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Two-Particle Forces for Relativistic Newtonian Equations of Motion*

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Forces are found that satisfy the covariance condition of Currie and Hill for Newtonian equations of motion of two particles in four-dimensional space-time. Examples are given that can decrease, at any prescribed rate, for large particle separation. Unfortunately these latter forces become complex and even singular for certain regions of velocity space. The technique used is to simplify the covariance condition by using four-vector variables as much as possible.

I. INTRODUCTION

In Newtonian equations of motion the forces are functions of the particle positions and velocities at one time. Since time is a scalar under Galilei transformations, Galilei covariance is achieved by requiring the forces to transform as tensors. But, since time is the component of a vector under Lorentz transformations, Lorentz covariance of Newtonian equations of motion cannot be gained by simply ascribing tensor properties to the forces. So, in the main, Newtonian equations of motion were abandoned at the advent of special relativity. Now, sixty years later we find renewed interest in the problem of Lorentz-covariant Newtonian equations of motion.[3] A major step was taken with the independent discoveries by Currie[1] and by Hill[2] of differential conditions, necessary and sufficient, for Lorentz covariance of Newtonian equations of motion. These conditions are coupled nonlinear partial differential equations for the forces and will be referred to in this text as the covariance condition.

Known solutions of the covariance condition do not include any physically interesting examples. Some are for two-dimensional space-time.[3,4] The solutions of Currie and Jordan[3] for two particles in four-dimensional space-time unfortunately have accelerations proportional to the relative velocity, so that for collinear motion there is no scattering.

The primary result of this work is the discovery of several new solutions of the covariance condition for two particles in four-dimensional space-time. The accelerations are not in the direction of the relative velocity and some examples have enough arbitrariness to require asymptotic decrease at any prescribed rate for large particle separation. However, accompanying the arbitrariness in asymptotics is a restriction to certain regions of velocity space outside of which the forces become complex and singularities appear.

An alternative program for obtaining Lorentz-covariant equations of motion has been proposed by Pearle.[5] The Pearle equations are second-order differential equations for the particle positions in terms of a Lorentz scalar variable s. For example, one could require that the proper times of the particles be equal providing one scalar s to parametrize all world lines. Then these clocks must be synchronized in order to correlate particular points on the respective particle world lines through the equations of motion. In the examples worked out by Pearle,[6] this is accomplished by setting $s=0$ at the instant the particles begin to interact. Unfortunately, the Pearle dynamics are such that the interaction will range from retarded or advanced to instantaneous, depending on the Lorentz frame of the observer. Thus his covariance condition is necessarily broad enough to require that these be equivalent descriptions of the dynamics. The Currie-Hill covariance condition is more severe in that it requires an interaction, instantaneous in one Lorentz frame, to have an equivalent instantaneous interaction description in any other Lorentz frame. Superimposed upon the Pearle framework, the Currie-Hill covariance condition would demand that particle clocks synchronized on equal-time surfaces in two different Lorentz frames facilitate identical instantaneous dynamics.

In the spirit of Pearle's approach, we use four-vector variables as much as possible and find a simpler set of equations for the covariance condition. For the two-body problem, there is a reduction from 18 to four coupled equations for the forces. The price paid is the loss of individual particle speeds as independent variables, so there are more solutions of the original Currie-Hill covariance condition than of these simpler equations.[7]

In the following sections we will rederive and solve the covariance condition with four-vector variables. Section II will introduce the notation and summarize the derivation of the covariance condition. The equa-

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[1] D. G. Currie, Phys. Rev. **142**, 817 (1966).
[2] R. N. Hill, J. Math. Phys. **8**, 201 (1967).
[3] For a discussion of the history and motivation of equal-time relativistic mechanics and examples of relativistic Newtonian forces, see D. G. Currie and T. F. Jordan, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA, p. 91.
[4] E. H. Kerner, Phys. Rev. Letters **16**, 667 (1966).
[5] P. Pearle, Phys. Rev. **168**, 1429 (1968).
[6] P. Pearle (Ref. 5), Appendix A.
[7] A reduction from 18 to six equations can be made without any loss in the number of independent variables: R. N. Hill (private communication). Thus by going to four-vector variables we are actually eliminating only two equations.

tions to be solved are reduced to a manageable form in Sec. III and the solutions stated in Sec. IV. The paper is most concisely summarized by the Newtonian equations of motion [Eqs. (2.1) and (2.5)], the covariance condition [Eqs. (2.12) and (2.13)], and the new solutions [Eqs. (4.9)–(4.14)].

II. COVARIANCE CONDITION

In this section, the covariance condition will be derived for the two-body problem in four-dimensional space-time. A major simplification of the resulting equations is achieved by a restriction to four-vector variables. This is not necessary to formulate the covariance condition, as is evident from the original papers of Currie[1] and of Hill.[2] The price for this simplification is the loss of individual particle speeds as independent variables. The generalization of this approach to the n-body problem is also given.

