}^2\gamma^2\theta^{-1}(p_{x2} - \beta\epsilon)], \quad (6)$$

$$u = \begin{bmatrix} C_1\bar{H}_n(\xi) \\ C_2\bar{H}_{n-1}(\xi) \\ C_3\bar{H}_n(\xi) \\ C_4\bar{H}_{n-1}(\xi) \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} C_4\bar{H}_{n-1}(\xi) \\ C_3\bar{H}_n(\xi) \\ C_2\bar{H}_{n-1}(\xi) \\ C_1\bar{H}_n(\xi) \end{bmatrix}, \quad (7)$$

$$C_1(n,s) \equiv C_1 = aA, \quad C_2 = saB, \quad C_3 = \eta sbA, \quad C_4 = \eta bB,$$
$$2a^2 = 1 + \eta m\epsilon^{-1}, \quad 2A^2 = 1 + sp_z(p_z^2 + 2m^2n\theta)^{-1/2}, \quad (8)$$
$$2b^2 = 1 - \eta m\epsilon^{-1}, \quad eB^2 = 1 - sp_z(p_z^2 + 2m^2n\theta)^{-1/2}.$$

$\bar{H}_n$ are the Hermite polynomials normalized to unity, and

$$N^2 = \theta^{1/2}(\lambda_c\gamma L_x L_z)^{-1}.$$

The matrix elements of the velocity operators α_x and α_y in the scheme of the eigenfunctions (5) are needed. Since we are interested in linear effects in the electric field E, we can expand the eigenfunction (5) to the first order in β. Putting

$$\alpha_\pm = \alpha_x \pm i\alpha_y,$$

we have, after some algebra,

$$\tfrac{1}{2}\langle nps\beta|\alpha_+|n'p's'\beta\rangle = \{P_n(s,s')\delta_{n',n+1} + \beta[R_n(s,s')\delta_{n',n} + Q_{n+2}(s,s')\delta_{n',n+2}]\}\delta_{p,p'}, \quad (9)$$

$$\tfrac{1}{2}\langle nps\beta|\alpha_-|n'p's'\beta\rangle = \{P_{n-1}(s',s)\delta_{n',n-1} + \beta[R_n(s',s)\delta_{n',n} + Q_{n-2}(s',s)\delta_{n',n-2}]\}\delta_{p,p'}, \quad (10)$$

where P, Q, and R are given in terms of the coefficients C_i appearing in the spinors. We mention here only the expressions needed in the following discussion. These are

$$P_n(s,s') = C_1(n,s)C_4(n+1,s') + C_3(n,s)C_2(n+1,s'), \quad R_n(s,s) = \tfrac{1}{2}. \quad (11)$$

III. EVALUATION OF DENSITY MATRIX

As stated in the Introduction, the density-matrix technique will be used to compute the average value of the electron current. In the independent-particle approximation, each electron is described by a normalized wave function $\psi(\mathbf{r},t)$ solution of the equation

$$i\partial\psi/\partial t = \mathcal{H}\psi.$$

Expanding $\psi(\mathbf{r},t)$ in a complete set of orthonormal time-independent functions $\varphi_n(\mathbf{r})$, we have

$$\psi(\mathbf{r},t) = \sum_n a_n(t)\varphi_n(\mathbf{r}).$$

The density matrix is defined as[9]

$$\rho_{mn} = \langle a_m(t) a_n{}^*(t)\rangle, \quad (12)$$

where the brackets indicate the operation of statistical average. The macroscopic value of any physical quantity Q is then given by

$$\langle Q\rangle = \mathrm{Tr}(\rho Q). \quad (13)$$

The time variation of the density matrix is governed by the equation

$$-i\frac{\partial\rho}{\partial t} = [\rho, \mathcal{H}].$$

To evaluate the matrix elements of ρ, we will pursue the same method given in Ref. 4, to which we refer for more details. The Hamiltonian $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_0 - eEy + V = \mathcal{H}_1 + V, \quad (14)$$

where $\mathcal{H}_0$ is the sum of the kinetic and magnetic parts of the Hamiltonian[2] and V is the electron-ion potential. Following a procedure first introduced in Ref. 10, E is replaced by $Ee\nu^t$, where ν is a small positive number which later will be allowed to be zero. This has the effect of disconnecting the electric field at $t = -\infty$, in such a way that at $t = -\infty$ the system can be assumed to be in a state of thermal equilibrium. Accordingly, ρ is written as

$$\rho = \rho_0 + \rho_1 e\nu^t, \quad (15)$$

where ρ_1 is the correction to the density matrix due to the scattering in the presence of the electric field. As explained in Ref. 4, the matrix elements of ρ_0 are given by

$$\langle nps\beta|\rho_0|n'p's'\beta\rangle = f_0(\epsilon_{np}{}^0)\delta_{n,n'}\delta_{p,p'}\delta_{s,s'}, \quad (16)$$

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where $f_0(\epsilon^0)$ is the Fermi distribution function. It then follows that the matrix elements of ρ_0 are independent of the electric field. Substituting Eqs. (14)–(16) into Eq. (9) and taking the matrix elements at $t=0$, we find

$$(\epsilon_{n,p}-\epsilon_{n',p'}-i\nu)\langle nps\beta|\rho_1|n'p's'\beta\rangle$$
$$=[f_0(\epsilon_{np}{}^0)-f_0(\epsilon_{n'p'}{}^0)]\langle nps\beta|V|n'p's'\beta\rangle$$
$$+\langle nps\beta|[\rho_1,V]|n'p's'\beta\rangle.$$

From this we derive, for $p=p'$,

$$(\epsilon_{np}{}^0-\epsilon_{n'p}{}^0-i\nu)\langle nsp\beta|\rho_1|n'ps'\beta\rangle$$
$$=[f_0(\epsilon_{np}{}^0)-f(\epsilon_{n'p}{}^0)]\langle nps\beta|V|n'ps'\beta\rangle$$
$$+\langle nps\beta|[\rho_1,V]|n'ps'\beta\rangle.$$

Solving the two preceding equations (to the lowest order in V) and taking the limit $\nu\to 0$, we finally get

$$\langle nps\beta|\rho_1|n'ps'\beta\rangle=i\pi(\epsilon_{n,p}{}^0-\epsilon_{n'p}{}^0)^{-1}$$
$$\sum_{n''p''s''}\langle nps\beta|V|n''p''s''\beta\rangle\langle n''p''s''\beta|V|n'ps'\beta\rangle$$
$$\times\{[f_0(\epsilon_{np}{}^0)-f_0(\epsilon_{n''p''}{}^0)]\delta(\epsilon_{np}-\epsilon_{n''p''})$$
$$+[f_0(\epsilon_{n'p'}{}^0)-f_0(\epsilon_{n''p''}{}^0)]\delta(\epsilon_{n'p}-\epsilon_{n''p''})\}. \quad (17)$$

The above expression depends on E through the δ functions and the matrix elements of V. Since we are interested in linear effects in the field, the matrix elements of ρ_1 can be expanded to the first order in β. In this way we obtain

$$\langle nps\beta|\rho_1|n'ps'\beta\rangle$$
$$=i\pi\beta(\epsilon_{np}{}^0-\epsilon_{n'p}{}^0)^{-1}\sum_{n''p''s''}\langle nps0|V|n''p''s''0\rangle$$
$$\times\langle n''p''s''0|V|n'ps'0\rangle(p_x-p_x'')\left(\frac{\partial f_0}{\partial\epsilon}\right)_{\epsilon_{n''p''}{}^0}$$
$$\times[\delta(\epsilon_{np}{}^0-\epsilon_{n''p''}{}^0)+\delta(\epsilon_{n'p}{}^0-\epsilon_{n''p''}{}^0)]. \quad (18)$$

To the same order in V, the matrix elements of ρ_1 with $n=n'$ and $p=p'$ vanish.

TABLE I. $f(\mu,\theta)$ as a function of $\mu-1$ (μ in units of mc^2), for $\theta=H/H_q=10^{-4}$. Here $2.50(-5)\equiv 2.50\times10^{-5}$, etc.

$\mu-1$	$f(\mu,\theta)$	$\mu-1$	$f(\mu,\theta)$
2.50(−5)	4.21(5)	7.25(−4)	2.54(6)
5.00(−5)	1.70(5)	7.50(−4)	2.10(6)
7.50(−5)	1.02(5)	7.75(−4)	1.88(6)
1.25(−4)	2.11(6)	8.25(−4)	2.62(6)
1.50(−4)	1.03(6)	8.50(−4)	2.21(6)
1.75(−4)	6.88(5)	8.75(−4)	2.00(6)
2.25(−4)	2.22(6)	9.25(−4)	2.70(6)
2.50(−4)	1.37(6)	9.50(−4)	2.31(6)
2.75(−4)	1.04(6)	9.75(−4)	2.11(6)
3.25(−4)	2.25(6)	1.02(−3)	2.78(6)
3.50(−4)	1.56(6)	1.05(−3)	2.40(6)
3.75(−4)	1.28(6)	1.07(−3)	2.21(6)
4.25(−4)	2.30(6)	1.12(−3)	2.86(6)
4.50(−4)	1.72(6)	1.15(−3)	2.49(6)
4.75(−4)	1.46(6)	1.17(−3)	2.31(6)
5.25(−4)	2.37(6)	1.22(−3)	2.94(6)
5.50(−4)	1.85(6)	1.25(−3)	2.58(6)
5.75(−4)	1.61(6)	1.27(−3)	2.40(6)
6.25(−4)	2.45(6)	1.32(−3)	3.01(6)
6.50(−4)	1.98(6)	1.35(−3)	2.70(6)
6.75(−4)	1.75(6)	1.37(−3)	2.54(6)

IV. TRANSVERSE CONDUCTIVITY

The average values of the current density components can now be calculated directly from the definition, Eq. (13). Introducing the quantities $J_\pm$ defined by

$$J_\pm=J_x\pm iJ_y,$$

we have

$$\langle J_\pm\rangle=(e/\Omega)\,\mathrm{Tr}(\rho\alpha_\pm)\quad(\Omega=\text{volume of the system}).$$