Consider Newtonian equations of motion for a system of two particles,

$$\boldsymbol{\alpha}_a(t_a)=\mathbf{F}_a(t_1,t_2)\,|_{t_1=t_2}\quad (a=1,2). \tag{2.1}$$

In this equation, the accelerations are

$$\boldsymbol{\alpha}_a=d\mathbf{u}_a/d\tau_a, \tag{2.2}$$

where τ_a is the proper time of the ath particle. The velocities $\mathbf{u}_a$ are the space components of the four vectors

$$u_a{}^\mu=dx_a{}^\mu/d\tau_a, \tag{2.3}$$

where

$$x_a{}^\mu=(t_a,\mathbf{x}_a) \tag{2.4}$$

is the space-time position of particle a. The forces $\mathbf{F}_a$ are written as functions of their implicit arguments, the time coordinates of the particles. More explicitly, to incorporate the correct tensor properties under translation and rotation we write

$$\mathbf{F}_a(t_1,t_2)=(-1)^{a+1}[\mathbf{x}_1(t_1)-\mathbf{x}_2(t_2)]f_a(t_1,t_2)+\mathbf{u}_1(t_1)g_a(t_1,t_2)+\mathbf{u}_2(t_2)h_a(t_1,t_2), \tag{2.5}$$

where f_a, g_a, and h_a are necessarily rotational scalars and therefore are functions of the rotational scalars made from the three-vectors $\mathbf{x}_1-\mathbf{x}_2$, $\mathbf{u}_1$, and $\mathbf{u}_2$. The innovation made here is to require the functions f_a, g_a, and h_a to depend upon only the four scalars

$$y_1=u_1\cdot x,\quad y_3=x\cdot x,\qquad y_2=u_2\cdot x,\quad y_4=u_1\cdot u_2, \tag{2.6}$$

obtained from the four-vectors $x^\mu\equiv x_1{}^\mu-x_2{}^\mu$, $u_1{}^\mu$, and $u_2{}^\mu$. The dot product means

$$a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}. \tag{2.7}$$

Since under this four-scalar generalization $\mathbf{u}_a{}^2\rightarrow u_a\cdot u_a=1$ $(a=1,2)$, the particle speeds are lost as independent variables as mentioned earlier. For simplicity of notation we have dropped the arguments t_a parametrizing the particle world lines. They are to be understood and will be explicitly indicated when convenient. It is noted that the individual particle speeds occur only in the variables $u_1\cdot u_2$, $\mathbf{u}_1\cdot\mathbf{x}$, and $\mathbf{u}_2\cdot\mathbf{x}$.

The dynamics given by Eq. (2.1) is Lorentz covariant if all inertial observers connected by Lorentz transformations have the same equations of motion. To convert this statement into an analytical expression, we follow Currie[1] and Hill[2] and Lorentz-transform Eqs. (2.1) to a frame moving with an infinitesimal velocity with respect to our original frame. Then, after converting to the coordinates of the new frame, we find that our equations are no longer at equal time. This is remedied by expanding in Taylor series about an equal-time surface in the new coordinate system. The equations of motion then dictate the vanishing of an expression which we call the covariance condition.

For completeness the details of this analysis are given in Appendix A. The resulting covariance condition for the two-body problem is

$$\delta_{ij}[\alpha_{a0}(t_a)-F_{a0}(t_1,t_2)]\,|_{t_1=t_2}=(-1)^{a+1}\left\{[x_{1i}(t_1)-x_{2i}(t_2)]\frac{d}{dt_b}F_{aj}(t_1,t_2)\right\}\Bigg|_{t_1=t_2;\,a\neq b}. \tag{2.8}$$

In the next section, we will see that this is equivalent to a set of four coupled equations for the functions f_a, g_a, and h_a. But first we must define the terms. The fourth component of the acceleration of the ath particle is

$$\alpha_{a0}=du_{a0}/d\tau_a \tag{2.9}$$

for $u_{a0}=(1+\mathbf{u}_a{}^2)^{1/2}$. The fourth component of the force F_{a0} is

$$F_{a0}(t_1,t_2)=(-1)^{a+1}x_0f_a(t_1,t_2)+u_{10}g_a(t_1,t_2)+u_{20}h_a(t_1,t_2) \tag{2.10}$$

and the derivative with respect to the implicit variable t_2 is, for example $(a=1)$,

$$\frac{d}{dt_2}F_{1j}(t_1,t_2)=\sum_{\mu=0}^{3}\left[\frac{dx_{2\mu}}{dt_2}\frac{\partial}{\partial x_{2\mu}}+\frac{du_{2\mu}}{dt_2}\frac{\partial}{\partial u_{2\mu}}\right]F_{1j}(t_1,t_2). \tag{2.11}$$

Now by taking $i\neq j$ in Eq. (2.8), we see that both sides of the equation must vanish separately, yielding

$$0=\alpha_{a0}(t_a)-F_{a0}(t_1,t_2)\,|_{t_1=t_2} \tag{2.12}$$

and

$$0=\frac{d}{dt_b}\mathbf{F}_a(t_1,t_2)\Bigg|_{t_1=t_2;\,a\neq b}. \tag{2.13}$$

For an n-particle system with equations of motion

$$\boldsymbol{\alpha}_a(t)=\mathbf{F}_a(t_1,t_2\cdots t_n)\,|_{t_1=t_2\cdots=t_n=t} \tag{2.14}$$

the generalization of Eq. (2.5) for the forces is

$$\mathbf{F}_a(t_1,t_2\cdots t_n)=\sum_{b=1}^{n}\Big[\sum_{c=1;\,b<c}^{n}(\mathbf{x}_b-\mathbf{x}_c)f_{abc}+\mathbf{u}_b g_{ab}\Big]. \quad (2.15)$$

The covariance condition, as shown in Appendix B, is just the n-particle generalization of Eq. (2.8),

$$\delta_{ij}[\alpha_{a0}-F_{a0}(t_1,t_2\cdots t_n)|_{t_1=t_2\cdots=t_n}]$$
$$=\left[\sum_{b=1}^{n}(x_a-x_b)_i\frac{d}{dt_b}F_{aj}(t_1,t_2\cdots t_b\cdots t_n)\right]\Bigg|_{t_1=t_2=\cdots=t_n}, \quad (2.16)$$

where $(x_a-x_b)_i$ is the ith component of $(\mathbf{x}_a-\mathbf{x}_b)$. Equation (2.16) exhibits a major difficulty of this approach to relativistic classical mechanics. For $n\leq 3$, both sides of Eq. (2.16) must vanish separately and one obtains the zeroth component of the force equation,

$$\alpha_{a0}(t_a)=F_{a0}(t_1,t_2\cdots t_n)|_{t_1=t_2=\cdots t_n} \quad (n\leq 3). \quad (2.17)$$

But for $n>3$, Eq. (2.17) is no longer necessary for the solution of Eq. (2.16). However, Eqs. (B3), (B8), and (B2), with the last rewritten in the 0′ frame, combine with Eq. (B10) to provide a general proof of Eq. (2.17) for all n. The restriction to four-scaler independent variables forces the equations of motion to take a four-vector form.

In the next sections we will further simplify Eqs. (2.12) and (2.13) and exhibit some solutions.

III. FURTHER SIMPLIFICATION

The covariance conditions for the two-body system with Lorentz-scalar variables for the functions f_a, g_a, and h_a are Eqs. (2.12) and (2.13). Equations (2.12) are purely algebraic and provide two linear constraints on the functions f_a, g_a, and h_a. The remaining six differential equations, (2.13), are readily reduced to four coupled equations for f_a, g_a, and h_a. In addition, a sufficient set of two coupled equations follows from the assumption of an additional linear constraint on f_a, g_a, and h_a. It is this last set of equations that has yielded the largest and most interesting set of solutions.

In order to reduce Eqs. (2.12) and (2.13), we first note that Eqs. (2.1) and (2.12) imply

$$0=u_1\cdot\alpha_1=u_1{}^\mu(x_\mu f_1+u_{1\mu}g_1+u_{2\mu}h_1)$$
$$=y_1f_1+g_1+y_4h_1$$

or

$$g_1=-(y_1f_1+y_4h_1). \quad (3.1)$$

Likewise,

$$u_2\cdot\alpha_2=0\rightarrow h_2=y_2f_2-y_4g_2. \quad (3.2)$$

For convenience we relabel

$$f_1\equiv f,\quad f_2\equiv F,\qquad h_1\equiv g,\quad g_2\equiv G, \quad (3.3)$$

such that Eqs. (2.1) become

$$\boldsymbol{\alpha}_1=\mathbf{F}_1(t_1,t_2)|_{t_1=t_2}=(\mathbf{x}-y_1\mathbf{u}_1)f+(\mathbf{u}_2-y_4\mathbf{u}_1)g, \quad (3.4)$$
$$\boldsymbol{\alpha}_2=\mathbf{F}_2(t_1,t_2)|_{t_1=t_2}=-(\mathbf{x}-y_2\mathbf{u}_2)F+(\mathbf{u}_1-y_4\mathbf{u}_2)G. \quad (3.5)$$

The differential condition (2.13) can be written for $a=1$:

$$0=u_{20}\frac{d}{dt_2}\mathbf{F}_1(t_1,t_2)\Big|_{t_1=t_2}$$
$$=\sum_{\mu=0}^{3}\left(-u_{2\mu}\frac{\partial}{\partial x_\mu}+F_{2\mu}\frac{\partial}{\partial u_{2\mu}}\right)\mathbf{F}_1 \quad (3.6)$$

for $F_{a\mu}\equiv F_{a\mu}(t_1,t_2)|_{t_1=t_2}$. Carrying out the differentiation of (3.6) and using (3.4) and (3.5), we obtain

$$0=x_k(Df+gF)-u_{1k}\{y_1Df+y_4Dg+y_4f$$
$$+g[G+F(y_1-y_2y_4)-G(1-y_4{}^2)]\}$$
$$+u_{2k}[Dg+f+g(y_4G-y_2F)] \quad (3.7)$$

for

$$D=y_4\frac{\partial}{\partial y_1}+[1+F(y_3-y_2{}^2)-G(y_1-y_2y_4)]\frac{\partial}{\partial y_2}$$
$$+2y_2\frac{\partial}{\partial y_3}+[F(y_1-y_2y_4)-G(1-y_4{}^2)]\frac{\partial}{\partial y_4}. \quad (3.8)$$

Now by looking at the projections of Eq. (3.7) in directions perpendicular to $\mathbf{x}$, $\mathbf{u}_1$, and $\mathbf{u}_2$, respectively, one finds that the coefficients of x_k, u_{1k}, and u_{2k} must vanish separately. Further, since the coefficient of u_{1k} vanishes as a consequence of the vanishing of the other two coefficients, we are left with

$$0=Df+gF, \quad (3.9)$$
$$0=Dg+f+g(y_4G-y_2F). \quad (3.10)$$

The corresponding equations for $a=2$ are

$$0=\mathfrak{D}F+Gf, \quad (3.11)$$
$$0=\mathfrak{D}G+F+G(y_4g+y_1f), \quad (3.12)$$

with

$$\mathfrak{D}=-[1+f(y_3-y_1{}^2)+g(y_2-y_1y_4)]\frac{\partial}{\partial y_1}-y_4\frac{\partial}{\partial y_2}$$
$$-2y_1\frac{\partial}{\partial y_3}-[f(y_2-y_1y_4)+g(1-y_4{}^2)]\frac{\partial}{\partial y_4}. \quad (3.13)$$

For "identical" particles, $f\rightarrow F$ and $g\rightarrow G$ under the exchange $y_1\rightarrow -y_2$ and Eqs. (3.9) and (3.10) become equivalent to Eqs. (3.11) and (3.12). The solutions discussed in Eqs. (4.1)–(4.7) are all of this type.