Selecting now the terms linear in β and using Eqs. (9)–(11) and (16), we obtain

$$\langle J_+\rangle=\frac{2e\beta}{\Omega}\sum_{nps}\sum_{n''p''s''}\langle nps0|\rho_0|n''p''s''0\rangle R_n(s,s'')$$
$$\times\delta_{n,n''}\delta_{pp''}+\frac{2e}{\Omega}\sum_{n,p,s}\sum_{n''p''s''}\langle nps0|\rho_1|n''p''s''0\rangle$$
$$\times P_n(s,s'')\delta_{n'',n+1}\delta_{p''p}$$
$$=\frac{e\beta N_e}{\Omega}+\frac{2e}{\Omega}\sum_{n,p}\sum_{s,s''}\langle nps0|\rho_1|n+1\,ps''0\rangle P_n(s,s''). \quad (19)$$

Analogously,

$$\langle J_-\rangle=\frac{e\beta N_e}{\Omega}+\frac{2e\beta}{\Omega}$$
$$\times\sum_{np}\sum_{ss''}\langle nps0|\rho_1|n-1\,ps''0\rangle P_{n-1}(s'',s), \quad (20)$$

where N_e is the total number of electrons. Changing $n\to n+1$ and interchanging s and s'' in Eq. (20), we finally get

$$\langle J_x\rangle=e\beta N_e/\Omega,\quad \sigma_{xy}=J_x/E=eN_e/\Omega H, \quad (21)$$

$$\langle J_y\rangle=-i\frac{e\beta}{\Omega}\sum_{np}\sum_{ss''}\{\langle nps0|\rho_1|n+1\,ps''0\rangle$$
$$-\langle n+1\,ps''|\rho_1|nps0\rangle\}P_n(s,s'')$$
$$=\frac{2e\beta}{\Omega}\sum_{np}\sum_{ss''}P_n(s,s'')\mathrm{Im}\langle nps0|\rho_1|n+1ps''0\rangle. \quad (22)$$

Introducing Eq. (18) into Eq. (22), we obtain

$$\sigma_{yy}=\frac{J_y}{E}=\frac{2\pi e}{\Omega H}\sum_{np}\sum_{n'p'}\omega_{n+1,p;n,p}^{-1}(p_x-p_x')\left(\frac{\partial f_0}{\partial\epsilon}\right)_{\epsilon_{n'p'}}$$
$$\times[\delta(\omega_{n+1,p;n',p'})+\delta(\omega_{n,p;n',p'})]\sum_{ss's''}P_n(s,s'')$$
$$\times\mathrm{Re}\{\langle nps0|V|n'p's'0\rangle\langle n'p's'0|V|n+1\ ps''0\rangle\},\quad(23)$$

where we have introduced the notation

$$\omega_{n,p;n',p'}\equiv\epsilon_{n,p}{}^0-\epsilon_{n',p'}{}^0.$$

Using Eqs. (21) and (23), the transverse conductivity $\sigma_\perp$, Eq. (1), can be computed once the scattering mechanism is specified. For Coulomb scattering, the relevant matrix elements have been computed in Ref. 2, and they are given by

$$\langle nps|V|n'p's'\rangle=\frac{4\pi e^2Z}{\Omega}\sum_{\alpha=1}^{N_i}\frac{1}{q^2}e^{-iq\cdot R_\alpha}I_{if},\quad(24)$$

$$I_{if}=\delta_{p_x-p_x'+q_x}\delta_{p_z-p_z'+q_z}[T_1\langle n|n'\rangle+T_2\langle n-1|n'-1\rangle],\quad(25)$$

$$T_1\equiv T_1(nn'ss')=C_1C_1'+C_3C_3',$$
$$T_2\equiv T_2(nn'ss')=C_2C_2'+C_4C_4',\quad(26)$$

$$\langle n|n'\rangle=\Psi(n,n')[e^{-i(n-n')\phi}(-)^{n'}\theta(n-n')$$
$$+e^{-i(n'-n)\phi}(-)^n\theta(n'-n)-(-)^n\delta_{n,n'}],\quad(27)$$

$$\Psi(n,n')=(n!n'!)^{-1/2}e^{-t/2}t^{(n+n')/2}\,{}_2F_0(-n,-n';-1/t),$$
$$2(H/H_q)\lambda_c^{-2}t=q_x^2+q_y^2,\quad\phi\equiv\mathrm{arc\,cot}(q_y/q_x),$$
$$\theta(x)=1,\quad x\geq0;\qquad=0,\quad x<0.\quad(28)$$

TABLE II. Same as Table I, for $\theta=H/H_q=10^{-3}$.

$\mu-1$	$f(\mu,\theta)$	$\mu-1$	$f(\mu,\theta)$
2.50(−4)	2.35(4)	7.22(−3)	2.08(5)
5.00(−4)	1.04(4)	7.47(−3)	1.72(5)
7.50(−4)	6.58(3)	7.72(−3)	1.56(5)
1.25(−3)	1.59(5)	8.22(−3)	2.10(5)
1.50(−3)	8.07(4)	8.46(−3)	1.78(5)
1.75(−3)	5.56(4)	8.71(−3)	1.63(5)
2.25(−3)	1.97(5)	9.21(−3)	2.12(5)
2.50(−3)	1.20(5)	9.46(−3)	1.83(5)
2.75(−3)	9.24(4)	9.70(−3)	1.69(5)
3.24(−3)	2.05(5)	1.02(−2)	2.14(5)
3.49(−3)	1.40(5)	1.04(−2)	1.87(5)
3.74(−3)	1.15(5)	1.07(−2)	1.74(5)
4.24(−3)	2.07(5)	1.12(−2)	2.17(5)
4.49(−3)	1.52(5)	1.14(−2)	1.92(5)
4.74(−3)	1.29(5)	1.17(−2)	1.80(5)
5.24(−3)	2.07(5)	1.22(−2)	2.20(5)
5.48(−3)	1.60(5)	1.24(−2)	1.96(5)
5.73(−3)	1.40(5)	1.27(−2)	1.85(5)
6.23(−3)	2.07(5)	1.32(−2)	2.23(5)
6.48(−3)	1.67(5)	1.34(−2)	2.01(5)
6.73(−3)	1.49(5)	1.37(−2)	1.90(5)

TABLE III. Same as Table I, for $\theta=H/H_q=10^{-2}$.

$\mu-1$	$f(\mu,\theta)$	$\mu-1$	$f(\mu,\theta)$
2.49(−3)	1.13(3)	7.00(−2)	1.81(4)
4.98(−3)	5.27(2)	7.42(−2)	1.41(4)
7.46(−3)	3.46(2)	7.47(−2)	1.25(4)
1.24(−2)	9.76(3)	7.93(−2)	1.81(4)
1.49(−2)	5.06(3)	8.17(−2)	1.45(4)
1.73(−2)	3.53(3)	8.40(−2)	1.30(4)
2.22(−2)	1.41(4)	8.86(−2)	1.81(4)
2.47(−2)	8.43(3)	9.09(−2)	1.49(4)
2.71(−2)	6.42(3)	9.32(−2)	1.35(4)
3.20(−2)	1.62(4)	9.77(−2)	1.82(4)
3.44(−2)	1.06(4)	10.00(−2)	1.52(4)
3.68(−2)	8.47(3)	0.10(0)	1.39(4)
4.16(−2)	1.73(4)	0.11(0)	1.82(4)
4.40(−2)	1.20(4)	0.11(0)	1.55(4)
4.64(−2)	9.93(3)	0.11(0)	1.43(4)
5.12(−2)	1.78(4)	0.12(0)	1.83(4)
5.36(−2)	1.29(4)	0.12(0)	1.57(4)
5.59(−2)	1.10(4)	0.12(0)	1.47(4)
6.07(−2)	1.80(4)	0.12(0)	1.84(4)
6.30(−2)	1.36(4)	0.13(0)	1.60(4)
6.54(−2)	1.18(4)	0.13(0)	1.50(4)

${}_2F_0$ is the generalized hypergeometric function.[11] Substituting Eq. (24) back in Eq. (23), summing over the final spin states, averaging over the initial spin states,

TABLE IV. Same as Table I, for $\theta=H/H_q=10^{-1}$.

$\mu-1$	$f(\mu,\theta)$	$\mu-1$	$f(\mu,\theta)$
2.39(−2)	4.88(1)	0.57(0)	2.10(3)
4.77(−2)	2.35(1)	0.58(0)	1.52(3)
7.16(−2)	1.57(1)	0.60(0)	1.30(3)
0.12(0)	5.81(2)	0.63(0)	2.19(3)
0.14(0)	3.06(2)	0.64(0)	1.63(3)
0.16(0)	2.15(2)	0.66(0)	1.41(3)
0.20(0)	1.03(3)	0.69(0)	2.26(3)
0.22(0)	6.00(2)	0.70(0)	1.72(3)
0.24(0)	4.52(2)	0.72(0)	1.52(3)
0.28(0)	1.37(3)	0.75(0)	2.32(3)
0.30(0)	8.52(2)	0.76(0)	1.81(3)
0.32(0)	6.68(2)	0.77(0)	1.61(3)
0.36(0)	1.63(3)	0.80(0)	2.38(3)
0.38(0)	1.06(3)	0.82(0)	1.88(3)
0.40(0)	8.59(2)	0.83(0)	1.69(3)
0.43(0)	1.83(3)	0.86(0)	2.42(3)
0.45(0)	1.24(3)	0.87(0)	1.96(3)
0.47(0)	1.03(3)	0.88(0)	1.77(3)
0.50(0)	1.98(3)	0.91(0)	2.46(3)
0.52(0)	1.39(3)	0.92(0)	2.02(3)
0.53(0)	1.17(3)	0.94(0)	1.85(3)

[11] *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, Chap. 6.

summing over the ion position through the relation

$$\sum_{\alpha=1}^{N_i}\sum_{\beta=1}^{N_i} e^{-i(\mathbf{q}\cdot\mathbf{R}_\alpha-\mathbf{q}'\cdot\mathbf{R}_\beta)}=N_i\delta_{\mathbf{q}-\mathbf{q}'}\,, \tag{29}$$

and, finally, integrating over the final momentum states, a long algebraic calculation, leads to

$$\sigma_{yy}=\frac{4Z^2\alpha^3}{\sqrt{2}\pi}\left(\frac{mc^2}{\hbar}\right)\lambda_c{}^3\frac{N_i}{\Omega}\left(\frac{H}{H_q}\right)^{-1/2} f(\mu,\theta)\,, \tag{30}$$

$$\sigma_{xy}=\alpha\left(\frac{mc^2}{\hbar}\right)\lambda_c{}^3\frac{N_e}{\Omega}\left(\frac{H}{H_q}\right)^{-1}, \tag{31}$$