Algebraic relations proved in Appendix C, which reduce the four equations (3.9)–(3.12) to two equations, are

$$f=c_1g, \quad (3.14)$$
$$F=c_2G \quad (3.15)$$

for

$$c_1=\frac{1}{y_1^2-y_3}(d^{1/2}-y_1y_4+y_2), \quad (3.16)$$

$$c_2=\frac{1}{y_2^2-y_3}(d^{1/2}-y_1+y_2y_4), \quad (3.17)$$

and

$$d=y_1^2+y_2^2+y_3(y_4^2-1)-2y_1y_2y_4. \quad (3.18)$$

Then Eqs. (3.9)–(3.12) become

$$0=(D+F/c_1)f, \quad (3.19)$$

$$0=(\mathfrak{D}+f/c_2)F, \quad (3.20)$$

where

$$D=y_4\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+2y_2\frac{\partial}{\partial y_3} -Fd^{1/2}\left(\frac{1}{c_2}\frac{\partial}{\partial y_2}+\frac{\partial}{\partial y_4}\right), \quad (3.21)$$

$$\mathfrak{D}=-\left(\frac{\partial}{\partial y_1}+y_4\frac{\partial}{\partial y_2}+2y_1\frac{\partial}{\partial y_3}\right) +fd^{1/2}\left(\frac{1}{c_1}\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_4}\right). \quad (3.22)$$

Thus Eqs. (3.14), (3.15), (3.19), and (3.20) provide us with sufficient conditions for the covariance of (3.4) and (3.5) with a system of two coupled equations as opposed to the four equations (3.9)–(3.12) which are both necessary and sufficient for covariance.

Unfortunately the radical $d^{1/2}$ appearing in Eqs. (3.16) and (3.17) is pure imaginary for a range of the particle velocities $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus, in general, these reduced equations have complex forces for solutions. Another set of equations reducing the four equations (3.9)–(3.12) to two equations is obtained by replacing c_1 and c_2 by their complex conjugate in Eqs. (3.14)–(3.22).

IV. SOLUTIONS

The existence of solutions of the covariance condition has already been demonstrated. For the case of one space and one time dimension, there is the simultaneously Galilei invariant interaction[3,4] and the constant force example.[3] For three space dimensions, Currie and Jordan[3] have published a set of solutions arising from the separate vanishing of the linear and nonlinear parts of their equations. Unfortunately, these solutions have accelerations parallel to the relative velocity of the two particles, so the direction of the relative velocity cannot change during the course of the interaction. Another set of solutions to the covariance condition is found when one of the two particles is given an infinite mass.[3]

The motivation for this study was a desire to incorporate the simplicity afforded by tensor notation and hopefully arrive at a more reasonable set of equations to solve. This has indeed been the case, and the following solutions are exhibited proving at least the self-consistency of the covariance condition with the additional requirement of four scalar variables. All of these examples are solutions of Eqs. (3.9)–(3.12), while only the third is also a solution of Eqs. (3.19) and (3.20).

$$(1)\quad f=F=0, \quad (4.1)$$

$$g=G=a/(y_4^2-1)^{1/2} \quad (4.2)$$

(here a is an arbitrary constant);

$$(2)\quad f=g=1/(1-y_4-y_1), \quad (4.3)$$

$$F=G=1/(1-y_4+y_2); \quad (4.4)$$

$$(3)\quad g=f/c_1, \quad (4.5)$$

$$f=a(c_2)(1+y_4)/d^{1/2}, \quad (4.6)$$

$$G=F/c_2, \quad (4.7)$$

$$F=b(c_1)(1+y_4)/d^{1/2}, \quad (4.8)$$

where a and b are arbitrary functions of c_1 and c_2, respectively. The functions c_1, c_2, and d are given by Eqs. (3.16)–(3.18).

For $d^{1/2}$ pure imaginary, an additional set of solutions is given by Eqs. (4.5)–(4.8), with c_1 and c_2 replaced by their complex conjugates.

These examples, though proving the existence of solutions of Eqs. (3.9)–(3.12), (3.19), and (3.20), are still pathological as far as physics is concerned. Examples (1) and (2) are nonvanishing for arbitrarily large particle separation. Although example (3) can be given any rate of asymptotic decrease through the arbitrary functions $a(c_1)$ and $b(c_2)$, the forces become singular for equal velocities through the vanishing of d at this point. In addition, the forces become complex, since for regions of velocity space the variable $d^{1/2}$ is pure imaginary. So the search for physical solutions of the covariance condition must continue.