$$f(\mu,\theta)=\sum_{n=0}^{\infty}(1-\tfrac{1}{2}\delta_{n0})\sum_{n'=0}^{\infty}(1-\tfrac{1}{2}\delta_{n'0})F(n,n')$$

$$\times\left[\int_0^\infty ds\, G(s,n,n')\Lambda_-(s,n,n')\right.$$

$$\left.+\int_0^\infty ds\, G_+(s,n,n')\Lambda_+(s,n,n')\right], \tag{32}$$

$$F(n,n')=\frac{\mu^2}{[(\mu^2-a_n{}^2)(\mu^2-a_{n'}{}^2)]^{1/2}} \tag{33}$$

$$a_n{}^2=1+2nH/H_q\equiv 1+2n\theta\,,$$

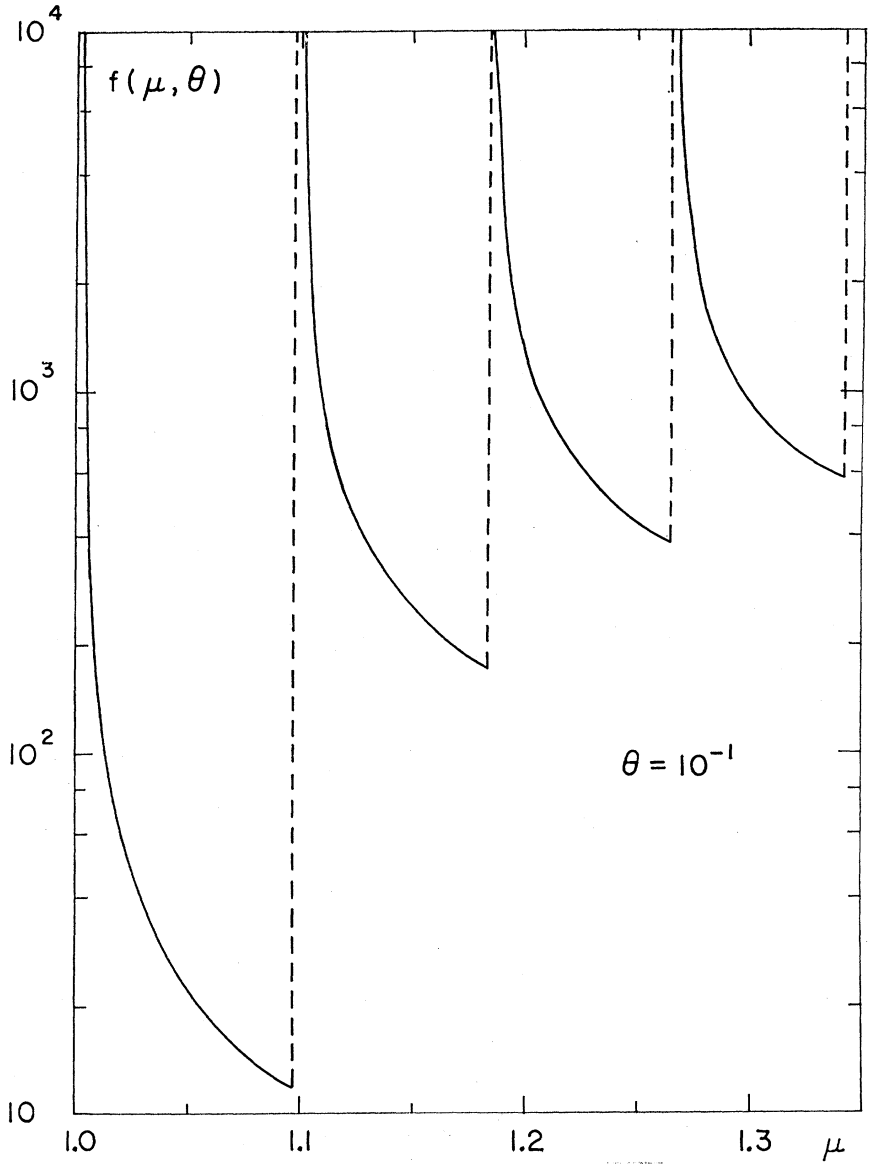

FIG. 1. The function $f(\mu,\theta)$ versus μ (in units of mc^2), for $\theta=H/H_q=10^{-1}$.

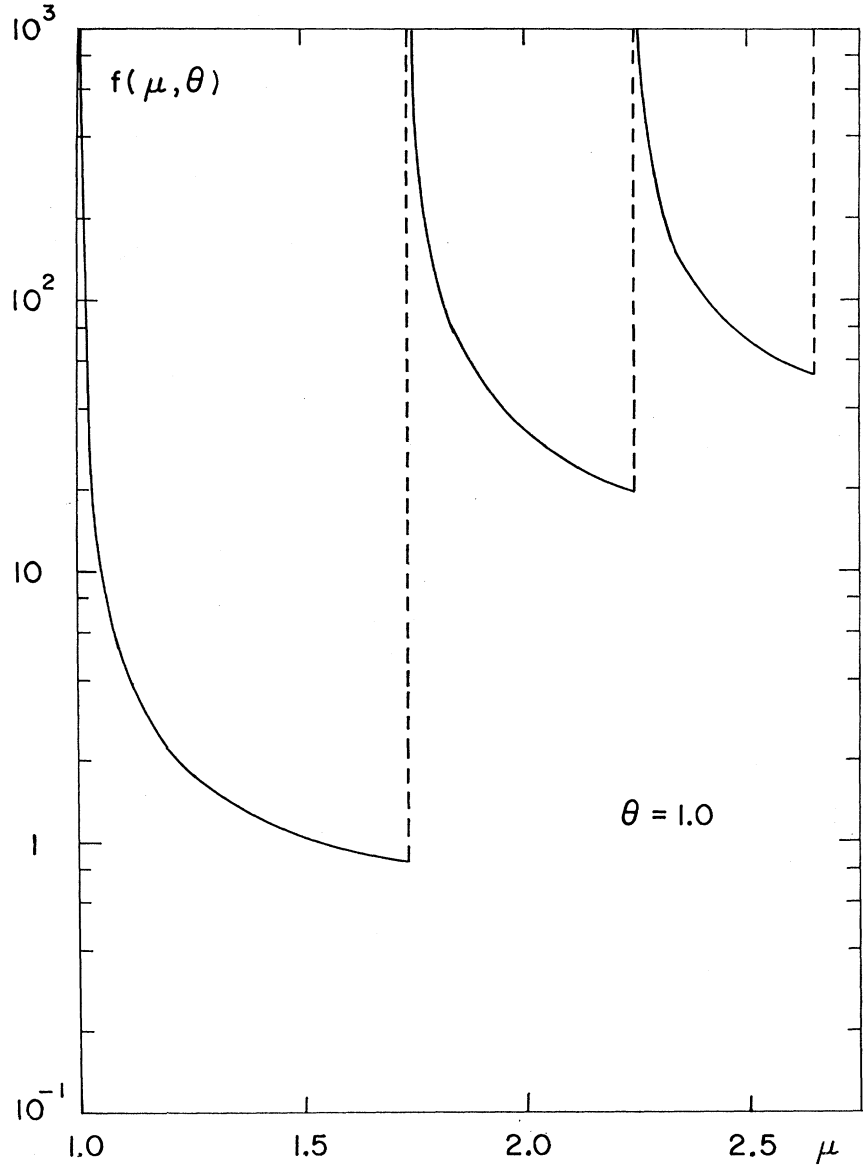

FIG. 2. The function $f(\mu,\theta)$ versus μ (in units of mc^2), for $\theta=H/H_q=1$.

$$G_{\pm}(s,n,n')=\frac{s^{1/2}}{\{s+(1/2\theta)[(\mu^2-a_n{}^2)^{1/2}\pm(\mu^2-a_{n'}{}^2)^{1/2}]^2\}^2}. \tag{34}$$

The appearance of the chemical potential $\mu\equiv\tilde{\mu}/mc^2$ is a consequence of the choice of the distribution function $f_0(\epsilon^0)$,

$$\frac{\partial}{\partial\epsilon^0}f_0(\epsilon^0)=-\delta(\epsilon^0-\tilde{\mu}),$$

which represents a degenerate electron gas. The analytic form of the functions $\Lambda_{\pm}$ is given in the Appendix.

V. NUMERICAL RESULTS AND CONCLUSIONS

The function $f(\mu,\theta)$ has been computed numerically, and the results are given in Tables I–V as a function of $\mu-1$ and $\theta=H/H_q$ in the range $10^{-4}<\theta<1$, $2\times10^{-5}<\mu-1<5$. The division of the results into groups of three is related to the discontinuous character of the function $f(\mu,\theta)$ [see Eq. (33)] due to the density of final states. In each jump the function $f(\mu,\theta)$ decreases smoothly with μ, and therefore, only three values were selected.

The behavior of $f(\mu,\theta)$ as a function of μ for fixed θ is illustrated in Figs. 1 and 2. Finally, in Table VI we compare the longitudinal[2] and transverse conductivity, $\sigma_{\parallel}$ and $\sigma_{\perp}$, and the one without magnetic field.[1] Because of the discontinuities shown in Figs. 1 and 2, the numbers selected in Table VI must be understood as average values.

The conclusion, therefore, is that longitudinal conductivity $\sigma_{\parallel}$ is greater than the one without a magnetic

TABLE V. Same as Table I, for $\theta=H/H_q=1$.

$\mu-1$	$f(\mu,\theta)$	$\mu-1$	$f(\mu,\theta)$
0.18(0)	2.36(0)	2.94(0)	4.69(2)
0.37(0)	1.30(0)	3.00(0)	3.27(2)
0.55(0)	0.99(0)	3.06(0)	2.75(2)
0.86(0)	6.23(1)	3.18(0)	5.05(2)
0.98(0)	3.36(1)	3.24(0)	3.62(2)
1.11(0)	2.41(1)	3.30(0)	3.09(2)
1.34(0)	1.48(2)	3.41(0)	5.35(2)
1.44(0)	8.50(1)	3.47(0)	3.94(2)
1.54(0)	6.37(1)	3.53(0)	3.40(2)
1.73(0)	2.31(2)	3.64(0)	5.60(2)
1.82(0)	1.40(2)	3.69(0)	4.22(2)
1.91(0)	1.09(2)	3.74(0)	3.69(2)
2.08(0)	3.06(2)	3.85(0)	5.81(2)
2.16(0)	1.94(2)	3.90(0)	4.47(2)
2.24(0)	1.54(2)	3.95(0)	3.96(2)
2.39(0)	3.70(2)	4.05(0)	6.00(2)
2.46(0)	2.43(2)	4.10(0)	4.71(2)
2.53(0)	1.97(2)	4.15(0)	4.21(2)
2.67(0)	4.24(2)	4.24(0)	6.17(2)
2.74(0)	2.88(2)	4.29(0)	4.93(2)
2.81(0)	2.38(2)	4.34(0)	4.45(2)