In conclusion, the examples satisfying the covariance condition will be converted back to particle velocities and positions and given in their Newtonian form. The particle accelerations $\mathbf{a}_b=d^2\mathbf{x}_b/dt^2$ are related to $\boldsymbol{\alpha}_b$ by

$$\boldsymbol{\alpha}_b=(1+\mathbf{u}_b^2)[(\mathbf{u}_b\cdot\mathbf{a}_b)\mathbf{u}_b+\mathbf{a}_b],$$

and the solutions (4.1)–(4.8) become

$$(1)\quad \boldsymbol{\alpha}_1=\frac{a}{[(u_1\cdot u_2)^2-1]^{1/2}}(\mathbf{u}_2-u_1\cdot u_2\mathbf{u}_1), \quad (4.9)$$

$$\boldsymbol{\alpha}_2=\frac{a}{[(u_1\cdot u_2)^2-1]^{1/2}}(\mathbf{u}_1-u_1\cdot u_2\mathbf{u}_2) \quad (4.10)$$

for a constant;

$$(2)\quad \boldsymbol{\alpha}_1=(1-u_1\cdot u_2+\mathbf{x}\cdot\mathbf{u}_1)^{-1} \times[\mathbf{x}+\mathbf{u}_1(\mathbf{x}\cdot\mathbf{u}_1\text{-}u_1\cdot u_2)+\mathbf{u}_2], \quad (4.11)$$

$$\boldsymbol{\alpha}_2=(1-u_1\cdot u_2-\mathbf{x}\cdot\mathbf{u}_2)^{-1} \times[-\mathbf{x}-\mathbf{u}_2(\mathbf{x}\cdot\mathbf{u}_2+u_1\cdot u_2)+\mathbf{u}_1]; \quad (4.12)$$

(3) $$\boldsymbol{\alpha}_1=(1/\xi)(1+u_1\cdot u_2)a(1/r\eta)\times[\hat{x}+\mathbf{u}_1(\hat{x}\cdot\mathbf{u}_1-\zeta u_1\cdot u_2)+\zeta\mathbf{u}_2], \quad (4.13)$$

$$\boldsymbol{\alpha}_2=(1/\xi)(1+u_1\cdot u_2)b(1/r\zeta)\times[-\hat{x}-\mathbf{u}_2(\hat{x}\cdot\mathbf{u}_2+\eta u_1\cdot u_2)+\eta\mathbf{u}_1] \quad (4.14)$$

for

$$\xi=[1+(\hat{x}\cdot\mathbf{u}_1)^2+(\hat{x}\cdot\mathbf{u}_2)^2+(\hat{x}\cdot\mathbf{u}_1\hat{x}\cdot\mathbf{u}_2)^2-(u_1\cdot u_2+\hat{x}\cdot\mathbf{u}_1\hat{x}\cdot\mathbf{u}_2)^2]^{1/2}, \quad (4.15)$$

$$\zeta=\frac{1}{1-(u_1\cdot u_2)^2}[\xi+\hat{x}\cdot(\mathbf{u}_2-u_1\cdot u_2\mathbf{u}_1)],$$

$$\eta=\frac{1}{1-(u_1\cdot u_2)^2}[\xi-\hat{x}\cdot(\mathbf{u}_1-u_1\cdot u_2\mathbf{u}_2)], \quad (4.16)$$

$$r=|\mathbf{x}|, \quad \mathbf{x}=r\hat{x},$$

and a, b are arbitrary functions of their respective arguments. An additional set of solutions is also obtained by substituting $-\xi$ for ξ in Eqs. (4.16).

V. SUMMARY

We have attempted to demonstrate the usefulness of a tensor formalism even when the equations are not manifestly covariant. The Newtonian equations of motion are cumbersome at best, when discussed in the framework of Minkowski space. But nonetheless, their very existence, and the possibility of an alternative, more easily solved relativistic dynamics than light-cone and other action-at-a-distance approaches,[3] makes the quest interesting and maybe even worthwhile.

The primary success here has been the reduction of the covariance condition for the two-body problem from 18 to four coupled nonlinear partial differential equations for the forces. This has been done at the expense of two independent variables, the speeds of the individual particles $\mathbf{u}_1^2$ and $\mathbf{u}_2^2$. In addition, an algebraic relation was discovered which further reduced these four coupled equations to a sufficient two equations. Solutions have been found for both the four and two coupled equations, though some are physically disqualified for lack of an asymptotic condition and others become complex or singular. It is emphasized that the equations solved here are in all cases sufficient but not necessary for the Currie-Hill covariance condition. This is because of the restriction to Lorentz-scalar variables that was used to simplify the equations. Hopefully, this simpler set of equations will yield to further analysis and lead to more interesting solutions of the covariance condition.

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APPENDIX A: COVARIANCE CONDITION FOR TWO-BODY SYSTEM

The covariance condition is the statement that Eqs. (2.1), as recorded by observer O of Fig. 1, also describe the particle dynamics for all observers related by Lorentz transformations. For example, observer O' of Fig. 1, related to O by a Lorentz transformation, sees the world lines $\mathbf{x}_a'(t')$ and velocities $\mathbf{u}_a'(t')$ described by the force equation

$$\boldsymbol{\alpha}_a'[t']=\mathbf{F}_a'[t_1',t_2']|_{t_1'=t_2'=t'}, \quad (A1)$$

where $\mathbf{F}_a'[t_1',t_2']|_{t_1'=t_2'}$ is the same functions of its arguments as $F_a(t_1,t_2)|_{t_1=t_2}$ but with the particle positions and velocities as observed in O' along an equal-time surface in O'. For the observer O, the acceleration at event M_1 on the world line of particle 1 is determined by the particle positions and velocities at M_1 and M_2. However, the observer O' sees an acceleration at M_1 that is determined by particle positions and velocities at M_1 and N. Thus, for $a=1$, the force of Eq. (A1) is obtained from the force of Eq. (2.1) by replacing all positions and velocities at the events M_1 and M_2 in O by their corresponding positions and velocities at M_1 and N in O'.