field by a factor ≈ 20, at least in the range $10\lesssim\rho_6\lesssim 50$. The latter is in turn greater than $\sigma_\perp$ by a factor of the order of 3, in the same range. Magnetic fields as high as 10^{13} G are now believed to be present in collapsed bodies like neutron stars, as suggested by the accepted models of pulsars. Slightly lower fields will exist in the outer part and on the surface of such stars. Our calculations are likely to be relevant in those regions, where the protons are not degenerate and the temperature of the electron gas is sufficiently low to make it completely degenerate.

TABLE VI. Comparison between the longitudinal conductivity $\sigma_{\parallel}$, the transverse conductivity $\sigma_\perp$, and the conductivity σ_0 without magnetic field, for $\theta=H/H_q=1$.

ρ_6	$\sigma_{\parallel}\times10^{-21}$ (sec^{-1})	$\sigma_0\times10^{-21}$ (sec^{-1})	$\sigma_\perp\times10^{-21}$ (sec^{-1})
8.0	69.37	3.14	1.68
14.7	74.08	3.12	1.13
22.6	71.93	3.32	1.00
31.6	72.61	3.49	0.97
41.6	76.48	3.59	0.98
52.4	81.56	3.45	1.01

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APPENDIX

Additional details on the functions appearing in the final formulas for the transverse conductivity, Eqs. (32)–(34), are presented below. The integral containing G_- in Eq. (32) is divergent if $n=n'$. To avoid this divergence we have introduced a screening factor in the Coulomb potential. This has the effect of changing the denominator of Eq. (34) in

$$s+(1/2\theta)\{[(\mu^2-a_n^2)^{1/2}\pm(\mu^2-a_{n'}^2)^{1/2}]^2+r_D^{-2}\}, \quad \text{(A1)}$$

where r_D^{-2}, for a degenerate gas, is given by[1]

$$r_D^{-2}=(2\alpha/\pi)\lambda_c^{-2}\mu(\mu^2-1)^{1/2}. \quad \text{(A2)}$$

The integrals appearing in Eq. (39) can be evaluated analytically in terms of the exponential integral function. The analytic form of the integrals, however, turns out to be exceedingly complicated, and numerical evaluation is preferable.

The analytic form of the functions $\Lambda_\pm$, Eq. (32), is given in Eq. (A3): The terms $K_i^{\pm}$, $i=1, 2, \ldots, 8$, appearing in the expression for $\Lambda_\pm$ come from the summations over the spin variables, and the functions $\Psi(n,n')$ are defined in Eq. (27):

$$\Lambda_\epsilon=[\mu-(\mu^2-\Delta^2)^{1/2}]^{-1}[K_1{}^\epsilon\Omega_1+K_2{}^\epsilon\Omega_2-K_3\Omega_3-K_4\Omega_4]H_1 +[-\mu+(\mu^2+\Delta^2)^{1/2}]^{-1}[K_5{}^\epsilon\Omega_1'+K_6{}^\epsilon\Omega_2' -K_7\Omega_3'-K_8\Omega_4']H_2, \quad \text{(A3)}$$

$$H_1\equiv H(n-n'-1)-H(n'-n),$$

$$H_2\equiv H(n-n')-H(n'-n-1),$$

$$H(x)=0 \quad \text{for} \quad x<0$$
$$=1 \quad \text{for} \quad x\geq 0,$$

$$\Omega_1=\Psi(n-1\,|\,n')\Psi(n'\,|\,n),$$

$$\Omega_2=\Psi(n-2\,|\,n'-1)\Psi(n'-1\,|\,n-1),$$

$$\Omega_3=\Psi(n'-1\,|\,n-1)\Psi(n-1\,|\,n'),$$

$$\Omega_4=\Psi(n-2\,|\,n'-1)\Psi(n'\,|\,n),$$

$$\Omega_k'=\Omega_k(n\rightarrow n+1\,|\,n'),$$

$$K_1{}^\epsilon=[a_n/4\mu^4(\mu^2-\Delta^2)] \times(\mu^2-1)^{1/2}\{\mu^2(\mu^2-\Delta^2)+\mu(\mu^2-\Delta^2)^{1/2} -\epsilon[(\mu^2-\Delta^2-1)(\mu^2-1)(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\},$$

$$K_2{}^\epsilon=[a_{n-1}/4\mu^4(\mu^2-\Delta^2)](\mu^2-\Delta^2-1)^{1/2} \times\{\mu^2(\mu^2+1)-\epsilon(\mu^2-1)[(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\},$$