Under an infinitesimal Lorentz transformation, all four-vectors transform like the particle positions,

$$\mathbf{x}_a \xrightarrow{\boldsymbol{\varepsilon}} \mathbf{x}_a'=\mathbf{x}_a+\boldsymbol{\varepsilon}t_a, \quad (A2)$$

$$t_a \xrightarrow{\boldsymbol{\varepsilon}} t_a'=t_a+\boldsymbol{\varepsilon}\cdot\mathbf{x}_a. \quad (A3)$$

Thus, if we consider the effect of an infinitesimal Lorentz transformation on Eq. (2.1), we obtain

$$\boldsymbol{\alpha}_1(t_1) \xrightarrow{\boldsymbol{\varepsilon}} \boldsymbol{\alpha}_1'(t_1)=\boldsymbol{\alpha}_1(t_1)+\boldsymbol{\varepsilon}\alpha_{10}(t_1), \quad (A4)$$

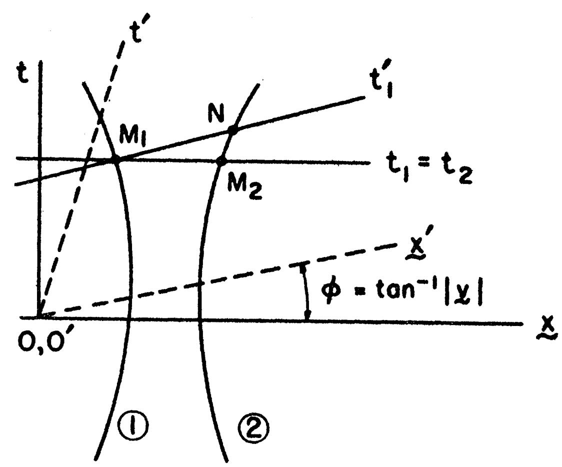

FIG. 1. Two-body scattering problem.

$$\mathbf{F}_1(t_1,t_2)|_{t_1=t_2} \xrightarrow{\boldsymbol{\varepsilon}} \mathbf{F}_1'(t_1,t_2)|_{t_1=t_2}$$
$$=\mathbf{F}_1(t_1,t_2)|_{t_1=t_2}+\boldsymbol{\varepsilon}F_{10}(t_1,t_2)|_{t_1=t_2}, \quad \text{(A5)}$$

where

$$\alpha_{10}(t_1)=(1/u_{10})\mathbf{u}_1\cdot\boldsymbol{\alpha}_1(t_1) \quad \text{(A6)}$$

and

$$F_{10}(t_1,t_2)|_{t_1=t_2}=(u_{10}g_1+u_{20}h_1)_{(t_1,t_2)}|_{t_1=t_2}. \quad \text{(A7)}$$

The Lorentz-transformed $\mathbf{F}_1'(t_1,t_2)$ of Eq. (A5) is dependent upon the event coordinates at M_1 and M_2 (see Fig. 1), as observed from O'. But the acceleration observed at M_1 by O' must be due to a force dependent upon the event coordinates at M_1 and N if our definition of covariance is to mean anything. In addition, the equation of motion as seen by O' must be parametrized by the time t' of O'. To carry out these instructions, we note first that a parameter transformation is facilitated by Eq. (A3),

$$\mathbf{F}_1'(t_1,t_2)=\mathbf{F}_1'(t_1(t_1'),t_2(t_2'))\equiv\mathbf{F}_1'[t_1',t_2'] \quad \text{(A8)}$$

for

$$t_a(t_a')=t_a'-\boldsymbol{\varepsilon}\cdot\mathbf{x}_a(t_a')+O(\boldsymbol{\varepsilon}^2).$$

The force obtained in this manner is still dependent upon events M_1 and M_2 only with the time, position, and velocity values as observed by O'. To obtain a force dependent upon the events M_1 and N in O', we express $\mathbf{F}_1'[t_1',t_2']$ as a Taylor series about the event N,

$$\begin{aligned}\mathbf{F}_1'[t_1',t_2']&=\mathbf{F}_1'[t_1',t_1'-\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1')}]\\&=\mathbf{F}_1'[t_1',t_1']-\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1')}\frac{d}{dt_2}\mathbf{F}_1'[t_1',t_2]\Big|_{t_2=t_1'}. \quad \text{(A9)}\end{aligned}$$

Thus, with Eqs. (A5), (A8), and (A9), we obtain a relation between the force dependent upon events M_1 and M_2 as seen from O and the force dependent upon events M_1 and N as observed by O',

$$\begin{aligned}\mathbf{F}_1'[t_1',t_1']&=\mathbf{F}_1'[t_1',t_2']+\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1')}\frac{d}{dt_2}\mathbf{F}_1'[t_1',t_2]\Big|_{t_2=t_1'}\\&=\mathbf{F}_1(t_1,t_1)+\boldsymbol{\varepsilon}F_{10}(t_1,t_1)\\&\quad+\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1')}\frac{d}{dt_2}\mathbf{F}_1'[t_1',t_2]\Big|_{t_2=t_1'}. \quad \text{(A10)}\end{aligned}$$

The acceleration at M_1 as observed by O' can be parametrized by t' again by use of (A3):

$$\boldsymbol{\alpha}_1'(t_1)=\boldsymbol{\alpha}_1'(t_1(t_1'))\equiv\boldsymbol{\alpha}_1'[t_1']. \quad \text{(A11)}$$

Now with (A4), (A10), and (A11) the covariance condition

$$\boldsymbol{\alpha}_1'[t_1']=\mathbf{F}_1'[t_1',t_1']$$

becomes

$$\begin{aligned}\boldsymbol{\alpha}_1(t_1)+\boldsymbol{\varepsilon}\alpha_{10}(t_1)&=\mathbf{F}_1(t_1,t_1)+\boldsymbol{\varepsilon}F_{10}(t_1,t_1)\\&\quad+\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1')}\frac{d}{dt_2}\mathbf{F}_1'[t_1',t_2]\Big|_{t_2=t_1'},\end{aligned}$$

which to $O(\boldsymbol{\varepsilon}^2)$ is equivalent to

$$\begin{aligned}\boldsymbol{\alpha}_1(t_1)+\boldsymbol{\varepsilon}\alpha_{10}(t_1)&=\mathbf{F}_1(t_1,t_1)+\boldsymbol{\varepsilon}F_{10}(t_1,t_1)\\&\quad+\boldsymbol{\varepsilon}\cdot(\mathbf{x}_1-\mathbf{x}_2)_{(t_1)}\frac{d}{dt_2}\mathbf{F}_1(t_1,t_2)\Big|_{t_1=t_2}. \quad \text{(A12)}\end{aligned}$$

Now eliminating $\boldsymbol{\varepsilon}$ from Eq. (A12), we obtain

$$\begin{aligned}\delta_{ij}[\alpha_{10}(t_1)-F_{10}(t_1,t_1)]\\=(x_{1i}-x_{2i})_{(t_1)}\frac{d}{dt_2}F_{1j}(t_1,t_2)\Big|_{t_1=t_2}, \quad \text{(A13)}\end{aligned}$$

the covariance condition for the two-body problem with the restriction to Lorentz-scalar variables.

APPENDIX B: COVARIANCE CONDITION FOR n-BODY SYSTEM

For the n-body system the Newtonian force is given by Eq. (2.15),

$$\mathbf{F}_a(t_1,t_2,\ldots,t_n)=\sum_{b=1}^{n}\Big[\sum_{c=1;\,b<c}^{n}(\mathbf{x}_b-\mathbf{x}_c)f_{abc}+\mathbf{u}_b g_{ab}\Big], \quad \text{(B1)}$$

where f_{abc} and g_{ab} are taken to be functions of the Lorentz scalars formed from the four-vectors

$$\{(x_e-x_f)^\mu(x_h-x_k)_\mu,\ (x_e-x_f)^\mu u_{h\mu},\ u_e{}^\mu u_{f\mu}\}\quad \text{for}\quad e,f,h,k=1,2,\ldots,n.$$

The equations of motion in the O coordinate frame of Fig. 2 are [Eq. (2.14)]

$$\boldsymbol{\alpha}_a(t)=\mathbf{F}_a(t_1,t_2\cdots t_n)|_{t_1=t_2=\cdots=t_n=t}\quad(a=1,\ldots,n). \quad \text{(B2)}$$

By covariance we must have the same equation in the O' coordinate frame of Fig. 2 connected by an infinitesi-

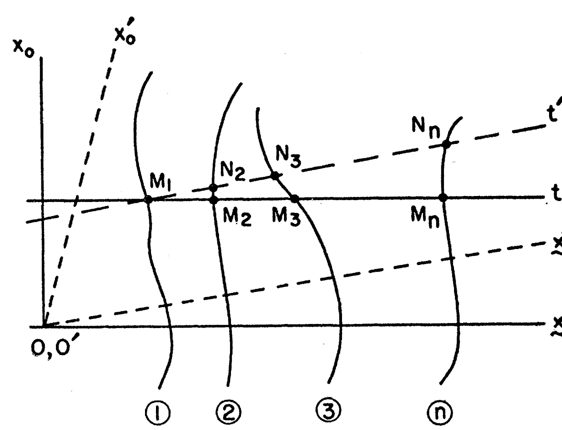

FIG. 2. n-body scattering problem.

mal Lorentz rotation to the O frame,

$$\boldsymbol{\alpha}_a'[t']=\mathbf{F}_a'[t_1',t_2',\ldots,t_n']_{t_1'=t'_2=\cdots t_n'=t'} \quad (a=1,\ldots,n). \quad \text{(B3)}$$

The force of Eq. (B2) is the same function of positions and velocities from the events $M_1, M_2, \ldots, M_n$ as seen from O as the force of Eq. (B3) is of the events $M_1, N_2,\ldots, N_n$ as seen from O'.