$$K_3^{\epsilon}=[a_{n'}/4\mu^4(\mu^2-\Delta^2)](\mu^2-1)^{1/2}$$
$$\times\{\mu^2(\mu^2-\Delta^2)-\mu(\mu^2-\Delta^2)^{1/2}-(\mu^2-a_n^2)$$
$$\times[(\mu^2-1)(\mu^2-\Delta^2-1)]^{1/2}\},$$
$$K_4=[a_n a_{n'} a_{n-1}/4\mu^4(\mu^2-\Delta^2)](\mu^2-1)(\mu^2-\Delta^2-1)^{1/2},$$
$$\Delta^2=2H/H_q,$$
$$K_5^{\epsilon}=[a_{n+1}/4\mu^4(\mu^2+\Delta^2)]\{\mu^2(\mu^2+1)-\epsilon(\mu^2-1)$$
$$\times[(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\}(\mu^2+\Delta^2-1)^{1/2},$$
$$K_6^{\epsilon}=[a_n(\mu^2-1)^{1/2}/4\mu^4(\mu^2+\Delta^2)]$$
$$\times\{\mu^2(\mu^2+\Delta^2)+\mu[\mu^2+\Delta^2]^{1/2}$$
$$-\epsilon[(\mu^2-1)(\mu^2+\Delta^2-1)(\mu^2-a_n^2)(\mu^2-a_{n'}^2)]^{1/2}\},$$
$$K_7=[a_{n'}(\mu^2-1)^{1/2}/4\mu^4(\mu^2+\Delta^2)]$$
$$\times\{\mu^2(\mu^2+\Delta^2)-\mu[\mu^2+\Delta^2]^{1/2}$$
$$-(\mu^2-a_n^2)[(\mu^2-1)(\mu^2+\Delta^2-1)]^{1/2}\},$$
$$K_8=[a_n a_{n'} a_{n+1}/4\mu^4(\mu^2+\Delta^2)](\mu^2-1)(\mu^2+\Delta^2-1)^{1/2}.$$

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Classification of Paraparticles*

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Hartle and Taylor have shown that the cluster law imposes certain restrictions on the allowed symmetries of first-quantized systems of identical particles. We extend their results to show that all particles which are neither bosons nor fermions and which obey the cluster law can be divided into two classes, those of finite order, and those of infinite order. For every positive integer p there are two types of finite-order particle, which we call parabosons and parafermions of order p. Ordinary bosons and fermions can be fitted into this scheme as particles of order 1. We conjecture that the finite-order particles can be identified with the parafermions and parabosons of the second-quantized, parafield theory. Infinite-order particles would seem to have no analog in the second-quantized theory, as presently formulated.

I. INTRODUCTION

IN this paper we establish a simple classification of the possible types of paraparticle (that is, particles which are neither bosons nor fermions). Our results, which extend the recent work of Hartle and Taylor,[1] are based on a quantum-mechanical (that is, first-quantized) point of view.[2]

II. CLUSTER LAW AND CONSEQUENT RESTRICTIONS ON ALLOWED SYMMETRY TYPES

The cluster law requires that two isolated experiments which are sufficiently well separated must not interfere. For example, an experiment involving n particles localized in London should be completely unaffected by the presence or absence of m more particles localized on the moon, and vice versa.

Hartle and Taylor apply the cluster law to a system of $n+1$ particles, one of which is far removed and isolated from the rest. By requiring that any observation localized near the n particles should see some allowed n-particle state, they establish the following result.

(a) If a particle has $(n+1)$-particle states corresponding to the irreducible representation (IR) $D(n+1,\lambda)$ of the permutation group[3] S_{n+1}, then it must have n-particle states corresponding to *all* IR $D(n,\nu)$ of S_n whose Young diagrams can be obtained from that of $D(n+1,\lambda)$ by removing one square.

We first generalize the result (a) so as to apply to systems of $n+m$ particles divided into two isolated groups of n and m particles. To this end we note that (a) can be rephrased to say that there must be n-particle states for all $D(n,\nu)$ of S_n which are obtained when $D(n+1,\lambda)$ is restricted to S_n by holding one variable fixed.[4] Applied to $(n+m)$-particle systems, the generalization of this result can be shown to be the following.

(b) If a particle has $(n+m)$-particle states with IR $D(n+m,\lambda)$, then it must have n-particle states corresponding to *all those* $D(n,\nu)$ and m-particle states corresponding to *all those* $D(m,\mu)$ such that the outer Kronecker product $D(n,\nu)\times D(m,\mu)$ is contained in the restriction of $D(n+m,\lambda)$ to $S_n\times S_m$.

In both of the results (a) and (b) one starts from the assumed existence of some $(n+m)$-particle symmetry and deduces the existence of certain symmetries for

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[1] J. B. Hartle and J. R. Taylor, Phys. Rev. **178**, 2043 (1969).

[2] By this we mean that we describe a system of n identical particles with certain of the wave functions $f(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ of the Hilbert space $L^2(R^{3n})$. The alternative is the second-quantized approach, in which one uses the vectors generated by the action of field operators on a vacuum state. We emphasize this distinction because, even though the two points of view are known to be equivalent for ordinary bosons and fermions, their relationship is not well understood for the general paraparticle. See, for example, Ref. 9 of A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **138**, B1155 (1965), and also the concluding remarks of this paper.

[3] As is well known—see Ref. 1 or A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**, B248 (1964)—every pure state of a system of n identical particles is associated with some IR of the permutation group S_n.

[4] Indeed, this is the form in which the result is first proved in Ref. 1.