Under an infinitesimal Lorentz transformation, we have for particle 1

$$\boldsymbol{\alpha}_1(t_1)\rightarrow \boldsymbol{\alpha}_1'(t_1)=\boldsymbol{\alpha}_1(t_1)+\boldsymbol{\varepsilon}\alpha_{10}(t_1) \quad \text{(B4)}$$

and

$$\mathbf{F}_1(t_1,t_2,\ldots,t_n)\rightarrow \mathbf{F}_1'(t_1,t_2,\ldots,t_n) =\mathbf{F}_1(t_1,t_2,\ldots,t_n)+\boldsymbol{\varepsilon}F_{10}(t_1,t_2,\ldots,t_n) \quad \text{(B5)}$$

for

$$F_{10}(t_1,t_2,\ldots,t_n) =\sum_{b=1}^{n}\Big[\sum_{c=1;\,b<c}^{n}(x_{b0}-x_{c0})f_{abc}+u_{b0}g_{ab}\Big]. \quad \text{(B6)}$$

In the same manner as for the two-body case we transform to the time parameters of O' by use of Eq. (A3), with $a=1, 2,\ldots, n$:

$$\mathbf{F}_1'(t_1(t_1'),t_2(t_2'),\ldots,t_n(t_n'))\equiv\mathbf{F}_1'[t_1',t_2',\ldots,t_n']. \quad \text{(B7)}$$

But the force of Eq. (B7) is still in terms of the positions and velocities at the events $M_1, M_2, \ldots, M_n$, though in the language of the O' coordinate frame. To obtain a force explicitly dependent upon the events $M_1, N_2, \ldots, N_n$ in O', we express $F_1'[t_1',t_2',\ldots,t_n']$ as a Taylor series about the events $N_2, N_3,\ldots, N_n$,

$$\mathbf{F}_1'[t_1',t_2',\ldots,t_n'] =\mathbf{F}_1'[t_1',t_1',\ldots,t_1']-\boldsymbol{\varepsilon}\cdot\sum_{k=2}^{n}(\mathbf{x}_1-\mathbf{x}_k)\frac{d}{dt_k} \times\mathbf{F}_1'[t_1',\ldots,t_1',t_k,t_1',\ldots,t_1']_{t_k=t_1'}. \quad \text{(B8)}$$

Thus with Eqs. (B3), (B5), and (B7) we have for Eq. (B8)

$$[\mathbf{F}_1(t_1,t_2,\ldots t_n)+\boldsymbol{\varepsilon}F_{10}(t_1,t_2,\ldots t_n)]\big|_{t_1=t_2=\cdots t_n} =\boldsymbol{\alpha}_1'[t_1']-\boldsymbol{\varepsilon}\cdot\sum_{k=2}^{n}(\mathbf{x}_1-\mathbf{x}_k)\frac{d}{dt_k} \times\mathbf{F}_1'[t_1',\ldots,t_1',t_k,t_1',\ldots,t_1']\Big|_{t_k=t_1'} \quad \text{(B9)}$$

or, with (B4) and (B2) and neglecting terms to $O(\boldsymbol{\varepsilon}^2)$,

$$\delta_{ij}[\alpha_{10}-F_{10}(t_1,t_2,\ldots,t_n)]_{t_1=t_2=\cdots=t_n} =\sum_{k=2}^{n}(x_{1i}-x_{ki})\frac{d}{dt_k} \times F_{1j}(t_1,t_2,\ldots,t_k,\ldots,t_n)_{t_1=t_2=\cdots=t_n}, \quad \text{(B10)}$$

the covariance condition for the n-body problem with the restriction to Lorentz-scalar variables.

APPENDIX C: ALGEBRAIC RELATION

In this appendix we will prove that Eqs. (3.9), (3.14), and (3.15) imply Eq. (3.10). The proof that Eqs. (3.11), (3.14), and (3.15) imply (3.12) goes through in the same manner. Thus we have the contention

$$\left.\begin{array}{l}0=Df+gF\\ f=c_1g\\ F=c_2G\end{array}\right\}\Rightarrow 0=Dg+f+g(y_4G-y_2F) \quad \text{(C1)}$$

for D given by Eq. (3.21).

The algebra of the proof is simplified by the following choice of variables:

$$\begin{aligned} r_1&=y_1^2-y_3, & s_1&=y_2^2-y_3,\\ r_2&=y_1y_4-y_2, & s_2&=y_1-y_2y_4,\\ z&=y_4^2-1, & w&=y_3y_4-y_1y_2. \end{aligned} \quad \text{(C2)}$$

In this notation we have for c_1 and c_2 of Eqs. (C1)

$$c_1=(1/r_1)(d^{1/2}-r_2),\quad c_2=(1/s_1)(d^{1/2}-s_2) \quad \text{for}\quad d=r_2^2-r_1z=s_2^2-s_1z. \quad \text{(C3)}$$

The right-hand side of Eq. (C1) can be written

$$\begin{aligned} Dg&=-f-g(y_4G-y_2F)\\ &=-g[c_1+(y_4-c_2y_2)G]. \end{aligned} \quad \text{(C4)}$$

In addition,

$$\begin{aligned} Df&=D(c_1g)=-gF=-c_2gG,\\ Dg&=-(g/c_1)(Dc_1+c_2G). \end{aligned} \quad \text{(C5)}$$

The proof of (C1) consists of demonstrating that Eqs. (C4) and (C5) are equivalent.

A direct calculation yields

$$Dc_1=c_1\left(c_1-\frac{c_2d^{1/2}G}{1+y_4}\right), \quad \text{(C6)}$$

which, upon substitution into Eq. (C5), gives

$$Dg=-g\left[c_1-c_2G\left(\frac{d^{1/2}}{1+y_4}-\frac{1}{c_1}\right)\right]. \quad \text{(C7)}$$

The algebraic relation

$$c_2y_2-y_4=c_2\left(\frac{d^{1/2}}{1+y_4}-\frac{1}{c_1}\right) \quad \text{(C8)}$$

can be proved to yield the final expression, Eq. (C4).